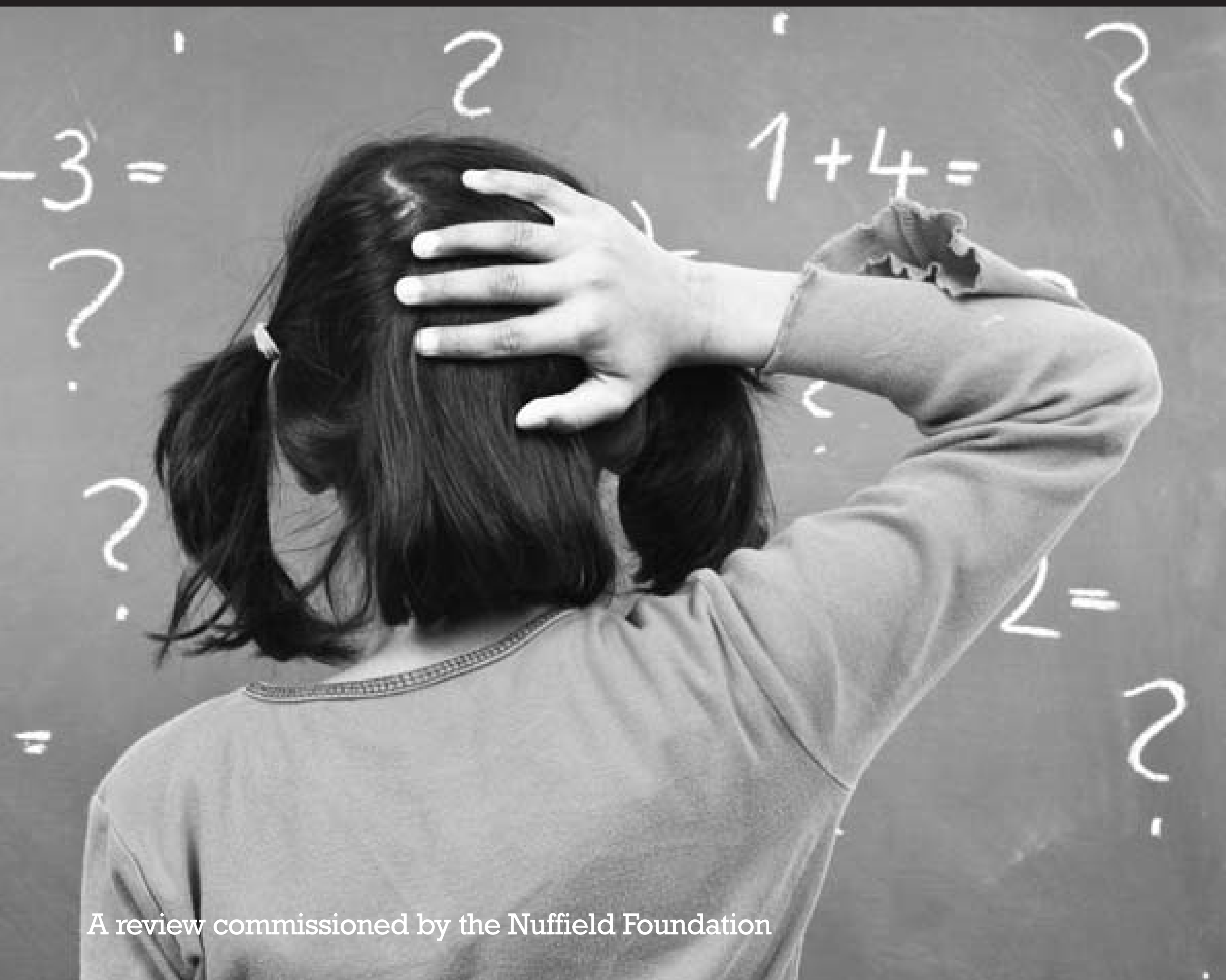
7

Key understandings in
mathematics learning

Paper 7: Modelling, problem-solving and integrating concepts

By Anne Watson, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

Contents

About the author

Anne Watson is Professor of Mathematics Education at the University of Oxford.

About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 7: Modelling, problem-solving and integrating concepts

Headlines

We have assumed a general educational context which encourages thinking and problem-solving across subjects. A key difference about mathematics is that empirical approaches may solve individual problems, and offer directions for reasoning, but do not themselves lead to *new* mathematical knowledge or mathematical reasoning, or to the power that comes from applying an abstract idea to a situation.

In secondary mathematics, the major issue is not how children learn elementary concepts, but what experiences they have had and how these enable or limit what else can be learnt. That is why we have combined several aspects of secondary mathematics which could be exemplified by particular topics.

- Students have to be fluent in understanding methods and confident about using them to know why and when to apply them, but such application does not automatically follow learning procedures. Students have to understand the situation as well as being able to call on a familiar repertoire of ideas and methods.

- Students have to know some elementary concepts well enough to apply them and combine them to form new concepts in secondary mathematics, but little is known from research about what concepts are essential in this way. Knowledge of a range of functions is necessary for modelling situations.

- Students have to learn when and how to use informal, experiential reasoning and when to use formal, conventional, mathematical reasoning. Without special attention to meanings many students tend to apply visual reasoning, or be triggered by verbal cues, rather than to analyse situations mathematically.

- In many mathematical situations in secondary mathematics, students have to look for relations between numbers and variables and relations between relations and properties of objects, and know how to represent them.

How secondary learners tackle new situations

In new situations students first respond to familiarity in appearance, or language, or context. They bring earlier understandings to bear on new situations, sometimes erroneously. They naturally generalise from what they are offered, and they often over-generalise and apply inappropriate ideas to new situations. They can learn new mathematical concepts either as extensions or integrations of earlier concepts, and/or as inductive generalisations from examples, and/or as abstractions from solutions to problems.

Routine or context?

One question is whether mathematics is learnt better from routines, or from complex contextual situations. Analysis of research which compares how children learn mathematics through being taught routines efficiently (such as with computerised and other learning packages designed to minimise cognitive load) to learning through problem-solving in complex situations (such as through Realistic Mathematics Education) shows that the significant difference is not about the speed and retention of learning but *what* is being learnt. In each approach the main question for progression is whether the student learns new concepts well enough to use and adapt them in future learning and outside mathematics. Both approaches have inherent weaknesses in this respect. These weaknesses will become clear in what follows. However, there are several studies which show that those who develop mathematical methods of enquiry over time can then learn procedures easily and do as well, or better, in general tests.

Problem-solving and modelling

To learn mathematics one has to learn to solve mathematical problems or model situations mathematically. Studies of students' problem solving mainly focus in the successful solution of contextually-worded problems using mathematical methods, rather than using problem solving as a context for learning new concepts and developing mathematical thinking. To solve unfamiliar problems in mathematics, a meta-analysis of 487 studies concluded that for students to be maximally successful:

- problems need to be fully stated with supportive diagrams
- students need to have previous extensive experience in using the representations used
- they have to have relevant basic mathematical skills to use
- teachers have to understand problem-solving methods.

This implies that fluency with representations and skills is important, but also depends on how clearly the problem is stated. In some studies the difficulty is also to do with the underlying concept, for example, in APU tests area problems were difficult with or without diagrams.

To be able to solve problems whose wording does not indicate what to do, students have to be able to read the problem in two ways: firstly, their technical reading skills and understanding of notation have to be good enough; secondly, they have to be able to interpret it to understand the contextual and mathematical meanings. They have to decide whether and how to bring informal knowledge to bear on the situation, or, if they approach it formally, what are the variables and how do they relate. If they are approaching it formally, they then have to represent the relationships in some way and decide how to operate on them.

International research into the use of ICT to provide new ways to represent situations and to see relationships, such as by comparing spreadsheets, graph plotters and dynamic images appear to speed up the process of relating representations through isomorphic reasoning about covariation, and hence the development of understanding about mathematical structures and relations.

Application of earlier learning

Knowing methods

Students who have only routine knowledge may not recognize that it is relevant to the situation. Or they can react to verbal or visual cues without reference to context, such as 'how much?' triggering multiplication rather than division, and 'how many?' always triggering addition. A further problem is that they may not understand the underlying relationships they are using and how these relate to each other. For example, a routine approach to $2 \times 1/3 \times 3/2$ may neither exploit the meaning of fractions nor the multiplicative relation.

Students who have only experience of applying generic problem-solving skills in a range of situations sometimes do not recognize underlying mathematical structures to which they can apply methods used in the past. Indeed, given the well-documented tendency for people to use *ad hoc* arithmetical trial-and-adjustment methods wherever these will lead to reasonable results, it is possible that problem-solving experience may not result in learning new mathematical concepts or working with mathematical structures, or in becoming fluent with efficient methods.

Knowing concepts

Students who have been helped to learn concepts, and can define, recognise and exemplify elementary ideas are better able to use and combine these ideas in new situations and while learning new concepts. However, many difficulties appear to be due to having too limited a range of understanding. Their understanding may be based on examples which have irrelevant features in common, such as the parallel sides of parallelograms always being parallel to the edges of a page. Understanding is also limited by examples being similar to a prototype, rather than extreme cases. Another problem is that students may recognize examples of a concept by focusing too much on visual or verbal aspects, rather than their properties, such as believing that it is possible to construct an equilateral triangle on a nine-pin geoboard because it 'looks like one'.

Robust connections between and within earlier ideas can make it easier to engage with new ideas, but can also hinder if the earlier ideas are limited and inflexible. For example, learning trigonometry involves understanding: the definition of triangle; right-angles; recognizing them in different orientations; what angle means and how it is measured; typical units for measuring lines; what ratio means; similarity of triangles; how ratio is written as a fraction; how to manipulate a multiplicative relationship; what 'sin' (etc.) means as a symbolic representation of a function and so on. Thus knowing about ratios can support learning trigonometry, but if the understanding of 'ratio' is limited to mixing cake recipes it won't help much. To be successful students have to have had enough experience to be fluent, and enough knowledge to use methods wisely.

They become better at problem-solving and modelling when they can:

- draw on knowledge of the contextual situation to identify variables and relationships and/or, through imagery, construct mathematical representations which can be manipulated further
- draw on a repertoire of representations, functions, and methods of operation on these
- have a purpose for the modelling process, so that the relationship between manipulations in the model and changes in the situation can be meaningfully understood and checked for reasonableness.

Knowing how to approach mathematical tasks

To be able to decide when and how to use informal or formal approaches, and how to use prior knowledge, students need to be able to think mathematically about all situations in mathematics lessons. This develops best as an all-encompassing perspective in mathematics lessons, rather than through isolated experiences.

Students have to:

- learn to avoid instant reactions based in superficial visual or verbal similarity
- practice using typical methods of mathematical enquiry explicitly over time
- have experience of mulling problems over time in order to gain insight.

With suitable environments, tools, images and encouragement, learners can and do use their general perceptual, comparative and reasoning powers in mathematics lessons to:

- generalise from what is offered and experienced
- look for analogies
- identify variables
- choose the most efficient variables, those with most connections
- see simultaneous variations
- understand change
- reason verbally before symbolising
- develop mental models and other imagery
- use past experience of successful and unsuccessful attempts
- accumulate knowledge of operations and situations to do all the above successfully

Of course, all the tendencies just described can also go in unhelpful directions and in particular people tend to:

- persist in using past methods and applying procedures without meaning, if that has been their previous mathematical experience
- get locked into the specific situation and do not, by themselves, know what new mathematical ideas can be abstracted from these experiences
- be unable to interpret symbols, text, and other representations in ways the teacher expects
- use additive methods; assume that if one variable increases so will another; assume that all change is linear; confuse quantities.

RECOMMENDATIONS

Research about mathematical learning	Recommendations for teaching
Learning routine methods and learning through complex exploration lead to different kinds of knowledge and cannot be directly compared; neither method necessarily enables learning new concepts or application of powerful mathematics ideas. However, those who have the habit of complex exploration are often able to learn procedures quickly.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of the importance of understanding new concepts, and avoid teaching solely to pass test questions, or using solely problem-solving mathematical activities which do not lead to new abstract understandings. Students should be helped to balance the need for fluency with the need to work with meaning.
Students naturally respond to familiar aspects of mathematics; try to apply prior knowledge and methods, and generalize from their experience.	Teaching should take into account students' natural ways of dealing with new perceptual and verbal information, and the likely misapplications. Schemes of work and assessment should allow enough time for students to adapt to new meanings and move on from earlier methods and conceptualizations.
Students are more successful if they have a fluent repertoire of conceptual knowledge and methods, including representations, on which to draw.	Developers of the curriculum, advisory schemes of work and teaching methods should give time for new experiences and mathematical ways of working to become familiar in several representations and contexts before moving on. Students need time and multiple experiences to develop a repertoire of appropriate functions, operations, representations and mathematical methods in order to solve problems and model situations. Teaching should ensure conceptual understanding as well as 'knowing about', 'knowing how to', and 'knowing how to use'.
Multiple experiences over time enable students to develop new ways to work on mathematical tasks, and to develop the ability to choose what and how to apply earlier learning.	Schemes of work should allow for students to have multiple experiences, with multiple representations over time to develop mathematically appropriate 'habits of mind'.
Students who work in computer-supported multiple representational contexts over time can understand and use graphs, variables, functions and the modelling process. Students who can choose to use available technology are better at problem solving, and have complex understanding of relations, and have more positive views of mathematics.	There are resource implications about the use of ICT. Students need to be in control of switching between representations and comparisons of symbolic expression in order to understand the syntax and the concept of functions. The United Kingdom may be lagging behind the developed world in exploring the use of spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

Recommendations for research

Application of research findings about problem-solving, modelling and conceptual learning to current curriculum developments in the United Kingdom suggests that there may be different outcomes in terms of students' ability to solve quantitative and spatial problems in realistic contexts. However, there is no evidence to convince us that the new National Curriculum in England will lead to better *conceptual* understanding of mathematics, either at the elementary levels, which are necessary to learn higher mathematics, or at higher levels which provide the confidence and foundation for further mathematical study. Where contextual and exploratory mathematics, integrated through the curriculum, *do* lead to further conceptual learning it is related to conceptual learning being a rigorous focus for curriculum and textbook design, and in teacher preparation, such as in China, Japan, Singapore, and the Netherlands, or in specifically designed projects based around such aims.

In the main body of *Paper 7: Modelling, solving problems and learning new concepts in secondary mathematics* there are several questions for future research, including the following.

- What are the key conceptual understandings for success in secondary mathematics, from the point of view of learning?
- How do students learn new ideas in mathematics at secondary level that depend on combinations of earlier concepts?
- What evidence is there of the characteristics of mathematics teaching at higher secondary level which contribute both to successful conceptual learning and application of mathematics?

Modelling, problem-solving and integrating concepts

Introduction

By the time students enter secondary school, they possess not only intuitive knowledge from outside mathematics and outside school, but also a range of quasi-intuitive understandings within mathematics, derived from earlier teaching and from their memory of generalisations, metaphors, images, metonymic associations and strategies that have served them well in the past. Many of these typical understandings are described in the previous chapters. Tall and Vinner (1981) called these understandings 'concept images', which are a ragbag of personal conceptual, quasi-conceptual, perceptual and other associations that relate to the language of the concept and are loosely connected by the language and observable artefacts associated with the concept. Faced with new situations, students will apply whatever familiar methods and associations come to mind relatively quickly – perhaps not realising that this can be a risky strategy. If 'doing what I think I know how to do' leads so easily to incorrect mathematics it is hardly surprising that many students end up seeing school mathematics as the acquisition and application of methods, and a site of failure, rather than as the development of a repertoire of adaptable intellectual tools.

