Using manipulatives in the foundations of arithmetic: Literature review

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A. The Nuffield Project

This literature review comprises the first part of a project funded by the Nuffield Foundation to develop guidance for teachers of 3 to 9 year olds in the use of manipulatives to teach arithmetic. The authors would like to thank the Nuffield Foundation for funding this work and trust that it will prove valuable to them and all teachers of elementary arithmetic who engage with its findings and outputs.

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B. Introduction and background

What are manipulatives?

The National Curriculum in England (Department for Education (DfE) 2013) requires learners 'to move fluently between representations of mathematical ideas' (DfE, 2013: 3). However, in the absence of guidance, it is not immediately obvious to mathematics educators how we help learners to do this. Teachers of younger children have traditionally begun by using practical resources or manipulatives, including everyday objects, counters or mathematically structured apparatus, of which there is a wide variety currently marketed. Particularly in relation to arithmetic, there is a lack of published consensus about their relative merits or use. Although government inspectors advocate the use of practical resources they have also criticized schools for the way they use them:

> 'Carefully chosen practical activities and resources … have two principal benefits: they aid conceptual understanding and make learning more interesting. Too few of the schools used these resources well.' (Office for Standards in Education (Ofsted), 2012: 27)

In the National Curriculum (DfE, 2013) 'concrete objects' and materials are mentioned only for Key Stage One, for 5 to 7 year olds: elsewhere references are to a 'range of representations' and to arrays and number lines in particular. Practical resources were not permitted in recent national tests, even for six and seven year olds, presumably deterring teachers further.

Therefore official guidance gives mixed messages about the desirability of the use of manipulatives, especially for the over sevens. As Brown (2014) noted, generally the use of manipulatives has declined in recent years.

For the purpose of this review, we define **manipulatives** as

> 'objects that can be handled and moved, and are used to develop understanding of a mathematical situation'.

This includes structured materials like hundreds, tens and ones apparatus and unstructured resources like counters or beans. Our definition of manipulatives reflects current literature. Whereas many studies just refer to 'concrete objects', some focus on structured materials:

> Manipulative materials are objects designed to represent explicitly and concretely mathematical ideas that are abstract. (Moyer, 2001:176)

Others, like Uttall et al. (2013), distinguish between 'informal' and 'formal' or structured manipulatives. Some also specify a pedagogical function, as for instance, with Swan and Marshall's (2010) definition:

> A mathematics manipulative material is an object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered. (2010:14)

This is also consistent with Carbonneau, Marley and Selig (2013:381), who refer to students 'manipulating concrete objects to represent mathematical concepts'. Like them, most authors exclude measuring tools and calculators, but include fingers. Virtual manipulatives are beyond the scope of this review, although they undoubtedly have advantages (as noted for instance by Dunphy, Dooley and Shiel, 2014).

In the UK, in recent years, the English government's numeracy projects have been highly influenced by the Dutch research into teaching mental arithmetic: beadstrings and empty number lines were recommended in official guidance (Department for Children, Schools and Families (DCSF), 2003; Beshuizen, 2010). However, according to Brown (2014:7), practical work has steadily declined, ' being a victim as much of the advent of interactive white boards and cuts in equipment budgets as of reduced teacher training periods'.

More recently, the English government has been influenced by high performing jurisdictions, such as Singapore. Brown (2014: 7) points out that the Singapore curriculum is one which the UK 'exported to Singapore in the 1950s, having then abandoned it ourselves as being widely dysfunctional'. Nevertheless, she points out that practical work has become fashionable there, commenting that this factor seems to have escaped the politicians. Current Singaporean influences in the UK, as evidenced by the government funded National Centre for Excellence in Teaching Mathematics (NCETM) include the bar model approach (reminiscent of Cuisenaire rods) and the use of Singapore textbooks advocating a Brunerian concrete- pictorial- abstract (CPA) approach (e.g. Fong, 2014). This has come to be interpreted as teachers presenting examples in different modes, in a linear sequence, which neither reflects original intentions for an active curriculum nor Bruner's pedagogy, according to Hoong, Kin and Pien (2015).

