Nuffield Foundation

Key understandings in mathematics learning

Introduction and summary of findings

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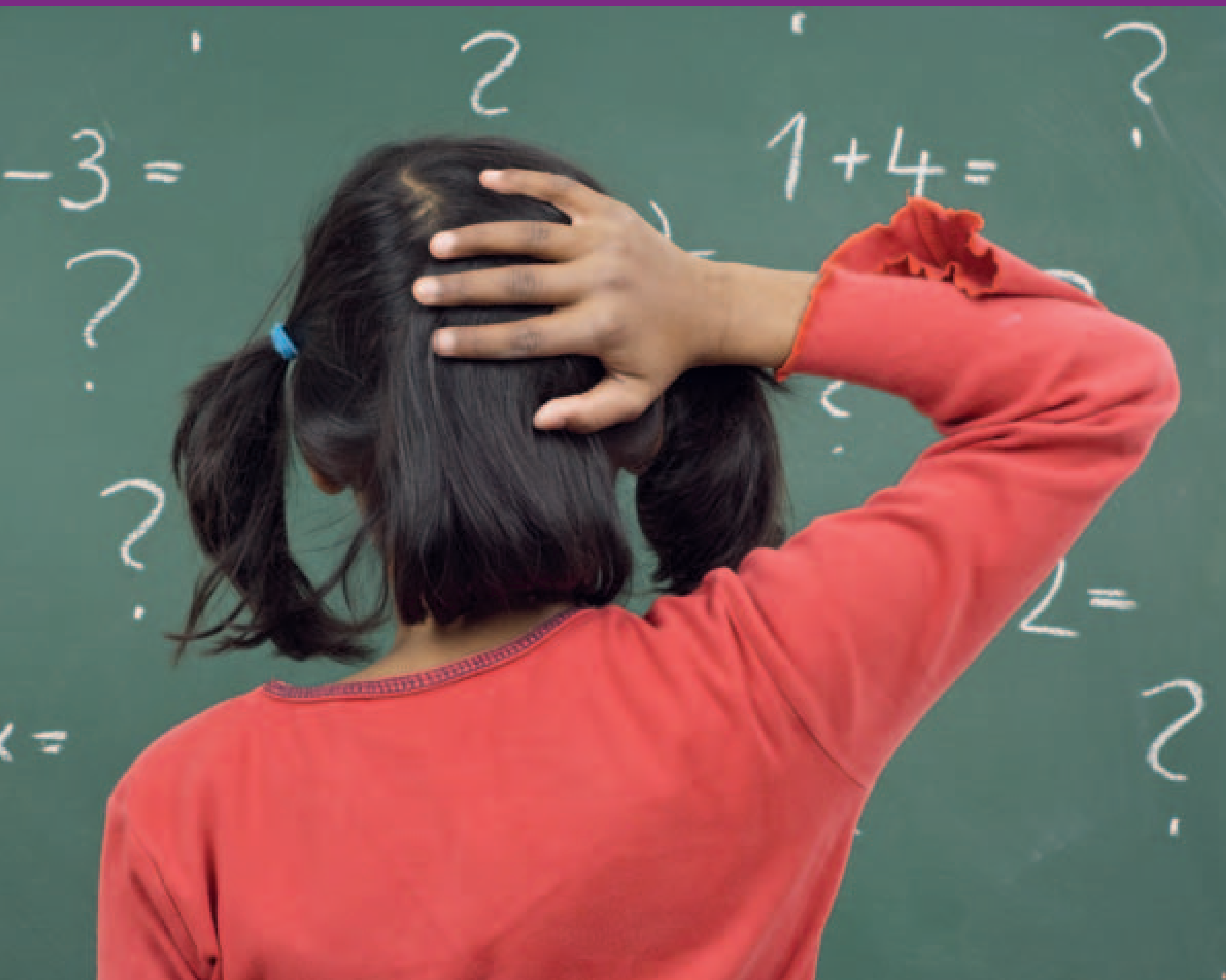

Introduction

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Summaries of papers 1-7 have been published together as *Key understandings in mathematics learning: Summary papers*.