At secondary level, new mathematical situations are usually ideas which arise through mathematics and can then be applied to other areas of activity; it is less likely that mathematics involves the formalisation of ideas which have arisen from outside experience as is common in the primary phase. Because of this difference, learning mathematics at secondary level cannot be understood only in terms of overall cognitive development. For this review, we developed a perspective which encompasses both the 'pure' and 'applied' aspects of learning at secondary level, and use research from both traditions to devise some common implications and overall recommendations for practice.

Characteristics of learning secondary mathematics

We justify the broad scope of this chapter by indicating similarities between the learning of the new concepts of secondary mathematics and learning how to apply mathematics to analyse, express and solve problems in mathematical and non-mathematical contexts. Both of these aspects of learning mathematics depend on interpreting new situations and bringing to mind a repertoire of mathematical concepts that are understood and fluent to some extent. In this review we will show that learning secondary mathematics presents core common difficulties, whatever the curriculum approach being taken, which need to be addressed through pedagogy.

In all teaching methods, when presented with a new stimulus such as a symbolic expression on the board, a physical situation, or a statement of a complex ill-defined 'real life' problem, the response of an engaged learner is to wonder:

What is this? This entails 'reading' situations, usually reading mathematical representations or words, and interpreting these in conventional mathematical ways. It involves *perception*, *attention*, understanding *representations* and being able to decipher *symbol systems*.

What is going on here? This entails identifying salient features including non-visual aspects, identifying variables, relating parts to each other, exploring what changes can be made and the effects of change, representing situations in mathematical ways, anticipating what might be the purpose of a mathematical object. It involves *attention, visualisation, modelling*, static and dynamic *representations*, understanding functional, statistical and geometrical *relationships*, focusing on what is mathematically *salient* and imagining the situation or a representation of it.

What do I know about this? This entails recognising similarities, seeking for recognisable structures beyond visual impact, identifying variables, proposing suitable functions, drawing on repertoire of past experiences and choosing what is likely to be useful. Research about *memory, problem-solving, concept images, modelling, functions, analysis* and *analogical reasoning* is likely to be helpful.

What can I do? This entails using past experience to try different approaches, heuristics, logic, controlling variables, switching between representations, transforming objects, applying manipulations and other techniques. It involves *analogical reasoning, problem-solving, tool-use, reasoning, generalisation* and *abstraction*, and so on.

Thus, students being presented with the task of understanding new ideas draw on past experience, if they engage with the task at all, just as they would if offered an unfamiliar situation and asked to express it mathematically. They may only get as far as the first step of 'reading' the stimulus. The alternative is to wait to be told what to do and treat everything as declarative, verbatim, knowledge. A full review of relevant research in all these areas is beyond the scope of this paper, and much of it is generic rather than concerned with mathematics.

We organise this Paper into three parts: Part 1 looks at what learners have to be able to do to be successful in these aspects of secondary mathematics; Part 2 considers what learners actually do when faced with new complex mathematical situations; and Part 3 reviews what happens with pedagogic intervention designed to address typical difficulties. We end with recommendations for future research, curriculum development and practice.

Part 1: What learners have to be able to do in secondary mathematics

In this chapter we describe what learners have to be able to do in order to learn new concepts, solve problems, model mathematical situations, and engage in mathematical thinking.

Learning 'new' concepts

Extension of meaning

Throughout school, students meet familiar ideas used in new contexts which include but extend their old use, often through integrating simpler concepts into more complex ideas. Sfard (1991) describes this process of development of meaning as consisting of 'interiorization' through acting on a new idea with some processes so that it becomes familiar and meaningful; understanding and expressing these processes and their effects as manageable units (condensation), and then this new structure becomes a thing in itself (reified) that can be acted on as a unit in future.

In this way, in *algebra*, letters standing for numbers become incorporated into terms and expressions which are number-like in some uses and yet cannot be calculated. Operations are combined to describe structures, and expressions of structures become objects which can be equated to each other. Variables can be related to each other in ways that represent relationships as functions, rules for mapping one variable domain to another (see the earlier chapters on functional relations and algebraic thinking).

Further, *number* develops from counting, whole numbers, and measures to include negatives, rationals, numbers of the form $a + b\sqrt{n}$ where a and b are rationals, irrationals and transcendentals, expressions, polynomials and functions which are number-like when used in expressions, and the uses of estimation which contradict earlier shifts towards accuracy. Eventually, two-dimensional complex numbers may also have to be understood. Possible discontinuities of meaning can arise between discrete and continuous quantities, monomials and polynomials, measuring and two-dimensionality, and different representations (digits, letters, expressions and functions).

Graphs are used first to compare values of various discrete categories, then are used to express two-dimensional discrete and continuous data as in scatter-graphs, or algebraic relationships between continuous variables, and later such relationships, especially linear ones, might be fitted to statistical representations.

Shapes which were familiar in primary school have to be defined and classified in new ways, and new properties explored, new geometric configurations become important and descriptive reasoning based on characteristics has to give way to logical deductive reasoning based on relational properties. Finally, all this has to be applied in the three-dimensional contexts of everyday life.

The processes of learning are sometimes said to follow historical development, but a better analogy would be to compare learning trajectories with the conceptual connections, inclusions and distinctions of mathematics itself.

Integration of concepts

As well as this kind of extension, there are new 'topics' that draw together a range of earlier mathematics. Typical examples of secondary topics are quadratic functions and trigonometry. Understanding each of these depends to some extent on understanding a range of concepts met earlier.

Quadratic functions: Learning about quadratic functions includes understanding:

- the meaning of letters and algebraic syntax;
- when letters are variables and when they can be treated as unknown numbers;
- algebraic terms and expressions;
- squaring and square rooting;
- the conventions of coordinates and graphing functions;
- the meaning of graphs as representing sets of points that follow an algebraic rule;
- the meaning of '=';
- translation of curves and the ways in which they can change shape;
- that for a product to equal zero at least one of its terms must equal zero and so on.

Trigonometry: Learning this includes knowing:

- the definition of triangle;
- about right-angles including recognizing them in different orientations;
- what angle means and how it is measured;
- typical units for measuring lines;
- what ratio means;
- similarity of triangles;
- how ratio is written as a fraction;
- how to manipulate a multiplicative relationship;
- what 'sin' (etc.) means as a symbolic representation of a function and so on.

New concepts therefore develop both through extension of meaning and combination of concepts. In each of these the knowledge learners bring to the new topic has to be adaptable and usable, not so strongly attached to previous contexts in which it has been used that it cannot be adapted. A hierarchical 'top down' view of learning mathematics would lead to thinking that all contributory concepts need to be fully understood before tackling new topics (this is the view taken in the NMAP review (2008) but is unsupported by research as far as we can tell from their document). By contrast, if we take learners' developing cognition into account we see that 'full understanding' is too vague an aim; it is the processes of applying and extending prior knowledge in the context of working on new ideas that contribute to understanding.

Whichever view is taken, learners have to bring existing understanding to bear on new mathematical contexts. There are conflicting research conclusions about the process of bringing existing ideas to bear on new stimuli: Halford (e.g. 1999) talks of conceptual chunking to describe how earlier ideas can be drawn on as packages, reducing to simpler objects ideas which are initially formed from more complex ideas, to develop further concepts and argues for such chunking to be robust before moving on. He focuses particularly on class inclusion (see Paper 5, Understanding space and its representation in mathematics) and transitivity, structures of relations between more than two objects, as ideas which are hard to deal with because they involve several levels of complexity. Examples of this difficulty were mentioned in Paper 4, Understanding relations and their graphical representation, showing how relations between relations cause problems. Chunking includes loss of access to lower level meanings, which may be useful in avoiding unnecessary detail of specific examples, but can obstruct meaningful use. Freudenthal (1991, p. 469) points out that automatic connections and actions can mask sources of insight, flexibility and creativity which arise from meanings. He observed that when students are in the flow of calculation they are not necessarily aware of what they are doing, and do not monitor their work. It is also the case that much of the chunking that has taken place in earlier mathematics is limited and hinders and obstructs future learning, leading to confusion with contradictory experiences. For example, the expectation that multiplication will make 'things' bigger can hinder learning that it only means this sometimes – multiplication scales quantities in a variety of ways. The difficulties faced by students whose understanding of the simpler concepts learnt in primary school is later needed for secondary mathematical ideas, are theorised by, among others, Trzcieniecka-Schneider (1993) who points out that entrenched and limited conceptual ideas (including what Fischbein calls 'intuitions' (1987) and what Tall calls 'metbefores' (2004)) can hinder a student's approach to unfamiliar examples and questions and create resistance, rather than a willingness to engage with new ideas which depend on adapting or giving up strongly-held notions. This leads not only to problems understanding new concepts which depend on earlier concepts, but also makes it hard for learners to see how to apply mathematics in unfamiliar situations. On the other hand, it is important that some knowledge is fluent and easily accessible, such as number bonds, recognition of multiples, equivalent algebraic forms, the shape of graphs of common functions and so on. Learners have to know when to apply 'old' understandings to be extended, and when to give them up for new and different understandings.

Inductive generalisation

Learners can also approach new ideas by inductive generalisation from several examples. English and Halford (1995 p. 50) see this inductive process[1] as the development of a mental model which fits the available data (the range of examples and instances learners have experienced) and from which procedures and conjectures can be generated. For example, learners' understanding about what a linear graph can look like is at first a generalisation of the linear graphs they have seen that have been named as such. Similarly, learners' conjectures about the relationship between the height and volume of water in a bottle, given as a data set, depends on reasoning both from the data and from general knowledge of such changes. Leading mathematicians often remark that mathematical generalisation also commonly arises from abductive reasoning on one generic example, such as conjectures about relationships based on static geometrical diagrams. For both these processes, the examples available as data, instances, and illustrations from teachers, textbooks and other sources play a crucial role in the process. Learners have to know what features are salient and generalise from them. Often such reasoning depends on metonymic association (Holyoak and Thagard, 1995), so that choices are based on visual, linguistic and cues which might be misleading (see also earlier chapter on number) rather than mathematical meaning. As examples: the prototypical parallelogram has its parallel edges horizontal to the page; and x^2 and $2x$ are confused because it is so common to use '$x = 2$' as an example to demonstrate algebraic meaning.

Abstraction of relationships

A further way to meet new concepts is through a process of 'vertical mathematisation' (Treffers, 1987) in which experience of solving complex problems can be followed by extracting general mathematical relationships. It is unlikely that this happens naturally for any but a few students, yet school mathematics often entails this kind of abstraction. The Freudenthal Institute has developed this approach through teaching experiments and national roll-out over a considerable time, and its Realistic Mathematics Education (e.g. Gravemeijer and Doorman, 1999) sees mathematical development as

- seeing what has to be done to solve the kinds of problems that involve mathematics
- from the solutions extracting new mathematical ideas and methods to add to the repertoire
- these methods now become available for future use in similar and new situations (as with Piaget's notion of reflective abstraction and Polya's 'looking back').

Gravemeijer and Doorman show that this approach, which was developed for primary mathematics, is also applicable to higher mathematics, in this case calculus. They refer to 'the role models can play in a shift from a model of situated activity to a model for mathematical reasoning. In light of this model-of/model-for shift, it is argued that discrete functions and their graphs play a key role as an intermediary between the context problems that have to be solved and the formal calculus that is developed.'

Gravemeijer and Doorman's observation explains why, in this paper, we are treating the learning of new abstract concepts as related to the use of problem-solving and modelling as forms of mathematical activity. In all of these examples of new learning, the fundamental shift learners are expected to make, through instruction, is from informal, experiential, engagement using their existing knowledge to formal, conventional, mathematical understanding. This shift appears to have three components: construction of meaning; recognition in new contexts; playing with new ideas to build further ideas (Hershkowitz, Schwartz and Dreyfus, 2001).

It would be wrong to claim, however, that learning can only take place through this route, because there is considerable evidence that learners can acquire routine skills through programmes of carefully constructed, graded, tasks designed to deal educatively with both right answers and common errors of reasoning, giving immediate feedback (Anderson , Corbett, Koedinger and Pelletier, 1995). The acquisition of routine skills without explicit work on their meaning is not the focus of this paper, but the automatisation of routines so that learners can focus on structure and meaning by reflection later on has been a successful route for some in mathematics.

There is recent evidence from controlled trials that learning routines from abstract presentations is a more efficient way to learn about underlying mathematical structure than from contextual, concrete and story-based learning tasks (Kaminski, Sloutsky and Heckler, 2008). There are several problems with their findings, for example in one study the sample consisted of undergraduates for whom the underlying arithmetical concept being taught would not have been new, even if it had never been explicitly formalised for them before. In a similar study with 11-year-olds, addition modulo 3 was being taught. For one group a model of filling jugs with three equal doses was used; for the other group abstract symbols were used. The test task consisted of spurious combinatorics involving three unrelated objects[2]. Those who had been taught using abstract unrelated symbols did better, those who had been taught using jug-filling did not so well. While these studies suggest that abstract knowledge about structures is not less applicable than experience and *ad hoc* knowledge, they also illuminate the interpretation difficulties that students have in learning how to model phenomena mathematically, and how familiar meanings (e.g. about jug-filling) dominate over abstract engagement. What Kaminski's results say to educationists is not 'abstract rules are better' but 'be clear about the learning outcomes you are hoping to achieve and do not expect easy transfer between abstract procedures and meaningful contexts'.

Summary

- Learners have to understand new concepts as extensions or integrations of earlier concepts, as inductive generalisations from examples, and as abstractions from solutions to problems.
- Robust chunking of earlier ideas can make it easier to engage with new ideas, but can also hinder if the earlier ideas are limited and inflexible.
- Routine skills can be adopted through practise to fluency, but this does not lead to conceptual understanding, or ability to adapt to unfamiliar situations, for many students.
- Learners have to know when and how to bring earlier understandings to bear on new situations.
- Learners have to know how and when to shift between informal, experiential activity to formal, conventional, mathematical activity.
- There is no 'best way' to teach mathematical structure: it depends whether the aim is to become fluent and apply methods in new contexts, or to learn how to express structures of given situations.

Problem-solving

The phrase 'problem-solving' has many meanings and the research literature often fails to make distinctions. In much research solving word problems is seen as an end in itself and it is not clear whether the problem introduces a mathematical idea, formalises an informal idea, or is about translation of words into mathematical instructions. There are several interpretations and the ways students learn, and can learn, differ accordingly. The following are the main uses of the phrase in the literature.

1 *Word problems with arithmetical steps* used to introduce elementary concepts by harnessing informal knowledge, or as situations in which learners have to apply their knowledge of operations and order (see Paper 4, Understanding relations and their graphical representation).

These situations may be modelled with concrete materials, diagrams or mental images, or might draw on experiences outside school. The purpose may be either to learn concepts through familiar situations, or to learn to apply formal or informal mathematical methods. For example, upper primary students studied by Squire, Davies and Bryant (2004) were found to handle commutativity much better than distributivity, which they could only do if there were contextual cues to help them. For teaching purposes this indicates that distributive situations are harder to recognise and handle, and a mathematical analysis of distributivity supports this because it entails encapsulation of one operation before applying the second and recognition of the importance of order of operations.