In addition, the UK government advocates the Shanghai approach, which does not include manipulatives. These confusing recommendations conflict with teachers' practice based on previous initiatives and compete with commercially marketed models, such as Numicon (Wing, 2001), which is accompanied by professional development and resources. This complex context underlines the need for research and guidance for teachers with regard to the use of a range of manipulatives.

The review begins by setting out our theoretical position, and is followed by a historical overview, followed by a review of current research regarding maths pedagogy.

Our theoretical position

Our approach to the role and significance of manipulatives in the teaching of arithmetic is based on a synthesis of cognitive and socio-cultural analyses of teaching and learning, in line with current thinking (e.g. Dunphy, Dooley and Shiel, 2014). We contend that a range of perspectives offers valuable insights into the processes of learning, of what goes on during lessons between the participants and also in terms of individuals' developing understandings of mathematical concepts. Here we give some examples regarding particular manipulatives, teachers and theorists who have contributed to our position.

Fingers: the first manipulatives

The advantages of drawing on a range of research are exemplified by considering what must be the most basic manipulative that is available to all children, their fingers. These important but sometimes overlooked manipulatives have been studied using various theoretical perspectives. From a cognitive view, Hughes (1986) found that three year olds used fingers as symbols to bridge between concrete and abstract. Others, such as Gray and Tall (1994), analysed the role of fingers as structured manipulatives; Marton and Neuman (1990) presented the process of abstraction from a phenomenological rather than a constructivist view of learning. According to Bills (2000:50) a 'phenomenographic' approach investigates 'people's understanding of phenomena, seeking to categorise and explain the qualitatively different ways in which people think about the phenomena'.

Marton and Neuman's study (1990) focused on children's responses and strategies, with detailed analysis of the challenges involved for low attainers, but also of the successful finger strategies of higher attainers. They cited Werner's description of abstraction as 'an organic transition from the level of concrete optical number groups to that of purely abstract number' (Marton and Neuman, 1990:69). They also emphasised the role of subitising (recognising the number in a group without counting) and of seeing numbers as wholes and parts simultaneously (for instance, recognising six fingers whilst also seeing five and one, or double three).

Marton and Neuman studied finger representations with 59 Swedish children aged 7 to 12, 31 of whom had mathematics difficulties. This group all counted on or back with their fingers, rather than subitising fingers. For 9 subtract 6, they would count backwards from 9, raising 6 fingers, involving the difficult task of double counting six, (that is, counting backwards, 8, 7, 6, 5, 4, 3 while keeping track of how many they had counted back). This left them with an unhelpful image of 6 raised fingers, rather than the answer, 3. More successful children represented the problem with 'finger numbers', by holding up 9, then folding down six, leaving 3 raised fingers. They also quickly discovered that they could solve a missing addend problem by taking away: eg 2 + ? = 9 could be solved by raising 9, then folding down 2 fingers. 'Used as part-part-whole patterns in finger numbers .. the parts as well as the whole were immediately grasped'. Children using 'finger numbers' realised that numbers like 7 could be grouped into a subitisable '5 plus something ' pattern (1990:66). Marton and Neuman referred to 'the unbreakable hand', as a unit of 5.

Children also readily used doubles: 'the whole number is analysed in order to see if it can be divided up into two similar parts: that is, in order to see if 'doubles' can be used' (1990:67). They used near doubles for 3 + ? =7 by thinking of 3 and 3 making 6, or solved 4 + ? = 10, by moving a thumb over from 5 and 5 fingers. For the children with mathematics difficulties who used counting strategies, 'the parts could not be subitised within the whole, because only the added or subtracted part was created by the fingers' (1990:71). Marton and Neuman concluded that double counting is 'a barrier to the development of abstract thinking of number' (1990:73), whereas 'finger numbers' helped children learn number relations within 10. The children with mathematics difficulties also attempted to solve multiplication problems with repeated counting of fingers, but lost track. This research therefore suggested that children should be taught to visualize finger numbers, which could help them to develop abstract, part-whole concepts of numbers and more efficient calculation strategies, and also that this might prevent mathematics difficulties.