The papers, together with the *Summary papers*, are available to download from our website, www.nuffieldfoundation.org

The review as a whole illuminates important aspects of mathematics learning from the perspectives of educational psychology and practice. It identifies important issues of significance to policy makers and practitioners as well as identifying significant gaps in our evidence base.

We are grateful to the authors for their commitment to this task and for producing such a comprehensive analysis of the extensive literature in this important field. We welcome the review and are confident it will usefully inform continuing debates about how best to improve curriculum design, teaching and learning for all students of elementary mathematics.

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of findings

Aims

Our aim in the review is to present a synthesis of research on mathematics learning by children from the age of five to the age of sixteen years and to identify the issues that are fundamental to understanding children's mathematics learning. In doing so, we concentrated on three main questions regarding key understandings in mathematics.

- What insights must students have in order to understand basic mathematical concepts?
- What are the sources of these insights and how does informal mathematics knowledge relate to school learning of mathematics?
- What understandings must students have in order to build new mathematical ideas using basic concepts?

Theoretical framework

While writing the review, we concluded that there are two distinct types of theory about how children learn mathematics.

Explanatory theories set out to explain how children's mathematical thinking and knowledge change. These theories are based on empirical research on children's solutions to mathematical problems as well as on experimental and longitudinal studies. Successful theories of this sort should provide insight into the causes of children's mathematical development and worthwhile suggestions about teaching and learning mathematics.

Pragmatic theories set out to investigate what children ought to learn and understand and also identify obstacles to learning in formal educational settings. Pragmatic theories are usually not tested for their consistency with empirical evidence, nor examined for the parsimony of their explanations vis-à-vis other existing theories; instead they are assessed in multiple contexts for their descriptive power, their credibility and their effectiveness in practice.

Our starting point in the review is that children need to learn about quantities and the relations between them and about mathematical symbols and their meanings. These meanings are based on sets of relations. Mathematics teaching should aim to ensure that students' understanding of quantities, relations and symbols go together.

Conclusions

This theoretical approach underlies the six main sections of the review. We now summarise the main conclusions of each of these sections.

Whole numbers

- Whole numbers represent both quantities and relations between quantities, such as differences and ratio. Primary school children must establish clear connections between numbers, quantities and relations.

- Children's initial understanding of quantitative relations is largely based on correspondence. One-to-one correspondence underlies their understanding of cardinality, and one-to-many correspondence gives them their first insights into multiplicative relations. Children should be encouraged to think of number in terms of these relations.

- Children start school with varying levels of ability in using different action schemes to solve arithmetic problems in the context of stories. They do not need to know arithmetic facts to solve these problems: they count in different ways depending on whether the problems they are solving involve the ideas of addition, subtraction, multiplication or division.

- Individual differences in the use of action schemes to solve problems predict children's progress in learning mathematics in school.

- Interventions that help children learn to use their action schemes to solve problems lead to better learning of mathematics in school.

- It is more difficult for children to use numbers to represent relations than to represent quantities.

Implications for the classroom

Teaching should make it possible for children to:

- connect their knowledge of counting with their knowledge of quantities
- understand additive composition and one-to-many correspondence
- understand the inverse relation between addition and subtraction
- solve problems that involve these key understandings
- develop their multiplicative understanding alongside additive reasoning.

Implications for further research

Long-term longitudinal and intervention studies with large samples are needed to support curriculum development and policy changes aimed at implementing these objectives. There is also a need for studies designed to promote children's competence in solving problems about relations.

Fractions

- Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. As with whole numbers, children need to make connections between quantities and their representations in fractions in order to be able to use fractions meaningfully.

- Two types of quantities that are taught in primary school must be represented by fractions. The first involves measurement: if you want to represent a quantity by means of a number and the quantity is smaller than the unit of measurement, you need a fraction; for example, a half cup or a quarter inch. The second involves division: if the dividend is smaller than the divisor, the result of the division is represented by a fraction; for example, three chocolates shared among four children.