2 *Worded contexts which require the learner to decide to use standard techniques*, such as calculating area, time, and so on. Diagrams, standard equations and graphs might offer a bridge towards deciding what to do. For instance, consider this word problem: 'The area of a triangular lawn is 20 square metres, and one side is 5 metres long. If I walk in a straight line from the vertex opposite this side, towards this side, to meet it at right angles, how far have I walked?' The student has to think of how area is calculated, recognise that she has been given a 'base' length and asked about 'height', and a diagram or mental image would help her to 'see' this. If a diagram is given some of these decisions do not have to be made, but recognition of the 'base' and 'height' (not necessarily named as such) and knowledge of area are still crucial.

   In these first two types of problem, Vergnaud's classification of three types of multiplicative problems (see Paper 4, Understanding relations and their graphical representation) can be of some help if they are straightforwardly multiplicative. But the second type often calls for application of a standard formula which requires factual knowledge about the situation, and understanding the derivation of formulae so that their components can be recognised.

3 *Worded contexts in which there is no standard relationship to apply, or algorithm to use, but an answer is expected.* Typically these require setting up an equation or formula which can then be applied and calculated. This depends on understanding the variables and relationships; these might be found using knowledge of the situation, knowledge of the meaning of operations, mental or graphical imagery. For example, consider the question, 'One side of a rectangle is reduced in length by 20%, the other side in increased by 20%; what change takes place in the area?' The student is not told exactly what to do, and has to develop a spatial, algebraic or numerical model of the situation in order to proceed. She might decide that this is about representing the changed lengths in terms of the old lengths, and that these lengths have to be multiplied to understand what happens to the area. She might ascribe some arbitrary numbers to help her do this, or some letters, or she might realise that these are not really relevant – but this realisation is quite sophisticated. Alternatively she might decide that this is an empirical problem and generate several numerical examples, then using inductive reasoning to give a general answer.

4 *Exploratory situations in which there is an ill-defined problem,* and the learner has to mathematise by identifying variables and conjecturing relationships, choosing likely representations and techniques. Knowledge of a range of possible functions may be helpful, as is mental or graphical imagery.

   In these situations the problem might have been posed as either quasi-abstract or situated. There may be no solution, for example: 'Describe the advantages and disadvantages of raising the price of cheese rolls at the school tuck shop by 5p, given that cheese prices have gone down by 5% but rolls have gone up by 6p each'. Students may even have posed the entire situation themselves. They have to treat this as a real situation, a real problem for them, and might use statistical, algebraic, logical or *ad hoc* methods.

5 *Mathematical problems* in which a situation is presented and a question posed for which there is no obvious method. This is what a mathematician means by 'problem' and the expected line of attack is to use the forms of enquiry and mathematical thinking specific to mathematics. For example: 'What happens to the relationship between the sum of squares of the two shorter sides of triangles and the square on the longer side if we allow the angle between them to vary?' We leave these kinds of question for the later section on mathematical thinking.

Learning about students' solution methods for elementary word problems has been a major focus in research on learning mathematics. This research focuses on two stages: translation into mathematical relations, and solution methods[3]. A synthesis can be found in Paper 4, Understanding relations and their graphical representation. It is not always clear in the research whether the aim is to solve the original problem, to become better at mathematising situations, or to demonstrate that the student can use algebra fluently or knows how to apply arithmetic.

Students have to understand that there will be several layers to working with worded problems and cannot expect to merely read and know immediately what to do. Problems in which linguistic structure matches mathematical structure are easier because they only require fluent replacement of words and numbers by algebra. For example, analysis according to cognitive load theory informs us that problems with fewer words, requiring fewer operations, and where the linguistic structure matches the mathematical structure closely, are easier for learners to solve algebraically (Kintsch, 1986), but this is tautologous as such problems are *necessarily* easier since they avoid the need for interpretation and translation. Such interpretation may or may not be related to mathematical understanding. This research does, however, alert us to the need for students to learn how to tackle problems which do not translate easily – simply knowing what to do with the algebraic representation is not enough. In Paper 4, Understanding relations and their graphical representation, evidence is given that rephrasing the words to make meaning more clear might hinder learning to transform the mathematical relationships in problems.

Students might start by looking at the numbers involved, thinking about what the variables are and how they relate, or by thinking of the situation and what they expect to happen in it. Whether the choice of approach is appropriate depends on curriculum aims, and this observation will crop up again and again in this chapter. It is illustrated in the assumptions behind the work of Bassler, Beers and Richardson (1975). They compared two approaches to teaching 15-year-olds how to solve verbal problems, one more conducive to constructing equations and the other more conducive to grasping the nature of the problem. Of course, different emphases in teaching led to different outcomes in the ways students approached word problems. If the aim is for students to construct symbolic equations, then strategies which involve identification of variables and relationships and understanding how to express them are the most appropriate. If the aim is for learners to solve the problem by whatever method then a more suitable approach might be for them to imagine the situation and choose from a range of representations (graphical, numerical, algebraic, diagrammatic) possibly shifting between them, which can be manipulated to achieve a solution.

Clements (1980) and others have found that with elementary students reading and comprehension account for about a quarter of the errors of lower achieving students. The initial access to such problems is therefore a separate issue before students have to anticipate and represent (as Boero (2001) describes the setting-up stage) the mathematics they are going to use. Ballew and Cunningham (1982) with a sample of 217 11-year-olds found that reading and computational weaknesses were to blame for difficulties alongside interpretation – but they may have underestimated the range of problems lurking within 'interpretation' because they did not probe any further than these two variables and the links between reading, understanding the relations, and deciding what to compute were not analysed. Verschaffel, De Corte and Vierstraete (1999) researched the problem-solving methods and difficulties experienced by 199 upper primary students with nine word problems which combined ordinal and cardinal numbers. Questions were carefully varied to require different kinds of interpretation. They found, among other characteristics, that students tended to choose operations according to the relative size of the numbers in the question and that choice of formal strategies tended to be erroneous while informal strategies were more likely to be correct. Interpretation therefore depends on understanding operations sufficiently to realize where to apply them, recognizing how variables are related, as well as reading and computational accuracy. Success also involves visualising, imagining, identifying relationships between variables. All these have to be employed before decisions about calculations can be made. (This process is described in detail in Paper 2 for the case of distinguishing between additive and multiplicative relations.) Then learners have to know which variable to choose as the independent variable, recognise how to express other variables in relation to it, have a repertoire of knowledge of operations and functions to draw on, and think to draw on them. Obviously elementary arithmetical skills are crucial, but automatisation of procedures only aids solution if the structural class is properly identified in the first place. Automatisation of techniques can hinder solution of problems that are slightly different to prior experience because it can lead to over-generalisation and misapplication, and attention to language and layout cues rather than the structural meaning of the stated problem. For example, if learners have decided that 'how many…?' questions always indicate a need to use multiplication (as in 'If five children have seven sweets each, how many do they have in total?') they may find it hard to answer the question 'If 13 players drink 10 litres of cola, how many should I buy for 22 players?' because the answer is not a straightforward application of multiplication. The 'automatic' association of 'how many' with a multiplication algorithm, whether it is taught or whether learners have somehow devised it for themselves, would lead to misapplication.

Learners may not know how and when to bring other knowledge into play; they may not have had enough experience of producing representations to think to use them; the problem may offer a representation (e.g. diagram) that does not for them have meaning which can match to the situation. If they cannot see what to do, they may decide to try possible numbers and see what happens. A difficulty with successive approximation is that young learners often limit themselves to natural numbers, and do not develop facility with fractions which appear as a result of division, nor with decimals which are necessary to deal with 'a little bit more than' and 'a little bit less than'. An area which is well-known to teachers but is under-researched is how learners shift from thinking about only about natural numbers in trial-and-adjustment situations.

Caldwell and Goldin (1987) extended what was already well-known for primary students into the secondary phase, and found that abstract problems were, as for primary, significantly harder than concrete ones for secondary students in general, but that the differences in difficulty became smaller for older students. 'Concrete problems' were those couched in terms of material objects and realistic situations, 'abstract' problems were those which contained only abstract objects and/or symbols. They analysed the scripts of over 1000 students who took a test consisting of 20 problems designed along the concrete-abstract dimension in addition to some other variables. Lower secondary students succeeded on 55% of the concrete problems and 43% abstract, whereas higher secondary students succeeded on 69% of concrete and 66% of abstract. Whether the narrowing of the gap is due to teaching (as Vygotsky might suggest) or natural maturation (as some interpretations of Piaget might suggest) we do not know. They also found that problems which required factual knowledge are easier than those requiring hypotheses for secondary students, whereas for primary students the reverse appeared to be true. This shift might be due to adolescents being less inclined to enter imaginary situations, or to adolescents knowing more facts, or it may be educative due to the emphasis teachers put on factual rather than imaginative mathematical activity. However, it is too simplistic to say 'applying facts is easy'. In this study, further analysis suggests that the questions posed may not have been comparable on a structural measure of difficulty, number of variables and operations for example, although comparing 'level of difficulty' in different question-types is not robust.

In a well-replicated result, the APU sample of 15-year-olds found area and perimeter problems equally hard both in abstract and diagrammatic presentations (Foxman *et al.* 1985). A contextual question scored 10% lower than abstract versions. The only presentation that was easier for area was 'find the number of squares in…' which virtually tells students to count squares and parts of squares. In a teaching context, this indication of method is not necessarily an over-simplification. Dickson's study of students' interpretation of area (in four schools) showed that, given the square as a measuring unit, students worked out how to evaluate area and in then went on to formalise their methods and even devise the rectangle area formula themselves (1989).

The research findings are therefore inconclusive about shifts between concrete and abstract approaches which can develop in the normal conditions of school mathematics, but the role of pedagogy indicates that more might be done to support abstract reasoning and hypothesising as important mathematical practices in secondary school.

Hembree's meta-analysis (1992) of 487 studies of problem-solving gives no surprises – the factors that contribute to success are:

- that problems are fully stated with supportive diagrams
- that students have previous extensive experience in using the representations used
- that they have relevant basic mathematical skills to use
- that teachers who understand problem-solving methods are better at teaching them
- that heuristics might help in lower secondary.

Hembree's analysis seems to say that learners get to the answer easiest if there is an obvious route to solution. While Hembree did a great service in producing this meta-analysis, it fails to help with the questions: How can students learn to create their own representations and choose between them? How can students learn to devise new methods to solve new problems? How can students learn to act mathematically in situations that are not fully defined?

How do students get the experience that makes them better at problem-solving? An alternative approach is to view problem-solving as far from clear-cut and instead to see each problem as a situation requiring modelling (see next section).

Summary

To solve problems posed for pedagogic purposes, secondary mathematics learners have to:

- be able to read and understand the problem
- know when they are expected to use formal methods
- know which methods to apply and in what order and how to carry them out
- identify variables and relationships, choosing which variable to treat as independent
- apply appropriate knowledge of situations and operations
- use mental, graphical and diagrammatic imagery
- choose representations and techniques and know how to operate with them
- know a range of useful facts, operations and functions
- decide whether to use statistical, algebraic, logical or *ad hoc* methods.

Modelling

In contrast to 'problem-solving' situations in which the aim and purpose is often ambiguous, modelling refers to the process of expressing situations in conventional mathematical representations which afford manipulation and exploration. Typically, learners are expected to construct an equation, function or diagram which represents the variables in the situation and then, perhaps, solve an equation or answer some other related question based on their model[4]. Thus modelling presents many of the opportunities and obstacles described under 'problem-solving' above but the emphasis of this section is to focus on the identification of variables and relationships and the translation of these into representations. Carpenter, Ansell, Franke, Fennema and Weisbeck (1993) show that even very young children can do far more sophisticated quantitative reasoning when modelling situations for themselves than is expected if we think of it solely as application of known operations, because they bring their knowledge of acting in similar situations to bear on their reasoning.

A typical modelling cycle involves representing a realistic situation in mathematical symbols and then using isomorphism between the model and the situation, manipulate variables either in the model or the situation and observe how such transformations re-translate between the model and the situation. This duality is encapsulated in the ideas of *model-of* and *model-for*. The situation is an instantiation of an abstract model. The abstract model becomes a model-for being used to provide new insights and possibilities for the original situation. This isomorphic duality is a more general version of Vergnaud's model described in Paper 4, Understanding relations and their graphical representation. For learners, the situation can provide insight into possibilities in the mathematics, or the mathematics can provide insights into the situation. For example, a graphical model of temperature changes can afford prediction of future temperatures, while actual temperature changes can afford understanding of continuous change as expressed by graphs.

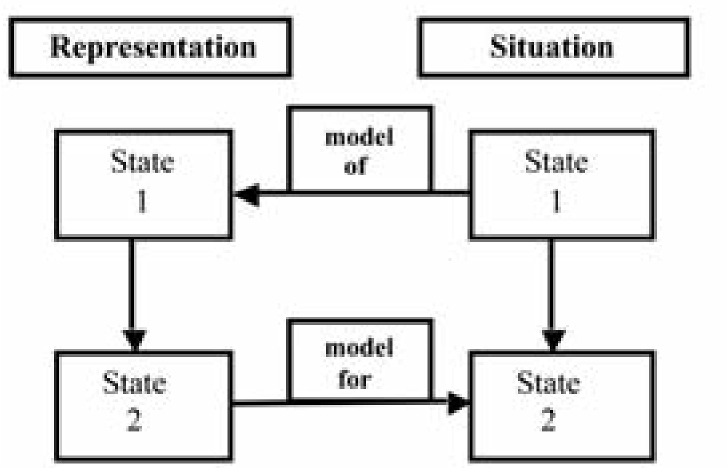


Figure 7.1: Typical modelling cycle with two-way relationship between situation and representations.

Research literature in this area gives primacy to different features. We are limited to looking at teaching experiments which are necessarily influenced by particular curriculum aims. Either the research looks at the learning of functions (that is extending the learners' repertoire of standard functions and their understanding of their features and properties) and sees modelling, interpreting and reifying functions as components of that learning (e.g. O'Callaghan, 1998), or the research sees skill in the modelling process as the goal of learning and sees knowledge of functions (their types and behaviour) as an essential component of that. In either approach there are similar difficulties. O'Callaghan, (1998) using a computer-intensive approach, found that while students did achieve a better understanding of functions through modelling than comparable students pursuing a traditional 'pure' course, and were more motivated and engaged in mathematics, they were no better at reifying what they had learnt than the traditional students. In pre- and post-tests students were asked to: model a situation using a function; interpret a function in a realistic situation; translate between representations; and use and transform algebraic functions which represent a financial situation. Students' answers improved in all but the last task which required them to understand the role of variables in the functions and the relation between the functions. In other words, they were good at modelling but not at knowing more about functions as objects in their own right.

MacGregor and Stacey (1993) (281 lower-secondary students in free response format and 1048 similar students who completed a multiple-choice item) show that the relationship between words, situations and making equations is not solely one of translating into symbols and correct algebra, rather it involves translating what is read into some kind of model developed from an existing schema and then representing the model – so there are two stages at which inappropriate relationships can be introduced, the mental model and the expressions of that model. The construction of mental models is dependent on:

- what learners know of the situation and how they imagine it
- how this influences their identification of variables, and
- their knowledge of possible ways in which variables can vary together.