Marton and Neuman described the development of abstract thinking in Piagetian terms as a natural progression, from activity via visualization to abstraction. This also echoed Bruner's modes of representation: finger numbers which are 'first used in a concrete and later in a concrete operational way, together with the strategies in which they are used, eventually become 'known number relations' or 'concepts of number'' (1990:69). They cited Werner: '"the concrete number groups become stripped of their picture-like properties .."' They argued that these 'are not counting skills, but rather "seeing" skills' or 'analytic skills' and that children 'analysed known numbers' (1990:69). From a phenomenological perspective, the fingers structured thinking, rather than knowledge being derived from concrete or mental actions alone. The qualities inherent in hands, such as 'the undivided hand ' and the way thumbs can be moved, allow children to see numbers in a concrete way and then in a visualized and 'body-anchored' way and finally just in a known or felt way (1990:72).

However, what is not explained by Marton and Neuman is why some children develop 'finger number' representations to begin with, while others do not, so remaining dependent on counting in ones, and whether this is simply a matter of not having been taught to do so.

Neuroscientific studies suggest other accounts for these processes in terms of representations and memory. Wood and Fischer (2008:355) propose 'embodied cognition' as an important theoretical approach 'according to which all cognition is grounded in perception and action'. According to Goswami and Bryant (2007), conceptual understanding consists of networks of associations between different representational modes, including tactile, visual and verbal modes. 'Finger numbers' have the advantage of being represented as personal body parts, in addition to being encoded in muscle memory as actions. Research has also suggested, for instance by Butterworth (1999), that small numbers are intuitively represented in the brain in an area adjacent to that which represents fingers.

More recent evidence endorses a close association between brain areas representing fingers and number magnitudes, which Wood and Fischer (2008) refer to as the 'manumerical cognition' hypothesis. This implies that fingers may provide some deeply intuitive and sensory embodiment of numbers in a way that other manipulatives do not. Neuroscientific evidence, according to Wood and Fischer (2008) also suggests that children use mental finger representations for numbers more than adults do, implying the importance of early finger use for developing numerical understanding. Gracia-Bafulluy and Noel (2008) report a predictive link between finger awareness, as shown by distinguishing individual fingers (for instance by identifying which have been touched) and numerical performance. They found that training young children in finger awareness improved their subitising, counting, comparing and recognition of numbers of fingers. Individual differences in intuitive finger awareness might therefore explain the different tendencies related by Marton and Neuman, which could be prevented by teaching.

Other theories might also explain some children's lack of use of finger numbers. For instance, Hannula and Lehtinen (2005) identified a general tendency of '*spontaneous focusing on numerosity*' (SFON) which varies significantly between individual children and is linked to mathematical achievement. This might also explain some children's lack of intuitive association of fingers with numbers. Similarly, '*awareness of mathematical pattern and structure'* (AMPS) has been identified by Mulligan and Mitchelmore (2009) as a spatial competence in young children which is linked to mathematical performance, suggesting that some children may more readily analyse the numerical composition of finger patterns. Mulligan & Mitchelmore also found that AMPS may be improved by teaching. Sinclair and Pimm (2015:108) report that using an electronic tablet application, 'Touchcounts', quickly resulted in very young children moving from counting on their fingers to 'all-at-once' subitised gestures for numbers to 10. All of this research suggests that young children might be taught more explicitly to use fingers to represent numbers.

Socio-cultural perspectives also help to explain differences in finger use. In the USA Jordan (2003) noted that children from low-income families tended not to use fingers to solve problems, suggesting that finger representations are learned at home and are culturally influenced. In England, Baker, Street and Tomlin (2003) reported the skills of Aysha, taught by her father to count three to a finger (counting each joint) according to their cultural tradition and who could therefore count all her classmates on two hands. However, she did not use this efficient strategy in school, apparently dissuaded by the Anglo-centric classroom culture. Recently, Bender and Beller (2012) reviewed the huge cultural diversity of 'finger numbers', with increasing evidence that these affected the way people thought about numbers, arguing for 'embodied cognition'. People from some cultures count fingers in fives, while others consider a hand as four fingers, with the thumb used for counting the segments of each finger. Bender and Beller noted that some systems might be more efficient than others or have different advantages: for instance four fingers with three segments allow calculation in base 12.