- Children use different schemes of action in these two different situations. In division situations, they use correspondences between the units in the numerator and the units in the denominator. In measurement situations, they use partitioning.

- Children are more successful in understanding equivalence of fractions and in ordering fractions by magnitude in situations that involve division than in measurement situations.

- It is crucial for children's understanding of fractions that they learn about fractions in both types of situation: most do not spontaneously transfer what they learned in one situation to the other.

- When a fraction is used to represent a quantity, children need to learn to think about how the numerator and the denominator relate to the value represented by the fraction. They must think about direct and inverse relations: the larger the numerator, the larger the quantity, but the larger the denominator, the smaller the quantity.

- Like whole numbers, fractions can be used to represent quantities and relations between quantities, but they are rarely used to represent relations in primary school. Older students often find it difficult to use fractions to represent relations.

Implications for the classroom

Teaching should make it possible for children to:
- use their understanding of quantities in division situations to understand equivalence and order of fractions
- make links between different types of reasoning in division and measurement situations
- make links between understanding fractional quantities and procedures
- learn to use fractions to represent relations between quantities, as well as quantities.

Implications for further research

Evidence from experimental studies with larger samples and long-term interventions in the classroom are needed to establish how division situations relate to learning fractions. Investigations on how links between situations can be built are needed to support curriculum development and classroom teaching.

There is also a need for longitudinal studies designed to clarify whether separation between procedures and meaning in fractions has consequences for further mathematics learning.

Given the importance of understanding and representing relations numerically, studies that investigate under what circumstances primary school students can use fractions to represent relations between quantities, such as in proportional reasoning, are urgently needed.

Relations and their mathematical representation

- Children have greater difficulty in understanding relations than in understanding quantities. This is true in the context of both additive and multiplicative reasoning problems.

- Primary and secondary school students often apply additive procedures to solve multiplicative problems and multiplicative procedures to solve additive problems.

- Teaching designed to help students become aware of relations in the context of additive reasoning problems can lead to significant improvement.

- The use of diagrams, tables and graphs to represent relations in multiplicative reasoning problems facilitates children's thinking about the nature of the relations between quantities.

- Excellent curriculum development work has been carried out to design programmes that help students develop awareness of their implicit knowledge of multiplicative relations. This work has not been systematically assessed so far.

- An alternative view is that students' implicit knowledge should not be the starting point for students to learn about proportional relations; teaching should focus on formalisations rather than informal knowledge and only later seek to connect mathematical formalisations with applied situations. This alternative approach has also not been systematically assessed yet.

- There is no research that compares the results of these diametrically opposed ideas.

Implications for the classroom

Teaching should make it possible for children to:

- distinguish between quantities and relations
- become explicitly aware of the different types of relations in different situations
- use different mathematical representations to focus on the relevant relations in specific problems
- relate informal knowledge and formal learning.

Implications for further research

Evidence from experimental and long-term longitudinal studies is needed on which approaches to making students aware of relations in problem situations improve problem solving. A study comparing the alternative approaches – starting from informal knowledge versus starting from formalisations – would make a significant contribution to the literature.

Space and its mathematical representation

- Children come to school with a great deal of informal and often implicit knowledge about spatial relations. One challenge in mathematical education is how best to harness this knowledge in lessons about space.

- This pre-school knowledge of space is mainly relational. For example, children use a stable background to remember the position and orientation of objects and lines.

- Measuring length and area poses particular problems for children, even though they are able to understand the underlying logic of measurement. Their difficulties concern iteration of standard units and the need to apply multiplicative reasoning to the measurement of area.

- From an early age children are able to extrapolate imaginary straight lines, which allows them to learn how to use Cartesian co-ordinates to plot specific positions in space with little difficulty. However, they need help from teachers on how to use co-ordinates to work out the relation between different positions.