What is it that students can see? Carlson and colleagues (2002) investigated students' perceptions and images of covariation, working mainly with undergraduates. The task is to work out how one variable varies in relation to another variable. Their findings have implications for younger students, because they find that their students can construct and manipulate images of how a dependent variable relates to the independent variable in dynamic events, such as when variation is positional, or visually identifiable, or can be seen to increase or decrease relative to the dependent variable, but the rate at which it changes change is harder to imagine. For our purposes, it is important to know that adolescents *can* construct images of relationships, but O'Callaghan's work shows that more is required for this facility to be used to develop knowledge of functions. When distinguishing between linear and quadratic functions, for example, rate of change is a useful indicator instead of some particular values, the turning point or symmetrical points, which may not be available in the data.

Looking at situations with a mathematical perspective is not something that can be directly taught as a topic, nor does it arise naturally out of school mathematical learning. Tanner and Jones (1994) worked with eight schools introducing modelling to their students. Their aim was not to provide a vehicle to learning about functions but to develop modelling skills as a form of mathematical enquiry. They found that modelling had to be developed over time so that learners developed a repertoire of experience of what kinds of things to focus on. Trelinski (1983) showed that of 223 graduate maths students only 9 could construct suitable mathematical models of non-mathematical situations – it was not that they did not know the relevant mathematics, but that they had never been expected to use it in modelling tasks before. It does not naturally follow that someone who is good at mathematics and knows a lot about functions automatically knows how to develop models.

So far we have only talked about what happens when learners are asked to produce models of situations. Having a use for the models, such as a problem to solve, might influence the modelling process. Campbell , Collis and Watson (1995) extended the findings of Kouba's research (1989) (reported in Paper 4) and analysed the visual images produced and used by four groups of 16-year-olds as aids to solving problems. The groups were selected to include students who had high and low scores on a test of vividness of visual imagery, and high and low scores on a test of reasoning about mathematical operations. They were then given three problems to solve: one involving drink-driving, one about cutting a painted cube into smaller cubes and one about three people consuming a large bag of apples by successively eating 1/3 of what was left in it. The images they developed differed in their levels of generality and abstraction, and success related more to students' ability to operate logically rather than to produce images, but even so there was a connection between the level of abstraction afforded by the images, logical operational facility and the use of visually based strategies. For example, graphical visualisation was a successful method in the drink-driving problem, whereas images of three men with beards sleeping in a hut and eating the apples were vivid but unhelpful. The creation of useful mathematical images needs to be learnt. In Campbell's study, questions were asked for which a model was needed, so this purpose, other than producing the model itself, may have influenced the modelling process. Models were both 'models of' and 'models for', the former being a representation to express structures and the latter being related to a further purpose (e.g. van den Heuvel-Panhuizen, 2003). Other writers have also pointed to the positive effects of purpose: Ainley, Pratt and Nardi (2001) and Friel, Curcio and Bright(2001) all found that having a purpose contributes to students' sense-making of graphs.

Summary

- Modelling can be seen as a subclass of problem-solving methods in which situations are represented in formal mathematical ways.

- Learners have to draw on knowledge of the situation to identify variables and relationships and, through imagery, construct mathematical representations which can be manipulated further.

- There is some evidence that learners are better at producing models for which they have a further purpose.
- To do this, they have to have a repertoire of mathematical representations, functions, and methods of operation on these.

- A modelling perspective develops over time and through multiple situational experiences, and can then be applied to given problems – the processes are similar to those learners do when faced with new mathematical concepts to understand.

- Modelling tasks do not necessarily lead to improved understanding of functions without the development of repertoire and deliberate pedagogy.

Functions

For learners to engage with secondary mathematics successfully they have to be able to decipher and interpret the stimuli they are offered, and this includes being orientated towards looking for relations between quantities, noticing structures, identifying change and generalising patterns of behaviour. Kieran (1992) lists these as good approaches to early algebra. They also have to know the difference between statistical and algebraic representations, such as the difference between a bar chart and an algebraic graph.

Understanding what a function is, a mapping that relates values from one space into values in another space, is not a straightforward matter for learners. In Paper 4, Understanding relations and their graphical representation, evidence that the experience of transforming between values in the same space is different from transforming between spaces is described, and for this paper we shall move on and assume that the purpose of simple additive, scalar and multiplicative functions is understood, and the task is now to understand their nature, a range of kinds of function, their uses, and the ways in which they arise and are expressed.

Whereas in early algebra learners need to shift from seeing expressions as things to be calculated to seeing them as expressing structures, they then have to shift further to seeing functions as relations between expressions, so that functions become mathematical objects in themselves and numerical 'answers' are likely to be pairs of related values (Yerushalmy and Schwartz, 1993). Similarly equations are no longer situations which hide an unknown number, but expressions of relationships between two (or more) variables. They have also to understand the difference between a point-wise view of functional relationships (as expressed by tables of values) and a holistic view (reinforced especially by graphs).

Yerushalmy and Gilead, in a teaching experiment with lower-secondary students over a few years (1999) found that knowledge of a range of functions and the nature of functions was a good basis for solving algebraic problems, particularly those that involved rate because a graph of a function allows rates to be observed and compared. Thus functions and their graphs support the focus on rate that Carlson's students found difficult in situations and diagrams. Functions appeared to provide a bridge that turned intractable word problems into modelling tasks by conjecturing which functions might 'fit' the situation. However, their students could misapply a functional approach. This seems to be an example of the well-known phenomenon of over-generalizing an approach beyond its appropriate domain of application, and arises from students paying too much attention to what has recently been taught and too little to the situation.

Students not only have to learn to think about functional relationships (and consider non-linear relationships as possibilities), which have an input to which a function is applied generating some specific output, but they also need to think about relations between relations in which there is no immediate output, rather a structure which may involve several variables. Halford's analysis (e.g.1999) closely follows Inhelder and Piaget's (1959) theories about the development of scientific reasoning in adolescence. He calls these 'quaternary' relationships because they often relate four components appearing as two pairs. Thus distributivity is quaternary, as it involves two binary operations; proportion is quaternary as it involves two ratios. So are rates of change, in which two variables are compared as they both vary in relation to something else (their functional relation, or time, for example). This complexity might contribute to explaining why Carlson and colleagues found that students could talk about covariation relationships from graphs of situations but not rates of change. Another reason could be the opacity of the way rate of change has to be read from graphs: distances in two directions have to be selected and compared to each other, a judgement or calculation made of their ratio, and then the same process has to be repeated around other points on the graphs and the ratios compared. White and Mitchelmore (1996) found that even after explicit instruction students could only identify rates of change in simple cases, and in complex cases tried to use algebraic algorithms (such as a given formula for gradient) rather than relate quantities directly.

One area for research might be to find out whether and how students connect the 'method of differences', in which rates of change are calculated from tables of values, to graphical gradients. One of the problems with understanding functions is that each representation brings certain features to the fore (Goldin, 2002). Graphical representations emphasise linearity, roots, symmetry, continuity, gradient; domain; ordered dataset representations emphasise discrete covariation and may distract students from starting conditions; algebraic representations emphasise the structure of relations between variables, and the family of functions to which a particular one might be related. To understand a function fully these have to be connected and, further, students have to think about features which are not so easy to visualize but have to be inferred from, or read into, the representation by knowing its properties, such as growth rate (Confrey and Smith, 1994; Slavit, 1997). Confrey and Smith used data sets to invite unit-by-unit comparison to focus on rate-of-change, and deduced that rate is different from ratio in the ways that it is learnt and understood (1994). Rate depends on understanding the covariation of variables, and being able to conceptualise the action of change, whereas ratio is the comparison of quantities.

Summary

To understand the use of functions to describe situations secondary mathematics learners have to:

- distinguish between statistical and algebraic representations
- extend knowledge of relations to understanding relations between relations
- extend knowledge of expressions as structures to expressions as objects
- extend knowledge of equations as defining unknown numbers to equations as expressing relationships between variables
- relate pointwise and holistic understandings and representations of functions
- see functions as a new kind of mathematical object
- emphasize mathematical meaning to avoid over-generalising
- have ways of understanding rate as covariation.

Mathematical thinking

In this section we mention mathematical problems – those that arise in the exploration of mathematics rather than problems presented to learners for them to exercise methods or develop 'problem-solving' skills. In mathematical problems, learners have to use mathematical methods of enquiry, some of which are also used in word problems and modelling situations, or in learning about new concepts. To learn mathematics in this context means two things:

to learn to use methods of mathematical enquiry and to learn mathematical ideas which arise in such enquiry.

Descriptions of what is entailed in mathematical thinking are based mainly on Polya's work (1957), in which mathematical thinking is described as a holistic habit of enquiry in which one might draw on any of about 70 tactics to make progress with a mathematical question. For example, the tactics include make an analogy, check a result, look for contradictions, change the problem, simplify, specialise, use symmetry, work backwards, and so on. Although some items in Polya's list appear in descriptions of problem-solving and modelling tactics, others are more likely to be helpful in purely mathematical contexts in which facts, logic, and known properties are more important than merely dealing with current data. Cuoco, Goldenberg and Mark (1997) have devised a typography of aspects of mathematical habits of mind. For example, mathematicians look at change, look at stability, enjoy symbolisation, invent, tinker, conjecture, experiment, relate small things to big things, and so on. The typography encompasses the perspectives which experts bring to bear on mathematics – that is they bring ideas and relationships to bear on situations rather than merely use current data and specific cases. Both of these lists contain dozens of different 'things to do' when faced with mathematics. Mason, Burton and Stacey (1982) condensed these into 'specialise-generalise; conjecture-convince' which focuses on the shifts between specific cases and general relationships and properties, and the reasoning shift between demonstrating and proving. All of these reflect the processes of mathematical enquiry undertaken by experienced mathematicians. Whereas in modelling there are clear stages of work to be done, 'mathematical thinking' is not an ordered list of procedures, rather it is a way of describing a cast of mind that views any stimulus as an object of mathematical interest, encapsulating relationships between relationships, relationships between properties, and the potential for more such relationships by varying variables, parameters and conditions.

Krutetskii (1976) conducted clinical interviews with 130 Soviet school children who had been identified as strong mathematicians. He tested them qualitatively and quantitatively on a wide range of mathematical tasks, looked for common factors in the way they tackled them, and found that those who are better at mathematics in general were faster at grasping the essence of a mathematical situation and seeing the structure through the particular surface features. They generalised more easily, omitted intermediate steps of reasoning, switched between solution methods quickly, tried to get elegant solutions, and were able to reverse trains of thought. They remembered relationships and principles of a problem and its solution rather than the details and tended to explain their actions rather than describe them. Krutetskii's methods were clinical and grounded and dependent on case studies within his sample, nevertheless his work over many years led him to form the view that such 'abilities' were educable as well as innate and drew strongly on natural propensities to reason spatially, perceptually, computationally, to make verbal analogies, mental associations with remembered experiences, and reasoning. Krutetskii, along with mathematicians reporting their own experiences, observed the need to mull, that is to leave unsolved questions alone for a while after effortful attempts, to sleep, or do other things, as this often leads to further insights when returning to them. This commonly observed phenomenon is studied in neuroscience which is beyond the scope of this paper, but does have implications for pedagogy.

Summary

- Successful mathematics learners engage in mathematical thinking in all aspects of classroom work. This means, for example, that they see what is varying and what is invariant, look for relationships, curtail or reverse chains of reasoning, switch between representations and solution methods, switch between examples and generalities, and strive for elegance.

- Mathematical 'habits of mind' draw on abilities or perception, reasoning, analogy, and mental association when the objects of study are mathematical, i.e. spatial, computational, relational, variable, invariant, structural, symbolic.

- Learners can get better at using typical methods of mathematical enquiry when these are explicitly developed over time in classrooms.

- It is a commonplace among mathematicians that mulling over time aids problem-solving and conceptualisation.

Part 2: What learners do when faced with complex situations in mathematics

In this section we collect research findings that indicate what school students typically do when faced with situations to model, solve, or make mathematical sense of.

Bringing outside knowledge to bear on mathematical problems

Real-life problems appear to *invite* solutions which are within a 'human sense' framework rather than a mathematical frame (Booth 1981). 'Wrong' approaches can therefore be seen not as errors, but as expressing a need for enculturation into what does and does not count in mathematical problem-solving. Cooper and Dunne (2000) show that in tests the appropriate use of outside knowledge and ways of reasoning, and when and when not to bring it into play, is easier for socially more advantaged students to understand than less advantaged students who may use their outside knowledge inappropriately. This is also true for students working in languages other than their first, who may only have access to formal approaches presented in standard ways. Cooper and Harries (2002) worked on this problem further and showed how typical test questions for 11- to 12-year olds could be rewritten in ways which encourage more of them to reason about the mathematics, rather than dive into using handy but inappropriate procedures. Vicente, Orrantia and Verschaffel (2007) studied over 200 primary school students' responses to word problems and found that elaborated information about the situation was much less effective in improving success than elaborating the conceptual information. Wording of questions, as well as the test environment, is therefore significant in determining whether students can or can not solve unfamiliar word problems in appropriate ways. Contrary to a common assumption that giving mathematical problems in some context helps learners understand the mathematics, analysis of learners' responses in these research studies shows that 'real-life' contexts can:

- lead to linguistic confusion
- create artificial problems that do not fit with their experience
- be hard to visualise because of unfamiliarity, social or emotional obstacles
- structure mathematical reasoning in ways which are different from abstract mathematics
- obscure the intended mathematical generalisation
- invite *ad hoc* rather than formal solution methods
- confuse students who are not skilled in deciding what 'outside' knowledge they can bring to the situation.

Clearly students (and their teachers) need to be clear about how to distinguish between situations in which everyday knowledge is, or is not, preferable to formal knowledge and how these relate. In Boaler's comparative study of two schools (1997) some students at the school, in which mathematics was taught in exploratory ways, were able to recognise these differences and decisions. However, it is also true that students' outside knowledge used appropriately might:

- enable them to visualise a situation and thus identify variables and relationships
- enable them to exemplify abstract relationships as they are manifested in reality
- enable them to see similar structures in different situations, and different structures in similar situations
- be engaged to generate practical, rather than formal, solutions
- be consciously put aside in order to perform as mathematically expected.

Information processing

In this section we will look at issues about cognitive load, attention, and mental representations. At the start of the paper we posed questions about what a learner has to do at first when faced with a new situation of any kind. Information processing theories and research are helpful but there is little research in this area within mathematics teaching except in terms of cognitive load, and as we have said before it is not helpful for cognitive load to be minimized if the aim is to learn how to work with complex situations. For example, Sweller and Leung-Martin (1997) used four experiments to find out what combinations of equations and words were more effective for students to deal successfully with equivalent information. Of course, in mathematics learning students have to be able to do both and all kinds of combinations, but the researchers did find that students who had achieved fluency with algebraic manipulations were slowed down by having to read text. If the aim is merely to do algebraic manipulations, then text is an extra load. Automaticity, such as fluency in algebraic manipulation, is achievable efficiently if differences between practice examples are minimized. Automaticity also frees up working memory for other tasks, but as Freudenthal and others have pointed out, automaticity is not always a suitable goal because it can lead to thoughtless application of methods. We would expect a learner to read text carefully if they are to choose methods meaningfully in the context. The information-processing tutors developed in the work of Anderson and his colleagues (e.g. 1995) focused mainly on mathematical techniques and processes, but included understanding the effects of such processes. We are not arguing for adopting his methods, but we do suggest that information processing has something to offer in the achievement of fluency, and the generation of multiple examples on which the learner can then reflect to understand the patterns generated by mathematical phenomena.