Fuson and Kwon (1992) linked Korean children's superior calculation skills with their early use of fingers to represent numbers to 20 by raising then folding them down, thus emphasizing 'bridging through ten' when adding or subtracting. Fuson and Kwon also pointed out that this strategy, which structures addition and subtraction through ten, was supported by the language structure, with numbers over ten named as 'ten-one', 'ten-two', in contrast to the more opaque English terms of 'eleven' and 'twelve'. Bender and Beller (2012) noted that finger systems may be supported or in conflict with verbal and symbolic number representations in a culture, with consequent possibilities for confusion. It also seems likely that elaborate finger use from an early age, as with the Korean example, may contribute to number understanding.

In terms of implications for teaching, it might also be helpful to teach efficient ways of using fingers, building on examples from different cultures. Systems that represent larger numbers may help to develop mental calculation strategies. Novack, Congdon, Hemani- Lopez and Goldin-Meadow (2014) found that children who represented meanings by gestures independently of action with objects, more successfully generalised their understanding. This underlines the potential significance of embodied cognition.

Neuroscientific and anthropological evidence about the way finger representation and vocabulary affect the development of number concepts suggests the need to consider manipulative use in conjunction with gesture and language from a social constructivist viewpoint. This combination of cognitive and socio-cultural analyses seems particularly fruitful in providing insights into the complexities of the use of manipulatives to develop children's arithmetic.

Teaching with structured equipment

Accounts of teachers who have successfully (and unsuccessfully) used manipulatives provide another important source of insights into factors affecting the use of manipulatives. The work of Madeleine Goutard (1964), teaching with Cuisenaire rods, exemplifies and provides insights into the pedagogical complexity involved. Following Gattegno and Cuisenaire's (1954) guidance, she developed her own pedagogy, firstly encouraging free play, then setting challenges, initially without referring to numbers. She analysed children's responses, then developed her approach accordingly, with high level outcomes in terms of children's abstract understanding, as shown for instance, by seven year olds' writing equivalent equations with fractions, brackets and indices. This demonstrated not only her knowledge of the mathematical learning potential of the rods, but of how children learn cognitively and affectively, in terms of ensuring motivation. Explicit in her approach are mathematical pedagogical principles, such as valuing insights into algebraic structures, encouraging processes such as generalisation and explanation, and developing children's positive self-belief as mathematical learners.

Central to her approach were the activities that she devised, based on children's responses and their ways of working, which were open-ended yet structured challenges. Another feature was her strong commitment to high expectations for all children, believing nothing was too difficult or abstract for six and seven year olds to learn. Her accounts of classroom discussions, valuing all children's contributions and encouraging reasoning, also exemplify the characteristics of a math-talk learning community (Hufferd-Ackles, Fuson and Sherin, 2004). The nature of activities and their aims, teachers' expectations of children's ability to learn mathematics, their interactions with pupils, and their 'central role in establishing the mathematical quality of the classroom environment' (Yackel and Cobb, 1996: 475) are all significant aspects of pedagogy, which are not usually identified in quantitative studies evaluating the success of manipulatives.

Cognitive representation theory: Bruner

In order to gain insights into factors affecting the pedagogical use of manipulatives, this review considers a range of studies from different perspectives. Theories of the role of representation have been central to any study of manipulatives, and Bruner (1964) remains an influential theorist in this field. Bruner's work resonates with Goutard (1964) in suggesting that even young children could learn sophisticated concepts, if they were presented in an accessible way. Bruner proposed an instructional sequence of representational modes with increasing abstraction, echoing Dienes' (1960) teaching cycle.

Bruner saw concepts as:

> 'representations resulting from the mental organization of experience: the end product of such a system of coding and processing is what we may speak of as representation'
> (Bruner, 1964: 2).

He proposed a sequence of enactive, iconic and symbolic modes of representation.