- Learning how to represent angle mathematically is a hard task for young children, even though angles are an important part of their everyday life. Initially children are more aware of angle in the context of movement (turns) than in other contexts. They need help from to teachers to be able to relate angles across different contexts.

- An important aspect of learning about geometry is to recognise the relation between transformed shapes (rotation, reflection, enlargement). This can be difficult, since children's preschool experiences lead them to recognise the same shapes as equivalent across such transformations, rather than to be aware of the nature of the transformation.

- Another aspect of the understanding of shape is the fact that one shape can be transformed into another by addition and subtraction of its subcomponents. For example, a parallelogram can be transformed into a rectangle of the same base and height by the addition and subtraction of equivalent triangles. Research demonstrates a danger that children learn these transformations as procedures without understanding their conceptual basis.

Implications for the classroom

Teaching should make it possible for children to:

- build on spatial relational knowledge from outside school
- relate their knowledge of relations and correspondence to the conceptual basis of measurement
- iterate with standard and non-standard units
- understand the difference between measurements which are/are not multiplicative
- relate co-ordinates to extrapolating imaginary straight lines
- distinguish between scale enlargements and area enlargements.

Implications for further research

There is a serious need for longitudinal research on the possible connections between children's pre-school spatial abilities and how well they learn about geometry at school.

Psychological research is needed on: children's ability to make and understand transformations and the additive relations in compound shapes; the exact cause of children's difficulties with iteration; how transitive inference, inversion and one-to-one correspondence relate to problems with geometry, such as measurement of length and area.

There is a need for intervention studies on methods of teaching children to work out the relation between different positions, using co-ordinates.

Algebra

- Algebra is the way we express generalisations about numbers, quantities, relations and functions. For this reason, good understanding of connections between numbers, quantities and relations is related to success in using algebra. In particular, understanding that addition and subtraction are inverses, and so are multiplication and division, helps students understand expressions and solve equations.

- To understand algebraic symbolisation, students have to (a) understand the underlying operations and (b) become fluent with the notational rules. These two kinds of learning, the meaning and the symbol, seem to be most successful when students know what is being expressed and have time to become fluent at using the notation.

- Students have to learn to recognise the different nature and roles of letters as: unknowns, variables, constants and parameters, and also the meanings of equality and equivalence. These meanings are not always distinct in algebra and do not relate unambiguously to arithmetical understandings.

- Students often get confused, misapply, or misremember rules for transforming expressions and solving equations. They often try to apply arithmetical meanings inappropriately to algebraic expressions. This is associated with over-emphasis on notational manipulation, or on 'generalised arithmetic', in which they may try to get concise answers.

Implications for the classroom

Teaching should make it possible for children to:

- read numerical and algebraic expressions relationally, rather than as instructions to calculate (as in substitution)

- describe generalisations based on properties (arithmetical rules, logical relations, structures) as well as inductive reasoning from sequences
- use symbolism to represent relations
- understand that letters and '=' have a range of meanings
- use hands-on ICT to relate representations
- use algebra purposefully in multiple experiences over time
- explore and use algebraic manipulation software.

Implications for further research

We need to know how explicit work on understanding relations between quantities enables students to move successfully between arithmetical to algebraic thinking.

Research on how expressing generality enables students to use algebra is mainly in small-scale teaching interventions, and the problems of large-scale implementation are not so well reported. We do not know the longer-term comparative effects of different teaching approaches to early algebra on students' later use of algebraic notation and thinking.

There is little research on higher algebra, except for teaching experiments involving functions. How learners synthesise their knowledge of elementary algebra to understand polynomial functions, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations is not known.

There is some research about the use of symbolic manipulators but more needs to be learned about the kinds of algebraic expertise that develops through their use.

Modelling, solving problems and learning new concepts in secondary mathematics

Students have to be fluent in understanding methods and confident about using them to know why and when to apply them, but such application does not automatically follow the learning of procedures. Students have to understand the situation as well as to be able to call on a familiar repertoire of facts, ideas and methods.