Most of the research on attention in mathematics education takes an affective and motivational view, which is beyond the scope of this paper (see NMAP, 2008). However, there is much that can be done about attention from a mathematical perspective. The deliberate use of variation in examples offered to students can guide their focus towards particular variables and differences. Learners have to know when to discern parts or wholes of what is offered and which parts are most critical; manipulation of variables and layouts can help direct attention. What is available to be learnt differs if different relations are emphasised by different variations. For example, students learning about gradients of straight line functions might be offered exercises as follows:

Gradient exercise 1: find the gradients between each of the following pairs of points.

(4, 3) and (8, 12)	(-2, -1) and (-10, 1)
(7, 4) and (-4, 8)	(8, -7) and (11, -1)
(6, -4) and (6, 7)	(-5, 2) and (10, 6)
(-5, 2) and (-3, -9)	(-6, -9) and (-6, -8)

Gradient exercise 2:

(4, 3) and (8, 12)	(4, 3) and (4, 12)
(4, 3) and (7, 12)	(4, 3) and (3, 12)
(4, 3) and (6, 12)	(4, 3) and (2, 12)
(4, 3) and (5, 12)	(4, 3) and (1, 12)

In the first type, learners will typically focus on the methods of calculation and dealing with negative numbers; in the second type, learners typically gesture to indicate the changes in gradient. Research in this area shows how learners can be directed towards different aspects by manipulating variables (Runesson and Mok, 2004; Chik and Lo, 2003).

Theories of mental representations claim that declarative knowledge, procedural knowledge and conceptual knowledge are stored in different ways in the brain and also draw distinctions between verbatim memory and gist memory (e.g. Brainerd and Reyna, 1993). Such theories are not much help with mathematics teaching and learning, because most mathematical knowledge is a combination of all three kinds, and in a typical mathematical situation both verbatim and gist memory would be employed. At best, this knowledge reminds us that providing 'knowledge' only in verbatim and declarative form is unlikely to help learners become adaptable mathematical problem-solvers. Learners have to handle different kinds of representation and know which different representations represent different ideas, different aspects of the same ideas, and afford different interpretations.

Summary

- Learners' attention to what is offered depends on variation in examples and experiences.
- Learners' attention can be focused on critical aspects by deliberate variation.
- Automaticity can be helpful, but can also hinder thought.
- If information is only presented as declarative knowledge then learners are unlikely to develop conceptual understanding, or adaptive reasoning.
- The form of representation is a critical influence on interpretation.

What learners do naturally that obstructs mathematical understanding

Most of the research in secondary mathematics is about student errors. These are persistent over time, those being found by Ryan and Williams (2007) being similar to those found by APU in the late 1970s. Errors do not autocorrect because of maturation, experience or assessment. Rather they are inherent in the ways learners engage with mathematics through its formal representations.

Persistence of 'child-methods' pervades mathematics at secondary level (Booth, 1981). Whether 'child-methods' are seen as intuitive, quasi-intuitive, educated, or as over-generalisations beyond the domain of applicability, the implication for teaching is that students have to experience, repeatedly, that new-to-them formal methods are more widely applicable and offer more possibilities, and that earlier ideas have to be extended and, perhaps, abandoned. If students have to adopt new methods without understanding why they need to abandon earlier ones, they are likely to become confused and even disaffected, but it is possible to demonstrate this need by offering particular examples that do not yield to child-methods. To change naïve conceptualisations is harder as the next four 'persistences' show.

Persistence of additive methods

This 'child method' is worthy of separate treatment because it is so pervasive. The negative effects of the persistence of additive methods show up again and again in research. Bednarz and Janvier (1996) conducted a teaching experiment with 135 12- to 13-year-olds before they had any algebra teaching to see what they would make of word problems which required several operations: those with multiplicative composition of relationships turned out to be much harder than those which involved composing mainly additive operations. The tendency to use additive reasoning is also found in reasoning about ratios and proportion (Hart, 1981), and in students' expectations about relationships between variables and sequential predictions. That it occurs naturally even when students know about a variety of other relationships is an example of how intuitive understandings persist even when more formal alternatives are available (Fischbein, 1987)

Persistence of more-more, same-same intuitions

Research on the interference from intuitive rules gives varied results. Tirosh and Stavy (1999) found that their identification of the intuitive rules 'more-more' and 'same-same' had a strong predictive power for students' errors and their deduction accords with the general finding that rules which generally work at primary level persist. For example, students assume that shapes with larger perimeters must have larger areas; decimals with more digits must be larger than decimals with fewer digits, and so on. Van Dooren, De Bok, Weyers and Verschaffel (2004), with a sample of 172 students from upper secondary found that, contrary to the findings of Tirosh and Stavy, students' errors were not in general due to consistent application of an intuitive rule of 'more-more' 'same-same'. Indeed the more errors a student made the less systematic their errors were. This was in a multiple-choice context, and we may question the assumption that students who make a large number of errors in such contexts are engaging in any mathematical reasoning. However, they also sampled written calculations and justifications and found that errors which looked as if they might be due to 'more-more' and 'same-same' intuitions were often due to other errors and misconceptions. Zazkis, however (1999), showed that this intuition persisted when thinking about how many factors a number might have, large numbers being assumed to have more factors.

Persistence of confusions between different kinds of quantity, counting and measuring

As well as the persistence of additive approaches to multiplication, being taught ideas and being subsequently able to use them are not immediately connected. Vergnaud (1983) explains that the conceptual field of intensive quantities, those expressed as ratio or in terms of other units (see Paper 3, Understanding rational numbers and intensive quantities), and multiplicative relationship development continues into adulthood. Nesher and Sukenik (1991) found that only 10% of students used a model based on understanding ratio after being taught to do so formally, and then only for harder examples.

Persistence of the linearity assumption

Throughout upper primary and secondary students act as if relationships are always linear, such as believing that if length is multiplied by *m* then so is area, or if the 10th term in a sequence is 32, then the 100th must be 320 (De Bock, Verschaffel and Janssens1998; Van Dooren, De Bock, Janssens and Verschaffel, 2004, 2007). Results of a teaching experiment with 93 upper-primary students in the Netherlands showed that, while linearity is persistent, a non-linear realistic context did not yield this error. Their conclusion was that the linguistic structure of word problems might invite linearity as a first, flawed response. They also found that a single experience is not enough to change this habit. A related assumption is that functions increase as the independent variable increases (Kieran, Boileau and Garancon 1996).Students' habitual ways of attacking mathematical questions and problems also cause problems.

Persistence with informal and language-based approaches

Macgregor and Stacey (1993) tested over 1300 upper-secondary students in total (in a range of studies) to see how they mathematised situations. They found that students tend first of all to try to express directly from the natural language of a situation, focusing on in equalities between quantities. Engaging with the underlying *mathematical* meaning is not a natural response. Students wanted the algebraic expression to be some kind of linguistic code, rather than a relational expression. Many researchers claim this is to do with translating word order inappropriately into symbols (such as 'there are six students to each professor, so $6s = p$') but Macgregor and Stacey suggest that the cause is more to with inadequate models of multiplicative relationships and ratio. However, it is easy to see that this is compounded by an unfortunate choice of letters as shorthand for objects rather than as variables. For example, Wollman (1983) and Clement (1982) demonstrate that students make this classic 'professor-student' error because of haste, failure to check that the meaning of the equation matches the meaning of the sentence, over-reliance on linguistic structure, use of non-algebraic symbols (such as p for professor instead of p for number of professors) and other reasons. At least one of these is a processing error which could be resolved with a 'read out loud' strategy for algebra.

Persistence of qualitative judgements in modelling

Lesh and Doerr (2003) reported on the modelling methods employed by students who had not had specific direction in what to do. In the absence of specific instructions, students repeat patterns of learning that have enabled them to succeed in other situations over time. They tend to start on each problem with qualitative judgements based on the particular context, then shift to additive reasoning, then form relationships by pattern recognition or repeated addition, and then shift to proportional and relational thinking. At each stage their students resorted to checking their arithmetic if answers conflicted rather than adapting their reasoning by seeing if answers made sense or not. This repetition of naïve strategies until they break down is inefficient and not what the most successful mathematics students do.

In modelling and problem-solving students confuse formal methods with contextual methods; they cling strongly to limited prototypes; they over-generalise; they read left-to-right instead of interpreting the meaning of symbolic expressions. In word problems they misread; miscomprehend; make errors in transformation into operations; errors in processes; and misinterpret the solution in the problem context (see also Ryan and Williams 2007).

Persistent application of procedures.

Students can progress from a manipulative approach to algebra to understanding it as a tool for problem-solving over time, but still tend over-rely on automatic procedures (Knuth, 2000). Knuth's sample of 178 first-year undergraduates' knowledge of the relationship between algebraic and graphical representations was superficial, and that they reached for algebra to do automatised manipulations rather than use graphical representations, even when the latter were more appropriate.

Summary

Learners can create obstacles for themselves by responding to stimuli in particular ways:

- persistence of past methods, child methods, and application of procedures without meaning
- not being able to interpret symbols and other representations
- having limited views of mathematics from their past experience
- confusion between formal and contextual aspects
- inadequate past experience of a range of examples and meanings
- over-reliance on visual or linguistic cues, and on application of procedures
- persistent assumptions about addition, more-more/same-same, linearity, confusions about quantities
- preferring arithmetical approaches to those based on meaning.

What learners do naturally that is useful

Students *can* be guided to explore situations in a systematic way, learning how to use a typically mathematical mode of enquiry, although it is hard to understand phenomena and change in dynamic situations. Carlson, Jacobs, Coe, Larson and Hsu. (2002) and Yerushalmy (e.g. 1997) have presented consistent bodies of work about modelling and covariation activities and their work, with that of Kaput (e.g. 1991), has found that this is not an inherently maturation problem, but that with suitable tools and representations such as those available in SimCalc children can learn not only to understand change by working with dynamic images and models, but also to create tools to analyse change. Carlson and her colleagues in teaching experiments have developed a framework for describing how students learn about this kind of co-variation. First they learn how to identify variables; then they form an image of how the variables simultaneously vary. Next, one variable has to be held still while the change in another is observed. This last move is at the core of mathematics and physics, and is essential in constructing mathematical models of multivariate situations, as Inhelder and Piaget also argued more generally.

In these supported situations, students appear to reason verbally before they can operate symbolically (Nathan and Koedinger, 2000). The usual 'order' of teaching suggested in most curricula (arithmetic, algebra, problem-solving) does not match students' development of competence in which verbal modes take precedence[5]. This fits well with Swafford and Langrall's study of ten 11-year-olds (2000) in which it was clear that even without formal teaching about algebra, students could identify variables and articulate the features of situations as equations where they were familiar with the underlying operations. Students were asked to work on six tasks in interviews. The tasks were realistic problems that could be represented by direct proportion, linear relations in a numerical context, linear relations in a geometric context, arithmetic sequences, exponential relations and inverse proportion. Each task consisted of subtasks which progressed from structured exploration of the situation, verbal description of how to find some unknown value, write an equation to express this given certain letters to represent variables, and use the equation to find out something else. The ability to express their verbal descriptions as equations was demonstrated across the tasks; everyone was able to do at least one of these successfully and most did more than one. The only situation for which no one produced an equation was the exponential one. The study also showed that, given suitably-structured tasks, students can avoid the usual assumptions of linearity. This shows some intuitive algebraic thinking, and that formal symbolisation can therefore be introduced as a tool to express relationships which are already understood from situations. Of course, as with all teaching experiments, this finding is specific to the teaching and task and would not automatically translate to other contexts, but as well as supporting the teaching of algebra as the way to express generality (see Paper 6 Algebraic reasoning) it contributes to the substantial practical knowledge of the value of starting with what students see and getting them to articulate this as a foundation for learning.

It is by looking at the capabilities of successful students that we learn more about what it takes to learn mathematics. In Krutetskii's study of such students, to which we referred earlier, (1976) he found that they exhibited what he called a 'mathematical cast of mind' which had analytical, geometric, and harmonic (a combination of the two) aspects. Successful students focused on structure and relationships rather than particular numbers of a situation. A key result is that memory about past successful mathematical work, and its associated structures, is a stronger indicator of mathematical success than memory about facts and techniques. He did not find any common aspects in their computational ability.

Silver (1981) reconstructed Krutetskii's claim that 67 lower-secondary high-achieving mathematics students remembered structural information about mathematics rather than contextual information. He asked students to sort 16 problems into groups that were mathematically-related. They were then given two problems to work on and asked to write down afterwards what they recalled about the problems. The 'writing down' task was repeated the next day, and again about four weeks later. There was a correlation between success in solving problems and a tendency to focus on underlying mathematical structure in the sorting task. In addition, students who recalled the structure of the problems were the more successful ones, but others who had performed near average on the problems could talk about them structurally immediately after discussion. The latter effect did not last in the four-week recall task however. Silver showed, by these and other similar tasks, that structural memory aided transfer of methods and solutions to new, mathematically similar, situations. A question arises, whether this is teachable or not, given the results of the four-week recall. Given that we know that mathematical strategies can be taught in general (Vos, 1976; Schoenfeld, 1979; 1982, and others) it seems likely that structural awareness might be teachable, however this may have to be sustained over time and students also need knowledge of a repertoire of structures to look for.

We also know something about how students identify relationships between variables. While many will choose a variable which has the most connections within the problem as the independent one, and tended also to start by dealing with the largest values, thus showing that they can anticipate efficiency, there are some who prefer the least value as the starting point. Nesher, Hershkovitz and Novotna, (2003) found these tendencies in the modelling strategies of 167 teachers and 132 15-year-old students in twelve situations which all had three variables and a comparative multiplication relationship with an additive constraint. This is a relatively large sample with a high number of slightly-varied situations for such studies and could provide a model for further research, rather than small studies with a few highly varied tasks.

Whatever the disposition towards identifying structures, variables and relationships, it is widely agreed that the more you know, the better equipped you are to tackle such tasks. Alexander and others (1997) worked with very young children (26 three- to five-year-olds) and found that they could reason analogically so long as they had the necessary conceptual knowledge of objects and situations to recognise possible patterns. Analogical reasoning appears to be a natural everyday power even for very young children (Holyoak and Thagard, 1995) and it is a valuable source of hypotheses, techniques, and possible translations and transformations. Construction of analogies appears to help with transfer, since seeking or constructing an analogy requires engagement with structure, and it is structure which is then sought in new situations thus enabling methods to be 'carried' into new uses. English and Sharry (1996) provide a good description of the processes of analogical reasoning: first seeing or working out what relations are entailed in the examples or instances being offered (abductively or inductively), this relational structure is extracted and represented as a model, mental, algebraic, graphical i.e. constructing an analogy in some familiar, relationally similar form. They observed, in a small sample, that some students act 'pseudo structurally' i.e. emphasising syntax hindered them seeking and recognising relational mappings. A critical shift is from focusing on visual or contextual similarity to structural similarity, and this has to be supported. Without this, the use of analogies can become two things to learn instead of one.