- Enactive representation is a 'mode of representing past events through appropriate motor response' (1964:2), or ' a set of actions' (1966; 44). The implication is that the action rather than the visual arrangement is remembered, and encoded in muscle memory. This might include action schemas, for instance 'sharing' by rhythmically dealing out objects one-to- one, as well as jumping along a number track. The enactive mode is therefore essentially about learner activity, rather than representation with concrete materials.
- Iconic representation entails selectively organized and 'summary images or graphics' (1966:44) which are created by the individual from their experience. This emphasis on schematic diagrams may be different to some current interpretations of 'iconic' as pictorial representation. It also implies sense- making activity by the learner, rather than just presenting children with pictures and images.
- Symbolic representation, the most abstract representation mode, involves 'a set of symbolic or logical propositions, drawn from a symbolic system which is governed by rules or laws for forming or transforming propositions' (1966: 45). This includes verbal generalisations to express relationships as well as written symbols, equations and formulae.

Bruner's theory is similar to Piaget's in emphasizing understanding as rooted in reflection on actions, but he refuted the idea of children going through developmental changes in thinking with maturation, instead suggesting that children could learn any concept by employing the sequence of modes, which provide increasingly economical and powerful representations of ideas. Hoong et al. (2015) pointed out that a common myth in mathematics education suggests that Bruner's modes are distinct and should be presented sequentially. This is common in some interpretations of the supposedly Brunerian concrete-pictorial-abstract (CPA) progression where different representations are presented to children, beginning with everyday contexts and culminating in symbolic equations. Hoong et al. (2015) point out that Bruner refers to introducing symbols 'along the way' (1966: 64) and in his pedagogy, 'elements of the symbolic mode, such as algebraic notations are developed alongside the primarily enactive and iconic stages of instruction, leading towards a proficiency of operation with the symbolic system' (2015: 6). Bruner's theory therefore suggests a more complex relationship between concrete and abstract representations with a more active role for the learner than some current interpretations may imply.

Mason and Johnston-Wilder (2006) present an alternative interpretation of Bruner from that of teacher-presented CPA images. They argue that Bruner's modes are essentially different worlds of experience, or different phases of activity in an internalizing process of coming to understand a mathematical concept or structure:

- Manipulating
- Getting a sense of
- Articulating

The first phase involves manipulating objects, which may not necessarily be physical, and dealing with specific examples of the relation being considered. The second phase involves getting a sense of, or a feel for 'some underlying structure, pattern or relationship' (2006:33) and this includes visualizing and drawing schematic diagrams to capture relationships. In the third phase learners articulate generalisations, can use formal and symbolic expressions and use words and symbols to represent concepts, which become reified so they can be acted on as mental objects, and manipulated in their turn. These phases therefore present a constructivist view of learning and teaching, involving creative activity on the part of the learner. Moving backwards and forwards through these three worlds of experience, between manipulable objects, mental imagery and drawing and abstract symbols helps children and adults to develop mathematical thinking and reasoning. The talk accompanying these worlds offers another layer in the complex process that allows the teacher to hear what the children are thinking.

Bruner's theory is therefore highly significant in proposing physical activity as the root of conceptual representation and learning, and also in suggesting a way of progressing away from the use of physical objects towards abstract expression. In suggesting a solution to the problem of how concrete understanding becomes abstract, it continues to be very influential. As interpreted by Mason, it becomes significant in presenting an active process for learners, via pattern spotting, visualization and articulation which includes their own drawing, writing and explanations.

Current early mathematics pedagogy

Dunphy et al. (2014), in reviewing the literature on mathematics education for three to eight year olds in preparation for a new Irish curriculum, provide a useful summary of recent theoretical developments, emphasizing learning as a social rather than purely cognitive process, following Lerman (2000). They point to the key role of language according to the different theories: from a cognitive view of learning, language is seen as representing ideas, whereas 'in sociocultural terms, children are enculturated into mathematics through social and discursive activity (2014:62). They cite Sfard (2007) as making a useful distinction between language as a tool and language as discourse, an activity in which the tool (one of several) is used or mediates. They identify the role of talk for discussing, challenging, reasoning and justifying, and also cite research linking the amount of math-related adult talk with pre-school children's learning (eg. Klibanoff, Levine, Huttenlocher, Hedges & Vasilyeva, 2006). They also emphasise situationist theories (eg. Lave & Wenger, 1991), which view learning as participation, with children developing identities as mathematicians, within 'math-talk' communities of learners, as proposed by Hufferd-Ackles et al. (2004), where learning is collaborative.