Students have to know some elementary concepts well enough to apply them and combine them to form new concepts in secondary mathematics. For example, knowing a range of functions and/or their representations seems to be necessary to understand the modelling process, and is certainly necessary to engage in modelling. Understanding relations is necessary to solve equations meaningfully.

Students have to learn when and how to use informal, experiential reasoning and when to use formal, conventional, mathematical reasoning. Without special attention to meanings, many students tend to apply visual reasoning, or be triggered by verbal cues, rather than analyse situations to identify variables and relations.

In many mathematical situations in secondary mathematics, students have to look for relations between numbers, and variables, and relations between relations, and properties of objects, and know how to represent them.

Implications for the classroom

Teaching should make it possible for children to:

- learn new abstract understandings, which is neither achieved through learning procedures, nor through problem-solving activities, without further intervention

- use their obvious reactions to perceptions and build on them, or understand conflicts with them
- adapt to new meanings and develop from earlier methods and conceptualizations over time
- understand the meaning of new concepts 'know about', 'know how to', and 'know how to use'
- control switching between, and comparing, representations of functions in order to understand them
- use spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

Implications for further research

Existing research suggests that where contextual and exploratory mathematics, integrated through the curriculum, *do* lead to further conceptual learning it is related to conceptual learning being a rigorous focus for curriculum and textbook design, and in teacher preparation, or in specifically designed projects based around such aims. There is therefore an urgent need for research to identify the key conceptual understandings for success in secondary mathematics. There is no evidence to convince us that the new U.K. curricula will necessarily lead to better *conceptual* understanding of mathematics, either at the elementary level which is necessary to learn higher mathematics, or at higher levels which provide the confidence and foundation for further mathematical study.

We need to understand the ways in which students learn new ideas in mathematics that depend on combinations of earlier concepts, in secondary school contexts, and the characteristics of mathematics teaching at higher secondary level which contribute both to successful conceptual learning and application of mathematics.

Common themes

We reviewed different areas of mathematical activity, and noted that many of them involve common themes, which are fundamental to learning mathematics: number, logical reasoning, reflection on knowledge and tools, understanding symbol systems and mathematical modes of enquiry.

Number

Number is not a unitary idea, which children learn in a linear fashion. Number develops in complementary strands, sometimes with discontinuities and changes of meaning. Emphasis on procedures and manipulation with numbers, rather than on understanding the underlying relations and mathematical meanings, can lead to over-reliance and misapplication of methods in arithmetic, algebra, and problem-solving. For example, if children form the idea that quantities are only equal if they are represented by the same number, a principle that they could deduce from learning to count, they will have difficulty understanding the equivalence of fractions. Learning to count and to understand quantities are separate strands of development. Teaching can play a major role in helping children co-ordinate these two forms of knowledge without making counting the only procedure that can be used to think about quantities.

Successful learning of mathematics includes understanding that number describes quantity; being able to make and use distinctions between different, but related, meanings of number; being able to use relations and meanings to inform application and calculation; being able to use number relations to move away from images of quantity and use number as a structured, abstract, concept.

Logical reasoning

The evidence demonstrates beyond doubt that children must rely on logic to learn mathematics and that many of their difficulties are due to failures to make the correct logical move that would have led them to the correct solution. Four different aspects of logic have a crucial role in learning about mathematics.

The logic of correspondence (one-to-one and one-to-many correspondence) The extension of the use of one-to-one correspondence from sharing to working out the numerical equivalence or non-equivalence of two or more spatial arrays is a vastly important step in early mathematical learning. Teaching multiplication in terms of one-to-many correspondence is more effective than teaching children about multiplication as repeated addition.

The logic of inversion Longitudinal evidence shows that understanding the inverse relation between addition and subtraction is a strong predictor of children's mathematical progress. A flexible understanding of inversion is an essential element in children's geometrical reasoning as well. The concept of inversion needs a great deal more prominence than it has now in the school curriculum.

