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# KEY UNDERSTANDINGS IN LEARNING MATHEMATICS[1]

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## Introduction

It is well known that very young children develop intuitive ideas that form the foundations of later formal mathematics. The pre-school notions of touching, matching, sharing, fitting, putting together and taking apart, are frequently drawn on in primary teaching to generate the counting of objects, adding, subtracting, dividing and multiplying. Eventually, children count in groups and develop an understanding that a grid of objects (an array) is a good way to understand multiplication by repeated counting. These ideas provide a basis of their understanding of later mathematics.

## Scaling

In secondary mathematics, however, the idea of multiplication as scaling is very important, and this does not follow easily from repeated addition and the array model. Also, ratio is important in many mathematical topics and scientific applications, but does not arise easily from counting and students often use additive methods to compare two quantities when the teacher expects them to be using ratio.

As an ex-secondary teacher, I am very well aware that multiplicative relations and comparisons are hard for students to learn, and over-reliance on addition can be the root cause of many difficulties. There is another path to multiplication which is often ignored and undeveloped in the curriculum. Imagine a small child, aged about two, looking up into her father's eyes. This image shows a physical understanding of ideas of ratio, proportion, angle, and similarity that will not be formally taught to her for many years. Between now and then, the child will grow but her father will not, so gradually their eyes will meet at a smaller angle of elevation, and their relative sizes will change. Holding hands will be easier too, because the scaling takes place throughout the whole body, not just by adding height to the top.

Intuitive ideas of scaling and similarity can also develop in the way we see things: small things close to us can obliterate large things further away, and children know how to line up their eyes and objects to make this happen. The idea of scaling is as natural as the idea of organising objects, and may be more natural than counting, but multiplication as scaling is not usually developed in the primary curriculum. Why is this? It is because attaching numbers to the idea is difficult – it is not a task that can always be done by counting directly.

## Relations

A review of research in children's learning for the Nuffield Foundation shows that understanding relations between quantities is at the heart of mathematical progress. In this section I shall demonstrate where the difficulties lie.

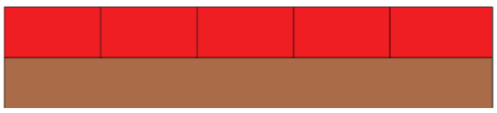
This is a representation of three rods arranged to show the additive relation. However, if I imagine myself doing the actions related to each representation I soon run into problems.

| | |
|---|---|
| $a+b=c$ | $c=a+b$ |
| $b+a=c$ | $c=b+a$ |
| $c-a=b$ | $b=c-a$ |
| $c-b=a$ | $a=c-b$ |

In the first representation on the left I would put one colour next to another, and then look for one that matched the whole length. In the first one on the right I would first take the long stick, and then look for two that matched it when put together. But how would I do the last one on the right? I might put one of the smaller sticks alongside a longer stick and 'see' a gap. In my mind I have to make a judgement about the size of the gap and reach for a stick that fits it – in other words, I have to use my imagination to appreciate 'difference'. Fortunately, in the additive relationship, 'difference' can be represented by the same kind of thing (a stick) as is used to represent the quantities themselves, but ‘difference’ is a relation between two quantities, and only becomes quantity itself through the representation. In a more extreme case, if I have £2 and you have £1.50 the ‘difference’ is only in our imagination – where is the 50p?

While all this rethinking of the additive relation is going on, you may have had in the back of your mind the knowledge that subtraction ‘undoes’ addition. Well so it does, and this is crucial for children to understand, but this demonstration with rods shows that it is not always a ‘doing and undoing’ situation; sometimes it needs imagination, and imagining the difference is, possibly, a good starting point for algebra.