Dunphy et al. (2014) agree with the USA's National Research Council (NRC) that the main aim of early mathematics education should be the 'development of mathematical proficiency, defined as including conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition' (Kilpatrick, Swafford, & Findell, 2001:10). This aim therefore sees both content and processes as important aspects of mathematics education, with the addition of positive attitudes.

The pedagogical approach recommended by the Irish review for early mathematics is *mathematization,* echoing the RME approach discussed previously. The suggestion is therefore that mathematical activity for young children should involve contextualized problem-solving, rather than, for instance, mathematical enquiry using patterns of colour rods. Ginsburg is cited as arguing that making meaning from everyday contexts and real world problems involves children in the key processes which develop mathematics proficiency.

Boaler (2009) discusses the creation of positive identities, suggesting it is not compatible with placing children in 'ability groups', but requires a pedagogical approach aimed at creating equity. This involves teachers having expectations that all children are capable of mathematical learning, and consciously using strategies such as 'assigning competence', or publicly highlighting contributions of children who might otherwise have low status, as part of the 'complex instruction' approach (Boaler, 2010). Creating positive mathematics identities also requires broader mathematics tasks, including problem solving, which focus discussion on children's solution methods and allow for more forms of success than finding the right answer. This approach is also endorsed by Sarama and Clements (2009), who conclude that discussing children's inventions of strategies to solve problems is the most effective starting point, even for teaching standard methods. Manipulatives may provide support in helping children to explain why strategies work.

From a socio-cultural view, manipulatives may be the focus or vehicle for learning through interaction:

> The core concern of sociocultural theories is the mediated nature of all human activity through interactions with others around tasks and activities and with material and symbolic tools.
> (Dunphy et al., 2014:48)

Manipulatives therefore may become the subject of dialogue and reasoning, rather than images to be internalized. Dunphy et al. also refer to constructionism, as proposed by Papert, the inventor of LOGO, and discussed with reference to IT programmes involving interactive geometry. They cite Papert and Harel, who suggest the value of constructing understanding through constructing 'a public entity, whether it is a sand castle on a beach or a theory of the universe' (2014:49). The collaborative element of learning through discussion about the constructions is emphasised within 'learning environments that encourage thoughtful reflection', suggesting that 'a key role of teachers is to foster the development of a reflective culture in their classrooms' (2014:51). This suggests that construction with manipulatives, such as children creating patterns with Cuisenaire, might also be considered as a constructionist way of learning, and points to the potential value of a more contemplative creative approach to mathematics activities.

Dunphy et al.'s (2014) synthesized theory sees children's learning as the development of:

- meaning making
- conceptual understanding
- skilled participation in mathematics-related activities
- a mathematics identity
- use of key tools such as language, symbols, materials and images
- participation in communities of learners engaged in mathematisation in small group and whole class conversations.

Within this synthesized theoretical stance, manipulatives have a role in the co-construction of meaning through modeling and argumentation. They are a focus for interaction, which mediates learning cognitively and socially.

This view is supported by current theories of representation and research into the effectiveness of manipulative use.

C. Historical overview

The use of objects to represent numerical relations goes back a long way- to the third millennium BC when early counting devices consisted of pebbles arranged in grooves in the sand or on tablets. In the history of children's education in Europe, following Froebel (1826/1885), Maria Montessori (1912) was amongst the earliest to devise specific materials for arithmetic, and was followed by others such as Stern in the 1930s, Cuisenaire, whose work was developed and disseminated by Gattegno in the 1950s, and Dienes in the 1960s. These developments aligned with Piagetian theory (1941/1952), that young children's number learning was based on manipulating objects. The popularity of these various structured materials spread internationally, waxing and waning throughout the 20th century, alongside other developments such as 'new mathematics' in the 1960s. Realistic Mathematics Education (RME) developed in the Netherlands in the 1970s, placing more emphasis on contextualised problems and diagrammatic models (Streefland, 1991).

With technological developments, new resources have become popular, such as interlocking plastic cubes, which could be used instead of colour rods and as an alternative to Dienes' base ten apparatus and have become ubiquitous in primary classrooms. Originally these linked on two faces only, while later versions linked on all six faces (Unifix and Multilink, respectively).