Past experience is also valuable in the interpretation of symbols and symbolic expressions, as well as what attracts their attention and the inter-relation between the two (Sfard and Linchevski, 1994). In addition to past experience and the effects of layout and familiarity, there is also a difference in readings made possible by whether the student perceives a statement to be operational (what has to be calculated), relational (what can be expressed algebraically) or structural (what can be generalized). Generalisation will depend on what students see and how they see it, what they look for and what they notice. Scheme-theory suggests that what they look for and notice is related to the ways they have already constructed connections between past mathematical experiences and the concept images and example spaces they have also constructed and which come to mind in the current situation. Thus generalisations intended by the teacher are not necessarily what will be noticed and constructed by students (Steele and Johanning, 2004).

Summary

There is evidence to show that, with suitable environments, tools, images and encouragement, learners can and do:

- generalise from what is offered and experienced
- look for analogies
- identify variables
- choose the most efficient variables, those with most connections
- see simultaneous variations
- observe and analyse change
- reason verbally before symbolising
- develop mental models and other imagery
- use past experience
- need knowledge of operations and situations to do all the above successfully
- particularly gifted mathematics students also:
    - quickly grasp the essence of a problem
    - see structure through surface features
    - switch between solution methods
    - reverse trains of thought
    - remembered the relationships and principles of a problem
    - do not necessarily display computational expertise.

Part 3: What happens with pedagogic intervention designed to address typical difficulties?

We have described what successful and unsuccessful learners do when faced with new and complex situations in mathematics. For this section we show how particular kinds of teaching aim to tackle the typical problems of teaching at this level. This depends on reports of teaching experiments and, as with the different approaches taken to algebra in the earlier paper, they show what it is possible for secondary students to learn in particular pedagogic contexts.

It is worth looking at the successes and new difficulties introduced by researchers and developers who have explored ways to influence learning without exacerbating the difficulties described above. We found broadly five approaches, though there are overlaps between them: focusing on development of mathematical thinking; task design; metacognitive strategies; the teaching of heuristics; and the use of ICT.

Focusing on mathematical thinking

Experts and novices see problems differently; and see different similarities and differences between problems, because experts have a wider repertoire of things to look for, and more experience about what is, and is not, worthwhile mathematically. Pedagogic intervention is needed to enable all learners to look for underlying structure or relationships, or to devise subgoals and reflect on the outcomes of pursuing these as successful students do. In a three-year course for 12- to 15-year-olds, Lamon educated learners to understand quantitative relationships and to mathematise experience by developing the habits of identifying quantities, making assumptions, describing relationships, representing relationships and classifying situations (1998). It is worth emphasising that this development of habits took place over three years, not over a few lessons or a few tasks.

- Students *can* develop habits of identifying quantities and relationships in situations, given extended experience.

Research which addresses development of mathematical thinking in school mathematics includes: descriptive longitudinal studies of cohorts of students who have been taught in ways which encourage mathematical enquiry and proof and comparative studies between classes taught in through enquiry methods and traditional methods. Most of these studies focus on the development of classroom practices and discourse, and how social aspects of the classroom influence the nature of mathematical knowledge. Other studies are of students being encouraged to use specific mathematical thinking skills, such as exemplification, conjecturing, and proof of the effects of a focus on mathematical thinking over time. These focused studies all suffer to some extent from the typical 'teaching experiment' problem of being designed to encourage X and students then are observed to do X. Research over time would be needed to demonstrate the effects of a focus on mathematical thinking on the nature of long-term learning. Longitudinal studies emphasise development of mathematical practices, but the value of these is assumed so they are outside the scope of this paper. However, it is worth mentioning that the CAME initiative appeared to influence the development of analytical and complex thinking both within mathematics and also in other subjects, evidenced in national test scores rather than only in study-specific tests (Johnson, Adhami and Shayer, 1988; Shayer, Johnson and Adhami, 1999). In this initiative teachers were trained to use materials which had been designed to encourage cycles of investigation: problem familiarity, investigating the problem, synthesising outcomes of investigation, abstracting the outcomes, applying this new abstraction to a further problem, and so on.

- Students can get better at thinking about and analysing mathematical situations, given suitable teaching.

Task design

Many studies of the complexity of tasks and the effect of this on solving appear to us to be the wrong way round when they state that problems are easier to solve if the tasks are stated more simply. For a mathematics curriculum the purpose of problem-solving is usually to learn how to mathematise, how to choose methods and representations, and how to contact big mathematical ideas – this cannot be achieved by simplifying problems so that it is obvious what to do to solve them.

Students who have spent time on complex mathematical activity, such as modelling and problem-solving, are not disadvantaged when they are tested on procedural questions against students who have had more preparation for these. This well-known result arises from several studies, such as that of Thompson and Senk (2001) in connection with the University of Chicago School Mathematics Project: those given a curriculum based on problems and a variety of exploratory activities did better on open-ended and complex, multistage tasks, than comparable groups taught in more conventional ways, and also did just as well on traditional questions. Senk and Thompson (2003) went on to collect similar results from eight mathematics teaching projects in the United States in which they looked specifically for students' development of 'basic skills' alongside problem-solving capabilities. The skills they looked for at secondary level included traditional areas of difficulty such as fractions computations and algebraic competence. Each project evaluated its findings differently, but overall the result was that students did as well or better than comparative students in basic mathematical skills at the appropriate level, and were better at applying their knowledge in complex situations. Additionally, several projects reported improved attainment for students of previously low attainment or who were 'at risk' in some sense. In one case, algebraic manipulation was not as advanced as a comparison group taught from a traditional textbook but teachers were able to make adjustments and restore this in subsequent cohorts without returning to a more limited approach. New research applying one of these curriculum projects in the United Kingdom is showing similar findings (Eade and Dickenson, 2006 *a*; Eade and Dickenson, Hough and Gough, 2006 *b*). A U.K. research project comparing two similar schools, in which the GCSE results of matched samples were compared, also showed that those who were taught through complex mathematical activity, solving problems and enquiring into mathematics, did better than students who were taught more procedurally and from a textbook. The GCSE scripts showed that the former group was more willing to tackle unfamiliar mathematics questions as problems to be solved, where the latter group tended to not attempt anything they had not been taught explicitly (Boaler, 1997). Other research also supports these results (Hembree, 1992; Watson and De Geest, 2005).

- Students who spend most of their time on complex problems can also work out how to do 'ordinary' maths questions.

Recent work by Swan (2006) shows how task design, based on introducing information which might conflict with students' current schema and which also includes pedagogic design to enable these conflicts to be explored collaboratively, can make a significant difference to learning. Students who had previously been failing in mathematics were able to resolve conflict through discussion with others in matching, sorting, relating and generating tasks. This led directly to improvements in conceptual understanding in a variety of traditionally problematic domains.

- Students can sort out conceptual confusions with others if the tasks encourage them to confront their confusion through contradiction.

Metacognitive strategies

Success in complex mathematical tasks is associated with a range of metacognitive orientation and execution decisions, but mostly with deliberate evaluating the effects of certain actions (Stillman and Galbraith, 1998). Reflecting on the effects of activity (to use Piaget's articulation) makes sense in the mathematics context, because often the ultimate goal is to understand relationships between independent and dependent variables. It makes sense, therefore, to wonder if teaching these strategies explicitly makes a difference to learning. Kramarski, Mevarech and Arami (2002) showed that explicitness about metacognitive strategies is important in success not only in complex authentic tasks but also in quite ordinary mathematical tasks. Kramarski (2004) went on to show that explicit metacognitive instruction to small groups provided them with ways to question their approach to graphing tasks. They were taught to discuss interpretations of the problem, predict the outcomes of using various strategies, and decide if their answers were reasonable. The groups who had been taught metacognitive methods engaged in discussions that were more mathematically focused, and did better on post-tests of graph interpretation and construction, than control groups. Discussion appeared to be a factor in their success. The value of metacognitive prompts also appears to be stronger if students are asked to write about their responses; students in a randomized trial tried more strategies if they were asked to write about them than those who were asked to engage in think-aloud strategies (Pugalee, 2004). In both these studies, the requirement and opportunity to express metacognitive observations turned out to be important. Kapa (2001) studied 441 students in four computer-instruction environments which offered different kinds of metacognitive prompting while they were working on mathematical questions: during the solution process, during and after the process, after the process, none at all. Those with prompts during the process were more successful, and the prompts made more difference to those with lower previous knowledge than to others. While this was an artificial environment with special problems to solve, the finding appears to support the view that teaching (in the form of metacognitive reminders and support) is important and that students with low prior knowledge can do better if encouraged to reflect on and monitor the effects of their activity. An alternative to explicit teaching and requests to apply metacognitive strategies is to incorporate them implicitly into the ways mathematics is done in classrooms. While there is research about this, it tends to be in studies enquiring into whether such habits are adopted by learners or not, rather than whether they lead to better learning of mathematics.

- Students can sometimes do better if they are helped to use metacognitive strategies.

- Use of metacognitive strategies may be enhanced: in small group discussion; if students are asked to write about them; and/or if they are prompted throughout the work.

Teaching problem-solving heuristics

The main way in which educators and researchers have explored the question of how students can get better at problem solving is by constructing descriptions of problem-solving heuristics, teaching these explicitly, and comparing the test performance of students who have and have not received this explicit teaching. In general, they have found that students do learn to apply such heuristics, and become better at problem-solving than those who have not had such teaching (e.g. Lucas, 1974). This should not surprise us.

A collection of clinical projects in the 1980s (e.g. Kantowski, 1977; Lee, 1982) which appear to show that students who are taught problem-solving heuristics get better at using them, and those who use problem-solving heuristics get better at problem solving. These results are not entirely tautologous if we question whether heuristics are useful for solving problems. The evidence suggests that they are (e.g. Webb, 1979 found that 13% of variance among 40 students was due to heuristic use), yet we do not know enough about how these help or hinder approaches to *unfamiliar* problems. For example, a heuristic which involves planning is no use if the situation is so unfamiliar that the students cannot plan. For this situation, a heuristic which involves collecting possible useful knowledge together (e.g. 'What do I know? What do I want?' Mason, Burton and Stacey , 1982) may be more useful but requires some initiative and effort and imagination to apply. The ultimate heuristic approach was probably Schoenfeld's (1982) study of seven students in which he elaborated heuristics in a multi-layered way, thus showing the things one can do while doing mathematical problem-solving to be fractal in nature, impossible to learn as a list, so that true mathematical problem-solving is a creative task involving a mathematical cast of mind (Krutetskii, 1976) and range of mathematical habits of mind (Cuoco, Goldenberg and Mark, 1997) rather than a list of processes.

Schoenfeld (1979; 1982) and Vos (1976) found that learners taught explicit problem-solving strategies are likely to use them in new situations compared to similar students who are expected to abstract processes for themselves in practice examples. There is a clear tension here between explicit teaching and the development of general mathematical awareness. Heuristics are little use without knowledge of when, why and how to use them. What is certainly true is that if learners perform learnt procedures, then we do not know if they are acting meaningfully or not. Vinner (1997) calls this 'the cognitive approach fallacy' – assuming that one can analyse learnt behavioural procedures as if they are meaningful, when perhaps they are only imitative or gap-filling processes.

Application of learnt heuristics can be seen as merely procedural if the heuristics do not require any interpretation that draws on mathematical repertoire, example spaces, concept images and so on. This means that too close a procedural approach to conceptualization and analysis of mathematical contexts is merely what Vinner calls 'pseudo'. There is no 'problem' if what is presented can be processed by heuristics which are so specific they can be applied like algorithmically. For example, finding formulae for typical spatial-numeric sequences (a common feature of the U.K. curriculum) is often taught using the heuristic 'generate a sequence of specific examples and look for patterns'. No initial analysis of the situation, its variables, and relevant choice of strategy is involved.

On the other hand, how are students to learn how to tackle problems if not given ideas about tactics and strategies? And if they are taught, then it is likely that some will misapply them as they do any learnt algorithm. This issue is unresolved, but working with unfamiliar situations and being helped to reflect on the effects of particular choices seem to be useful ways forwards.

There is little research evidence that students taught a new topic using problems with the explicit use of taught heuristics learn better, but Lucas (1974) did this with 30 students learning early calculus and they did do significantly better that a 'normal' group when tested. Learning core curriculum concepts through problems is under-researched. A recent finding reported by Kaminski, Sloutsky and Heckler (2006; 2008) is that learning procedurally can give faster access to underlying structure than working through problems. Our reading of their study suggests that this is not a robust result, since the way they categorise contextual problems and formal approaches differs from those used by the research they seek to refute.

- Students can apply taught problem-solving heuristics, but this is not always helpful in unfamiliar situations if their learning has been procedural.

One puzzle which arose in the U.S. Task Panel's review of comparative studies of students taught in different ways (NMAP, 2008) is that those who have pursued what is often called a 'problem-solving curriculum' turn out to be better at tackling unfamiliar situations using problem-solving strategies, but not better at dealing with 'simple' given word problems. How students can be better at mathematising real world problems and resolving them, but not better at solving given word problems? This comment conceals three important issues: firstly, 'word problems', as we have shown, can be of a variety of kinds, and the 'simple' kinds call on different skills than complex realistic situations; secondly, that according to the studies reported in Senk and Thompson (2003) performance on 'other aspects' of mathematics such as solving word problems may not have improved, but neither did it decline; thirdly, that interpretation of these findings as good or bad depends on curriculum aims[6]. Furthermore, the panel confined its enquiries to the U.S. context and did not take into account the Netherlands research in which the outcomes of 'realistic' activity are scaffolded towards formality. The familiar phrase 'use of real world problems' is vague and can include a range of practices.

The importance of the difference in curriculum aims is illustrated by Huntley, Rasmussen, Villarubi, Sangtog and Fey (2000) who show, along with other studies, that students following a curriculum focusing on algebraic problem solving are better at problem solving, especially with support of graphical calculators, but comparable students who have followed a traditional course did better in a test for which there were no graphical calculators available and were also more fluent at manipulating expressions and working algebraically without a context. In the Boaler (1997) study, one school educated students to take a problem-solving view of all mathematical tasks so that what students 'transferred' from one task to another was not knowledge of facts and methods but a general approach to mathematics. We described earlier how this helped them in examinations.

- There is no unique answer to the questions of why and when students can or cannot solve problems – it depends on the type of problem, the curriculum aim, the tools and resources, the experience, and what the teacher emphasises.

How can students become more systematic at identifying variables and applying operations and inverses to solve problems? One aspect is to be clear about whether the aim is for a formal method of solution or not. Another is experience so that heuristics can be used flexibly because of exposure to a range of situations in which this has to be done – not just being given equations to be solved; not just constructing general expressions from sequences; etc. The value of repeated experience might be what is behind a finding from Blume and Schoen (1988) in which 27 14-year-old students who had learnt to programme in Basic were tested against 27 others in their ability to solve typical mathematical word problems in a pen and paper environment. Their ability to write equations was no different but their ability to solve problems systematically and with frequent review was significantly stronger for the Basic group. Presumably the frequent review was an attempt to replicate the quick feedback they would get from the computer activity. However, another Basic study which had broader aims (Hatfield and Kieren, 1972) implied that strengths in problem-solving while using Basic as a tool were not universal across all kinds of mathematics or suitable for all kinds of learning goal.

A subset of common problem-solving heuristics are those that relate specifically to modelling, and modelling can be used as a problem-solving strategy. Verschaffel and De Corte (1997) working with 11-year-olds show that rather than seeing modelling-of and modelling-for as two separate kinds of activity, a combination of the two, getting learners to frame real problems as word problems through modelling, enables learners to do as well as other groups in both 'realistic' mathematical problem-solving and with word problems when compared to other groups. Their students developed a disposition towards modelling in all situational problems.