The logic of class inclusion and additive composition Class inclusion is the basis of the understanding of ordinal number and the number system. Children's ability to use this form of inclusion in learning about number and in solving mathematical problems is at first rather weak, and needs some support.

The logic of transitivity All ordered series, including number, and also forms of measurement involve transitivity ($a > c$ if $a > b$ and $b > c$: $a = c$ if $a = b$ and $b = c$). Learning how to use transitive relations in numerical measurements (for example, of area) is difficult. One reason is that children often do not grasp the importance of iteration (repeated units of measurement).

The results of longitudinal research (although there is not an exhaustive body of such work) support the idea that children's logic plays a critical part in their mathematical learning.

Reflection on knowledge and tools

Children need to re-conceptualise their intuitive models about the world in order to access the mathematical models that have been developed in the discipline. Some of the intuitive models used by children lead them to appropriate mathematical problem solving, and yet they may not know why they succeeded. Implicit models can interfere with problem solving when students rely on assumptions that lead them astray.

The fact that students use intuitive models when learning mathematics, whether the teacher recognises the models or not, is a reason for helping them to develop an awareness of their models. Students can explore their intuitive models and extend them to concepts that are less intuitive, more abstract. This pragmatic theory has been shown to have an impact in practice.

Understanding symbol systems

Systems of symbols are human inventions and thus are cultural tools that have to be taught. Mathematical symbols are human-made tools that improve our ability to control and adapt to the environment. Each system makes specific cognitive demands on the learner, who has to understand the systems of representation and relations that are being represented; for example place-value notation is based on additive composition, functions depict covariance. Students can behave as if they understand how the symbols work while they do not understand them completely: they can learn routines for symbol manipulation that remain disconnected from meaning. This is true of rational numbers, for example.

Students acquire informal knowledge in their everyday lives, which can be used to give meaning to mathematical symbols learned in the classroom. Curriculum development work that takes this knowledge into account is not as widespread as one would expect given discoveries from past research.

Mathematical modes of enquiry

Some important mathematical modes of enquiry arise in the topics covered in this synthesis.

Comparison helps us make new distinctions and create new objects and relations Comparisons are related to making distinctions, sorting and classifying; students need to learn to make these distinctions based on mathematical relations and properties, rather than perceptual similarities.

Reasoning about properties and relations rather than perceptions Throughout mathematics, students have to learn to interpret representations before they think about how to respond. They need to think about the relations between different objects in the systems and schemes that are being represented.

Making and using representations Conventional number symbols, algebraic syntax, coordinate geometry, and graphing methods, all afford manipulations which might otherwise be impossible. Coordinating different representations to explore and extend meaning is a fundamental mathematical skill.

Action and reflection-on-action In mathematics, actions may be physical manipulation, or symbolic rearrangement, or our observations of a dynamic image, or use of a tool. In all these contexts, we observe what changes and what stays the same as a result of actions, and make inferences about the connections between action and effect.

Direct and inverse relations It is important in all aspects of mathematics to be able to construct and use inverse reasoning. As well as enabling more understanding of relations between quantities, this also establishes the importance of reverse chains of reasoning throughout mathematical problem-solving, algebraic and geometrical reasoning.

Informal and formal reasoning At first young children bring everyday understandings into school and mathematics can allow them to formalise these and make them more precise. Mathematics also provides formal tools, which do not describe everyday experience, but enable students to solve problems in mathematics and in the world which would be unnoticed without a mathematical perspective.

Epilogue

We have made recommendations about teaching and learning, and hope to have made the reasoning behind these recommendations clear to educationalists (in the extended review). We have also recognised that there are weaknesses in research and gaps in current knowledge, some of which can be easily solved by research enabled by significant contributions of past research. Other gaps may not be so easily solved, and we have described some pragmatic theories that are or can be used by teachers when they plan their teaching. Classroom research stemming from the exploration of these theories can provide new insights for further research in the future, alongside longitudinal studies which focus on learning mathematics from a psychological perspective.

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