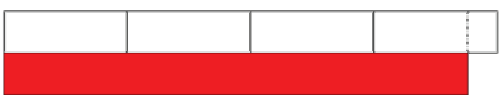
So what about multiplication? Try the same kind of thing with this arrangement in the eight representations (there are even more if you think about reciprocals as well). Again, try to imagine the physical actions you would make that ‘show’ each representation.



$$a = bc \qquad bc = a$$
$$a = cb \qquad cb = a$$
$$b = \frac{a}{c} \qquad \frac{a}{c} = b$$
$$c = \frac{a}{b} \qquad \frac{a}{b} = c$$

Very soon I find that the ‘5’ is a problem – it is only in my imagination. This means that the bottom two lines represent two very different kinds of division, one being ‘how many red lengths equal the brown length?’ and the other being ‘what is the length of a rod so that five of it laid end to end match the length of the brown rod?’ Knowing that division is the inverse of multiplication is not quite enough. Ratio is in the imagination and cannot be represented in the same way as the original quantities, unlike ‘difference’ in the additive relationship. But the illustration is of an exact multiplicative relation which could be represented as repeated addition, or as a 5 by $x$ array, where $x$ is the quantity represented by the red stick – again a useful root of algebra.

What about inexact multiplicative relations such as this one that cannot be represented as an array? How can the red ‘measure’ the white, or the white ‘measure’ the red?



Problems like this relate closely to the meaning of measurement, and at first seem to yield only to approximation. However, if young children are given similar problems with pouring between larger and smaller vessels they soon work out ways to find exact answers. They can often transfer their methods to lengths. Interestingly, their solutions take one of two forms, either finding a smaller length that fits exactly into both the white and red lengths and hence becomes a measuring unit, or making strings of white and red until they find a matching length. These approaches lend themselves very well to algebraic representation and both rely on some intuitive understanding of ratio. Each method also gives a way in to expressing such a relation as a rational number. From a secondary mathematics standpoint it is worth noting that the first method depends on finding a common factor, the second method on finding a common multiple. The concept of repeated addition on its own does nothing to help solve the problem; the comparison between two quantities is achieved through the idea of ratio, which is not represented by the materials themselves.

**Relations and scaling**

All the research we have found points towards the importance of relations rather than calculations, and the underdevelopment of children’s ideas of comparing quantities.

The approach to considering relations I have described focuses on how learners have to move away from materials and into their imagination to understand multiplication fully, and in so doing lay the foundations for number theory, measurement and ratio. What is the link with scaling? For me the connection is in the idea of stretching, because when we scale things each unit length of the original extends to become a new unit length of the new one. I am not suggesting that this is a way to teach, nor that the connection is developmental, but that pouring between vessels can lead to expressions such as: if one fill of one cup is the same amount as two fills of another, then five fills of the first is the same amount as ten fills of the second. This retains the 2:1 relationship of their capacity when the quantities are enlarged or reduced. With the rods, if 5 of one are the same length as 9 of another, then 10 of the first are the same length as 18 of the other; the 9:5 relationship of length is preserved. In neither of these situations do we have to know the actual measures as numerical quantities, because the relations speak for themselves.

In secondary and primary school much can be done by ramping up the difficulty of questions about pouring water and sand and finding ways to express relations between different quantities. Another way forward would be to focus strongly on relations by having special non-computational arithmetic lessons in which it is only relations that are discussed. This shifts learners to thinking about how numbers and operations can be expressed and manifested in different ways, and provides a foundation for algebra as well as an important view of multiplication. While there are very young children learning mathematics this way in Russia and USA and doing very well as a result, it is also possible to spend valuable time in secondary school revisiting both the ideas of scaling and of non-computational arithmetic as starting points for many secondary mathematics concepts. This, I believe, is a far more productive approach than trying to adapt learners' existing ideas of repeated addition and array models, because it draws on learners' memories of pouring, stretching, and looking up at adults when they were tiny.

**Note**

1 This article is based on ideas generated in a review of research under the same title, written by Terezinha Nunes, Peter Bryant and Anne Watson. The full review can be found at www.nuffieldfoundation.org. The contents of this article are, however, solely authored by me and are not necessarily the views of my co-authors nor of the Nuffield Foundation.

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