- Students may understand the modelling process better if they have to construct models of situations which then are used as models for new situations.

- Students may solve word problems more easily if they have experience of expressing realistic problems as word problems themselves.

Using ICT

Students who are educated to use available handheld technology appear to be better problem solvers. The availability of such technology removes the need to do calculations, gives immediate feedback, makes reverse checking less tedious, allows different possibilities to be explored, and gives more support for risk taking. If the purpose of complex tasks is to show assessors that students can 'do' calculations than this result is negative; if the purpose is to educate students to deal with non-routine mathematical situations, then this result is positive.

Evidence of the positive effects of access to and use of calculators is provided by Hembree and Dessart (1986) whose meta-analysis of 79 research studies showed conclusively that students who had sustained access to calculators had better pencil-and-paper and problem-solving skills and more positive attitudes to mathematics than those without. The only years in which this result was not found was grade 4 in the United States, and we assume that this is because calculator use may make students reluctant to learn some algorithmic approaches when this is the main focus of the curriculum. In the United Kingdom, these positive results were also found in the 1980s in the CAN project, with the added finding that students who could choose which method to use, paper, calculator or mental, had better mental skills than others.

We need to look more closely at why this is, what normal obstacles to learning are overcome by using technology and what other forms of learning are afforded? Doerr and Zangor (2000) recognized that handheld calculators offered speed and facility in computation, transformation of tasks, data collection and analysis, visualisation, switching representations, checking at an individual level but hindered communication between students. Graham and Thomas (2000) achieved significant success using graphical calculators in helping students understand the idea of variable. The number of situations, observation of variation, facility for experimentation, visual display, instant feedback, dynamic representation and so on contributed to this.

- Students who can use available handheld technology are better at problem solving and have more positive views of mathematics.

We do not know if it is only in interactive computer environments that school students can develop a deep, flexible and applicable knowledge of functions, but we do know that the affordances of such ICT environments allow all students access to a wide variety of examples of functions, and gives them the exploratory power to see what these mean in relation to other representations and to see the effects on one of changing the other. These possibilities are simply not available within the normal school time and place constraints without hands-on ICT. For example, Godwin and Beswetherick (2002) used graphical software to enhance learners' understanding of quadratic functions and point out that the ICT enables the learning environment to be structured in ways that draw learners' attention to key characteristics and variation. Schwarz and Hershkowitz (1999) find that students who have consistent access to such tools and tasks develop a strong repertoire of prototypical functions, but rather than being limited by these can use these as levers to develop other functions, apply their knowledge in other contexts and learn about the attributes of functions as objects in themselves.

Software that allows learners to model dynamic experiences was developed by Kaput (1999) and the integration of a range of physical situations, represented through ICT, with mental modelling encouraged very young students to use algebra to pose questions, model and solve questions. Entering algebraic formulae gave them immediate feedback both from graphs and from the representations of situations. In extended teaching experiments with upper primary students, Yerushalmy encouraged them to think in terms of the events and processes inherent in situations. The software she used emphasised change over small intervals as well as overall shape. This approach helped them to understand representations of quantities, relationships among quantities, and relationships among the representations of quantities in single variable functions (Yerushalmy, 1997). Yerushalmy claims that the shifts between pointwise and holistic views of functions are more easily made in technological environments because, perhaps, of the easy availability of several examples and feedback showing translation between graphs, equations and data sets. She then gave them situations which had more than one input variable, for example the cost of car rental which is made up of a daily rate and a mileage rate. This kind of situation is much harder to analyse and represent than those which have one independent and one dependent variable. To describe the effects of the first variable the second variable has to be invariant, and *vice versa*. In discussion, a small sample of students tried out relations between various pairs of variables and decided, for themselves, that two of the variables were independent and the final cost depended on both of them. They then tried to draw separate graphs in which one of the variables was controlled. We are not claiming that all students can do this by themselves, but that these students could do it, is remarkable. This study suggests that students for whom the ideas of variables, functions, graphs and situations are seen as connected have the skills to analyse unfamiliar and more complex situations mathematically. Nemirovsky (1996) suggests another reason is that students can relate different representations to understand the story the graph is representing. He undertook a multiple representation teaching experiment with 15- and 16-year-olds in which graphs were generated using a toy car and a motion detector. Having seen the connection between one kind of movement and the graph, students were then asked to predict graphs for other movements, showing how their telling of the story of the movement related to the graphs they were drawing. Students could analyse continuous movement that varied in speed and direction by seeing it to be a sequence of segments, then relate segments of movement to time, and then integrate the segments to construct a continuous graph. Additionally, comparing the real movement, their descriptions of it, and graphs also enabled them to correct and adjust their descriptions. Nemirovsky found that switching between these representations helped them to see that graphs told a continuous story about situations. Rather than expressing instances of distance at particular times, the students were talking about speed, an interpretation of rate from the graph rather than a pointwise use of it. Using a similar approach with nine- and ten-year-olds Nemirovsky found that these students were more likely to 'read' symbolic expressions as relationships between variables rather than merely reading them from left to right as children taught traditionally often do.

Students at all levels can achieve deep understanding of concepts and also learn relevant graphing and function skills themselves, given the power to see the effects of changes in multiple representations, taking much less time than students taught only skills and procedures through pencil-and-paper methods (Heid, 1988; Ainley and her colleagues, e.g. 1994).

- Computer-supported multiple representational contexts can help students understand and use graphs, variables, functions and the modelling process.

Recommendations

For curriculum and practice

The following recommendations for secondary mathematics teaching draw on the conclusions summarized above.

Learning new concepts

- Teaching should take into account students' natural ways of dealing with new perceptual and verbal information (see summaries above), including those ways that are helpful for new mathematical ideas and those that obstruct their learning.

- Schemes of work and assessment should allow enough time for students to adapt to new meanings and move on from earlier methods and conceptualisations; they should give time for new experiences and mathematical ways of working to become familiar in several representations and contexts before moving on.

- Choice of tasks and examples should be purposeful, and they should be constructed to help students shift towards understanding new variations, relations and properties. Such guidance includes thinking about learners' initial perceptions of the mathematics and the examples offered. Students can be guided to focus on critical aspects by the use of controlled variation, sorting and matching tasks, and multiple representations.

- Students should be helped to balance the need for fluency with the need to work with meaning.

Applications, problem-solving, modelling, mathematical thinking

- As above, teaching should take into account students' natural ways of dealing with new perceptual and verbal information (see summaries above), including those ways that are helpful for new mathematical ideas and those that obstruct their learning.

- Schemes of work should allow for students to have multiple experiences, with multiple representations, over time to develop mathematically appropriate 'habits of mind'.

- The learning aims and purpose of tasks should be clear: whether they are to develop a broader mathematical repertoire; to learn modelling and problem-solving skills; to understand the issues within the context better etc.

- Students need help and experience to know when to apply formal, informal or situated methods.

- Students need a repertoire of appropriate functions, operations, representations and mathematical methods in order to become good applied mathematicians. This can be gained through multiple experiences over time.

- Student-controlled ICT supports the development of knowledge about mathematics and its applications; student-controlled ICT also provides authentic working methods.

For policy

- These recommendations indicate a training requirement based on international research about learning, rather than merely on implementation of a new curricula.

- There are resource implications about the use of ICT. Students need to be in control of switching between representations and comparisons of symbolic expression in order to understand the syntax and the concept of functions. The United Kingdom may be lagging behind the developed world in exploring the use of spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

- The United Kingdom is in the forefront of new school mathematics curricula which aim to prepare learners better for using mathematics in their economic, intellectual and social lives. Uninformed teaching which focuses only on methods and test-training is unlikely to achieve these goals.

- Symbolic manipulators, graph plotters and other algebraic software are widely available and used to allow people to focus on meaning, application and implications. Students should know how to use these and how to incorporate them into mathematical explorations and extended tasks.

- A strong message emerging about learning mathematics at this level is that students need multiple experiences over time for new-to-them ways of thinking and working to become habitual.

For research

- There are few studies focusing on the introduction of specific new ideas, based on students' existing knowledge and experience, at the higher secondary level. This would be a valuable research area. This relates particularly to topics which combine concepts met earlier in new ways, such as: trigonometry, quadratics and polynomials, and solving simultaneous equations. (There is substantial research about calculus beyond the scope of this paper.)

- There are many studies on the development of modelling and problem-solving skills, but a valuable area for research, particularly in the new U.K. context at 14–19, would be the relationship between these and mathematical conceptual development which, as we have shown above, involves similar – not separate – learning processes if it is to be more than trial-and-error.

- There is little research which focuses on the technicalities of good mathematics teaching, and it would be valuable to know more about: use of imagery, the role of visual and verbal presentations, development of mathematical thinking, development of geometrical reasoning, how representations commonly used in secondary mathematics influence learning, and how and why some students manage to avoid over-generalising about facts, methods, and approaches.

- There is very little research on statistical reasoning, non-algebraic modelling, and learning mathematics with and without symbolic manipulators.

Endnotes

1 'Induction' here is the process of devising plausible generalisations from several examples, not mathematical inductive reasoning.

2 They claimed that the post-test was contextual because objects were used, but the relations between the objects were spurious so the objects functioned as symbols rather than as contextual tools.

3 There a little research on interpreting problems in statistical terms, but this is beyond the scope of this paper.

4 Modelling has other meanings as well in mathematics education, such as the provision or creation of visual and tactile models of mathematical ideas, but here we are sticking to what mathematicians mean by modelling.

5 Also, as is recognised in the Realistic Mathematics Education and some other projects, students are able to engage in ad hoc problem solving from a young age.

6 Meta-analysis of the studies they used is beyond the scope of this review.

Acknowledgements

This paper was produced with the help of Nusrat Rizvi who did much of the technical work.

References

Ainley, J. (1994) Building on children's intuitions about line graphs. In J. da Ponte and J. Matos (eds.) *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education* vol. 1, pp. 1-8, University of Lisbon, Portugal.

Ainley, J., Pratt, D., and Nardi, E. (2001). Normalising: children's activity to construct meanings for trend.' *Educational Studies in Mathematics,* 45 (1-3), 131-146.

Alexander, P., White, C. and Daugherty, M. (1997) Analogical reasoning and early mathematics learning. Pp. 117-148. In English (ed.) *Mathematical reasoning: analogies, metaphors and images*. Mahwah, NJ: Erlbaum.

Anderson, J. R., Corbett, A. T., Koedinger, K., and Pelletier, R. (1995). Cognitive tutors: Lessons learned. *The Journal of Learning Sciences, 4,* 167-207.

Ballew, Hunter; Cunningham, James W. (1982) Diagnosing strengths and weaknesses of sixth-grade students in solving word problems *Journal for Research in Mathematics Education* 13(3) 202-210

Bassler, O., Beers, M. and Richardson, L. (1975) Comparison of two instructional strategies for teaching the solution of verbal problems *Journal for Research in Mathematics Education* 6(3)170-177.

Bednarz, N. and Janvier, B. (1996). Emergence and development of algebra as a problem solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran and L. Lee, L. (Eds.), *Approaches to Algebra* (pp. 115-136). Dordrecht/Boston/London: Kluwer Academic Publishers.

Blume, G. and Schoen, H. (1988).Mathematical Problem-Solving Performance of Eighth-Grade Programmers and Nonprogrammers *Journal for Research in Mathematics Education* 19(2) 142-156

Boaler, J. (1997) *Experiencing School Mathematics: Teaching Styles, Sex and Setting.* Open University Press: Buckingham, England

Bock, D. De; Verschaffel, L.; Janssens, D. (1998) The Predominance of the Linear Model in Secondary School Students' Solutions of Word Problems Involving Length and Area of Similar Plane Figures *Educational Studies in Mathematics* 35(1) 65-83

Boero, P. (2001). Transformation and Anticipation as Key Processes in Algebraic Problem Solving. In Sutherland, R., Rojano, T., Bell, A. and Lins, R.(eds.) *Perspectives on School Algebra*. pp. 99-119 Dordrecht, Kluwer.

Booth, Lesley R.(1981) Child-Methods in Secondary Mathematics *Educational Studies in Mathematics* 12(1) 29-41

Brainerd, C. J., and Reyna, V. F. (1993). Memory independence and memory interference in cognitive development. *Psychological Review*, 100, 42-67.

Caldwell, J., Goldin, G. (1987) Variables Affecting Word Problem Difficulty in Secondary School Mathematics *Journal for Research in Mathematics Education* 18(3) 187-196.

Campbell, K.; Collis, K.; Watson, J. (1995) Visual Processing during Mathematical Problem Solving *Educational Studies in Mathematics* 28(2) 177-194

Carlson, M., Jacobs, S., Coe, E., Larsen, S. and Hsu, E. (2002) Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study *Journal for Research in Mathematics* 33(5) 352-378

Carpenter, T.; Ansell, E.; Franke, M.; Fennema, E.; Weisbeck, L. (1993) Models of Problem Solving: A Study of Kindergarten Children's Problem-Solving Processes *Journal for Research in Mathematics Education* 24(5) 428-441

Chik, P. and Lo. M. (2003) Simultaneity and the Enacted Object of Learning. In F. Marton and A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp.89-110) Mahwah, New Jersey: Lawrence

Clement, J. (1982) Algebra Word Problem Solutions: Thought Processes Underlying a Common Misconception *Journal for Research in Mathematics Education* 13(1) 16-30

Clements, M. (1980) Analyzing children's errors on written mathematical tasks. *Educational Studies in Mathematics*, 11 (1), 1-21

Confrey, J. and Smith, E. (1994) Exponential Functions, Rates of Change, and the Multiplicative Unit *Educational Studies in Mathematics* 26(3) 135-164

Cooper, B. and Dunne, M) (2000) *Assessing Children's Mathematical Knowledge: Social Class, Sex and Problem-Solving*, Maidenhead: Open University Press

Cooper, B. & Harries, T. (2002) Children's Responses to Contrasting 'Realistic' Mathematics Problems: Just How Realistic Are Children Ready to Be? *Educational Studies in Mathematics* 49(1) 1-23

Cuoco, A., Goldenberg, E. P., and J. Mark. (1997). 'Habits of Mind: an organizing principle for mathematics curriculum.' *Journal of Mathematical Behavior*, 15(4), 375-402.

Dickson, L. (1989) Equations. In D. Johnson (ed.) *Children's Mathematical Frameworks 8 to 13: a study of classroom teaching.* (p.151-190) Windsor: NFER-nelson

Doerr, H. and Zangor, R. (2000) Creating meaning for and with the graphing calculator *Educational Studies in Mathematics* 41(2) 143-163

Eade, F. and Dickinson, P. (2006a) *Exploring Realistic Mathematics Education in English Secondary Schools*. Proceedings PME July 2006

Eade, F., Dickinson, P., Hough S. and Gough S (2006b) *Developing Mathematics through Contexts: year 2*. Annual Report September 2006 www.partnership.mmu.ac.uk/cme

English, L. and Halford, G. (1995) *Mathematics Education: Models and Processes*. Mahwah, NJ: Lawrence Erlbaum Associates

English, L. and Sharry, P. (1996) Analogical Reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics*, Vol. 30, No. 2.135-157.

Fischbein, E. (1987) *Intuition in science and mathematics : an educational approach*. Dordrecht; D. Reidel

Foxman, D., Ruddock, G., Joffe, L., Mason, K., Mitchell, P., and Sexton, B. (1985) *A Review of Monitoring in mathematics 1978 to 1982,* Assessment of Performance Unit, Department of Education and Science, London.

Freudenthal, H. (1991) *Didactical Phenomenology of Mathematical Structures*, Dordrecht: D. Reidel.

Friel, S., Curcio, F. and Bright, G. (2001) Making Sense of Graphs: Critical Factors Influencing Comprehension and Instructional Implications *Journal for Research in Mathematics Education* 32(2), 124-158.

Godwin,S. and Beswetherick, R. (2002) Reflections on the role of task structure in learning the properties of quadratic functions with ICT - a design initiative. In S. Pope (ed.) *Proceedings of the British Society for Research into Learning Mathematics day conference* 16 November, Nottingham University, British Society for Research into Learning Mathematics. pp. 43-48.

Goldin, G. (2002) Representation in mathematical learning in English, L (ed.) *Handbook of international research in mathematics education* (pp. 197-218) Mahwah, NJ: Erlbaum.

Graham, A. and ; Thomas, M. (2000) Building a versatile understanding of algebraic variables with a graphic calculator *Educational Studies in Mathematics* 41(3) 265-282

Gravemeijer, K. and Doorman, M. (1999) Context Problems in Realistic Mathematics Education: A Calculus Course as an Example *Educational Studies in Mathematics* 39(1-3) 111-129

Halford, G. (1999) The Development of Intelligence Includes the Capacity to Process Relations of Greater Complexity.in M. Anderson (ed.) *The Development of Intelligence* pp. 193-213 Hove: Psychology Press

Hart, K. (Ed.) (1981) *Children's Understanding of Mathematics 11-16*, Murray, London..

Hatfield, L. and Kieren, T. (1972) Computer-Assisted Problem Solving in School Mathematics *Journal for Research in Mathematics Education* 3(2) 99-112

Heid, M. (1988) Resequencing Skills and Concepts in Applied Calculus Using the Computer as a Tool *Journal for Research in Mathematics Education* 19(1) 3-25

Hembree, Ray (1992) Experiments and Relational Studies in Problem Solving: A Meta-Analysis *Journal for Research in Mathematics Education* 23(3) 242-273

Hembree, Ray; Dessart, Donald J. (1986) Effects of Hand-Held Calculators in Precollege Mathematics Education: A Meta-Analysis *Journal for Research in Mathematics Education* 17(2) 83-99

Hershkowitz, R., Schwarz, B.and Dreyfus, T. (2001) Abstraction in Context: Epistemic Actions. *Journal for Research in Mathematics Education*, 32 (2) 195-222

Heuvel-Panhuizen, M. van den (2003) The Didactical Use of Models in Realistic Mathematics Education: An Example from a Longitudinal Trajectory on Percentage *Educational Studies in Mathematics* 54(1) 9-35

Holyoak, K. and Thagard, P. (1995) *Mental Leaps: Analogy in Creative Thought.* Cambridge, MA: Bradford Book, MIT Press

Huntley, M., Rasmussen, C., Villarubi, R., Sangtong, J. and Fey, J. (2000) Effects of Standards-Based Mathematics Education: A Study of the Core-plus Mathematics Project Algebra and Functions Strand *Journal for Research in Mathematics Education* 31(3) 328-361

Inhelder, B. & Piaget, J. (1959). *The Growth of Logical Thinking From Childhood to Adolescence*. NY: Basic Books

Johnson, D.C., Adhami, M. and Shayer, M. (1998). Does CAME work? Summary report on phase 2 of the Cognitive Acceleration in Mathematics Education, CAME, project. In *Proceedings of the British Society for Research into Learning Mathematics Day Conference, Bristol, 15 Nov. 1997*. London: BSRLM, 26-31.

Kaminski, J., Sloutsky, V. and Heckler, A. (2006) Do Children Need Concrete Instantiations to Learn an Abstract Concept? *Proceedings of the XXVIII Annual Conference of the Cognitive Science Society*, 1167-1172. Mahwah, NJ: Erlbaum

Kaminski, J., Sloutsky, V. and Heckler, A. (2008) The Advantage of Abstract Examples in Learning Math *Science* 320 (454-455).

Kantowski, M. (1977) Processes Involved in Mathematical Problem Solving *Journal for Research in Mathematics Education* 8(3) 163-180

Kapa, E. (2001) A Metacognitive Support during the Process of Problem Solving in a Computerized Environment *Educational Studies in Mathematics* 47(3) 317-336

Kaput, J. (1999) *Teaching and Learning a New Algebra With Understanding* Dartmouth: University of Massachusetts.

Kaput, J. J. (1991). Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp. 53-74). Netherlands: Kluwer Academic Publishers.

Kieran, C. (1992). The learning and teaching of algebra, in D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, Macmillan, New York, p390-419.

Kieran, C., Boileau, A.and Garancon, M. (1996) Introducing algebra by means of a technology-supported, functional approach. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. Pp. 257-294 Kluwer, Dordrecht

Kintsch, W. (1986) Learning from text. *Cognition and Instruction* 3(2).

Knuth, E. J. (2000). Student understanding of the cartesian connection: An exploratory study. *Journal for Research in Mathematics Education,* 31(4), 500-508.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division problems. *Journal for Research in Mathematics Education, 20*, 147-158.

Kramarski, B. (2004) Making sense of graphs: Does metacognitive instruction make a difference on students' mathematical conceptions and alternative conceptions? *Learning and Instruction* 14, 593-619.

Kramarski, B., Mevarech, Z. and Arami, M. (2002) The Effects of Metacognitive Instruction on Solving Mathematical Authentic Tasks *Educational Studies in Mathematics* 49(2) 225-250

Krutetskii, j, V. (1976). *The psychology of mathematical abilities in school children.* Chicago: University of Chicago Press.

Lamon, S. (1998) Algebra: Meaning through modeling. In A. Oliver and K. Newstead (eds.) Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, vol. 3, pp.167-174, University of Stellenbosch: Stellenbosch.

Lee, K. (1982) Fourth Graders' Heuristic Problem-Solving Behavior *Journal for Research in Mathematics Education* 13(2) 110-123

Lesh, R. and Doerr, H. (2003). Foundations of a models and modelling perspective on mathematics teaching, learning and problem solving. In R. Lesh and H. Doerr (Eds.) *Beyond constructivisim: A models and modeling perspectives on mathematics problem solving, learning and teaching.*. Mahwah, NJ: Lawrence Earlbaum.

Lucas, J. (1974) The Teaching of Heuristic Problem-Solving Strategies in Elementary Calculus *Journal for Research in Mathematics Education* 5(1) 36-46.

MacGregor, M. and Stacey, K. (1993) Cognitive Models Underlying Students' Formulation of Simple Linear Equations *Journal for Research in Mathematics Education*, Vol. 24, No. 3.,217-232.

Mason, J. Burton L. and Stacey K. (1982) *Thinking Mathematically*, Addison Wesley, London.

Nathan, M. and Koedinger, K. (2000) Teachers' and Researchers' Beliefs about the Development of Algebraic Reasoning *Journal for Research in Mathematics Education* 168-190

Nemirovsky, R. (1996) Mathematical narratives. In N. Bednarz, C. Kieran and L. Lee, L. (Eds.), *Approaches to Algebra* (pp.197-223) Dordrecht/Boston/London: Kluwer Academic Publishers.

Nesher, P. and M. Sukenik (1991). The Effect of Formal Representation on the Learning of Ratio Concepts. *Learning and Instruction* 1: 161 - 175.

Nesher, P., Hershkovitz, S. and Novotna, Jarmila (2003) Situation Model, Text Base and What Else? Factors Affecting Problem Solving *Educational Studies in Mathematics* 52(2) 151-176

NMAP, National Mathematics Advisory Panel (2008) downloaded April 2008 from: http://www.ed.gov/about/bdscomm/list/mathpanel/index.html

O'Callaghan, B. (1998) Computer-Intensive Algebra and Students' Conceptual Knowledge of Functions *Journal for Research in Mathematics Education* 29(1) 21-40

Polya, G. (1957) *How to Solve It*, 2nd ed., Princeton University Press.

Pugalee, D. (2004) A Comparison of Verbal and Written Descriptions of Students' Problem Solving Processes Educational Studies in Mathematics 55(1) 27-47

Runesson, U., and Mok, I. A. C. (2004). Discernment and the Question, 'What can be learned?' In F. Marton and A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 63-87). Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc., Publishers.

Ryan, J. and Williams, J. (2007) *Children's Mathematics 4 -15: learning from errors and misconceptions* Maidenhead Open University Press.

Schoenfeld, A. (1979) Explicit Heuristic Training as a Variable in Problem-Solving Performance *Journal for Research in Mathematics Education* 10(3) 173-187.

Schoenfeld, A. (1982) Measures of Problem-Solving Performance and of Problem-Solving Instruction. *Journal for Research in Mathematics Education* 13 (1) 31-49

Schwarz, B., Hershkowitz, R> (1999) Prototypes: Brakes or Levers in Learning the Function Concept? The Role of Computer Tools *Journal for Research in Mathematics Education* 30(4) 362-389

Senk, S. L. & Thompson, D. R. (*2003*). *Standards-based school mathematics curricula: What are they? What do students learn?* Mahwah, NJ: Lawrence Erlbaum

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.

Sfard, A. and Linchevski, L. (1994) The Gains and the Pitfalls of Reification: The Case of Algebra *Educational Studies in Mathematics* 26(2-3) 191-228

Shayer, M., Johnson, D.C. and Adhami, M. (1999). Does CAME work (2)? Report on Key Stage 3 results following the use of the Cognitive Acceleration in Mathematics Education, CAME, project in Year 7 and 8. *In Proceedings of the British Society for Research into Learning Mathematics Day Conference, St. Martin's College, Lancaster, 5 June* London: BSRLM, 79-84.

Silver, E. (1981) Recall of Mathematical Problem Information: Solving Related Problems *Journal for Research in Mathematics Education* 12(1) 54-64

Slavit, D. (1997) An Alternate Route to the Reification of Function *Educational Studies in Mathematics* 33(3) 259-281

Squire, S., Davies, C. and Bryant, P. (2004) Does the cue help? Children's understanding of multiplicative concepts in different problem contexts *British Journal of Educational Psychology,* 515-532

Steele, D. and Johanning, D. (2004) A Schematic-Theoretic View of Problem Solving and Development of Algebraic Thinking *Educational Studies in Mathematics* 57(1) 65-90

Stillman, G. and Galbraith, P. (1998) Applying Mathematics with Real World Connections: Metacognitive Characteristics of Secondary Students *Educational Studies in Mathematics* 36(2) 157-195

Swafford, J. and Langrall, C. (2000).Preinstructional Use of Equations to Describe and Represent Problem Situations *Journal for Research in Mathematics Education* 31(1) 89-112

Swan, M. (2006) *Collaborative Learning in Mathematics: A Challenge to Our Beliefs and Practices* London: National Institute of Adult Continuing Education.

Sweller, J. and Leung-Martin, L. (1997) Learning from equations or words.
Instructional Science, 25(1) 37-70.

Tall, D. O., (2004), Thinking through three worlds of mathematics, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen, Norway.

Tall, D.O. & Vinner S. (1981), Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-169.

Tanner, H. and Jones, S. (1994) Using Peer and Self-Assessment to Develop Modelling Skills with Students Aged 11 to 16: A Socio-Constructive View *Educational Studies in Mathematics* 27(4) 413-431

Thompson, D. and Senk, S. (2001) The Effects of Curriculum on Achievement in Second-Year Algebra: The Example of the University of Chicago School Mathematics Project *Journal for Research in Mathematics Education* 32(1) 58-84

Tirosh, D. and Stavy, R. (1999) Intuitive rules: A way to explain and predict students' reasoning *Educational Studies in Mathematics* 38(1-3) 51-66

Treffers, A. (1987) *Three dimensions. A model of goal and theory description in mathematics instruction – The Wiskobas project.* Dordrecht: Reidel

Trelinski, G (1983) Spontaneous mathematization of rsituations outside mathematics *Educational Studies in Mathematics* 14(3) 275-284

Trzcieniecka-Schneider, I. (1993) Some remarks on creating mathematical concepts *Educational Studies in Mathematics* 24(3) 257-264.

Van Dooren, W., De Bock, D., Janssens, D. and Verschaffel, L. (2007) Pupils' over-reliance on linearity: A scholastic effect? *British Journal of Educational Psychology*, 77 (2) 307-321.

Van Dooren, W.; De Bock, D., Weyers, D. and Verschaffel, L. (2004) The predictive power of intuitive rules: A critical analysis of the impact of 'More A-More B' and 'Same A-Same B' *Educational Studies in Mathematics* 56(2/3) 179-207

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh and M. Landau (Eds.), *Acquisition of mathematics concepts and processes*. New York: Academic Press.

Verschaffel, L. and De Corte, E. (1997) Teaching Realistic Mathematical Modeling in the Elementary School: A Teaching Experiment with Fifth Graders *Journal for Research in Mathematics Education* 28(5) 577-601

Verschaffel, L., De Corte, E. and Vierstraete, H. (1999) Upper Elementary School Pupils' Difficulties in Modeling and Solving Nonstandard Additive Word Problems Involving Ordinal Numbers *Journal for Research in Mathematics Education* 30(3) 265-285

Vicente, S., Orrantia, J. and Verschaffel, L. (2007) Influence of situational and conceptual rewording on word problem solving *British Journal of Educational Psychology*, 77(4) 829-848

Vinner, S. (1997) The Pseudo-Conceptual and the Pseudo-Analytical Thought Processes in Mathematics Learning *Educational Studies in Mathematics* 34(2) 97-129

Vos, K. (1976) The Effects of Three Instructional Strategies on Problem-Solving Behaviors in Secondary School Mathematics *Journal for Research in Mathematics Education* 7(5) 264-275

Watson, A. & De Geest, E. (2005) Principled teaching for deep progress: improving mathematical learning beyond methods and materials. *Educational Studies in Mathematics,* 58 (2), 209-234.

Webb, N. (1979) Processes, Conceptual Knowledge, and Mathematical Problem-Solving Ability *Journal for Research in Mathematics Education* 10(2) 83-93.

White, P. and Mitchelmore, M. (1996) Conceptual Knowledge in Introductory Calculus *Journal for Research in Mathematics Education* 27(1) 79-95.

Wollman, W. (1983) Determining the Sources of Error in a Translation from Sentence to Equation *Journal for Research in Mathematics Education* 14(3) 169-181

Yerushalmy, M. (1997) Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education* 28(4), 431-466

Yerushalmy, M. and Gilead, S. (1999) Structures of Constant Rate Word Problems: A Functional Approach Analysis *Educational Studies in Mathematics* 39(1-3) 185-203

Yerushalmy, M.. and Schwartz, J. (1993), Seizing the Opportunity to Make Algebra Mathematically and Pedagogically Interesting, in T. Romberg, E. Fennema and T. Carpenter, (Eds.), *Integrating Research on the Graphical Representation of Functions*, Lawrence Erlbaum Associates, Hillsdale, New Jersey.

Zazkis, R. (1999).Intuitive rules in number theory: example of the more of A, the more of B rule implementation. *Educational Studies in Mathematics*, 40(2)197-209

Published by the Nuffield Foundation, 28 Bedford Square, London WC1B 3JS
Telephone 020 7631 0566

ISBN 978-0-904956-74-0

www.nuffieldfoundation.org