Some older resources have been rediscovered, such as Stern's number plates now available in range of guises including Numicon. These shapes are based on a 'tens frame' image, which is much more common outside the UK, providing easily recognised images for numbers up to 10 and emphasising odd and even properties. More recently, virtual manipulatives have become widely available, with advantages over their physical equivalents for access and classroom management (Mildenhall, Swan, Northcote & Marshall, 2008) but with the major disadvantages that children do not have the chance to handle the materials and cannot work on their own or in pairs with the interactive version.

In England from the late 1990s, government initiatives including the National Numeracy Strategy focused on a variety of images including more abstract models such as number lines and number squares (DCSF, 2003). More recently, interest in Singapore's mathematics teaching has refocused English attention on Bruner's (1966) theory of an enactive mode of learning (eg. Fong, 2014) but also on the use of the diagrammatic 'bar model' for solving number problems with young children (National Centre for Excellence in Teaching Mathematics (NCETM), 2015). Teachers of young children have therefore continually been exposed to arguments about the relative merits of practical resources and abstract models and images for arithmetic. Throughout these developments, there has been ongoing debate about how exactly children are supposed to learn abstract mathematical ideas from manipulating objects (Dufour-Janvier, Bednarz & Belanger, 1987; Goldin, 1998). Therefore, it is perhaps not surprising if teachers are inconsistent in their use of manipulatives and vague about the learning processes involved (Moyer, 2001; Swann and Marshall, 2010; Ofsted 2012).

Froebel (1826/1885), in Germany, saw children as 'endowed with inner spiritual powers that unfolded in an educational environment' (Gutek 2004:11). He devised teaching materials, including a set of 500 wooden building blocks representing numbers to 12:

> The material for building in the beginning should consist of a number of wooden blocks whose base is always one inch square and whose length varies from one to twelve inches. If, then, we take twelve pieces of each length, two sets e.g., the pieces one and eleven, the pieces two and ten inches long, etc.- will always make up a layer an inch thick and covering one foot of square surface; so that all the pieces, together with a few larger pieces, occupy a space of somewhat more than half a cubic foot. It is best to keep these in a box that has exactly these dimensions; such a box may be used in many ways in instruction, as will appear in the progress of a child's development. (Froebel, 1826/1885:283)

These blocks were the precursors of rods such as Cuisenaire's.

Montessori (1912/1965), from Italy, also devised specific resources to help children learn arithmetic, with an approach based on discovery, facilitated by teachers who observed children's development. She advocated that 'things' were 'the best teachers' (Mayer, 1965). According to Mayer, she followed previous 'child-centred' educationalists such as Rousseau, Pestalozzi and Froebel, but rejected Rousseau's total freedom, found Pestalozzi too mechanical and all of them too philosophical. Her methods were developed by working intensively with children then described as 'mentally defective', and helped them to pass public examinations. Montessori saw the teacher as the 'directress' who was to 'guide the children as they taught themselves to learn' (Gutek, 2004:17). Children used self-correcting, self-selected materials, so teachers were no longer in controlling roles. However her approach was more structured than that of contemporary US progressives such as Dewey and consequently her approach was less favoured in the USA.

The Montessori number materials, which are still used with children from the age of three in Montessori schools, emphasise numeral values, including place value, and relations between numbers. They include segmented number rods and 'golden' beads strung in tens which can be arranged to form a 'cube' of 1000. Activities include matching objects to large numerals, and pairing rods to investigate ways of adding and subtracting to produce numbers within 20. Some Montessori activities pre-empt materials developed later: for instance number cards where single digit cards were superimposed over cards with multiples of 10, to form 2 digit numbers. 'Arrow cards' or place value cards fulfill a similar function.

Stern (1949/1953) developed her materials in the late 1920s in Germany in her training nursery. She was concerned that children's understanding of arithmetic should not be based on counting, according to Sawyer(1953), who describes 'counting on' as a 'disastrous habit'. Stern had become dissatisfied with Montessori apparatus, where two 5 strings of beads were longer than a 10 string, which forced children to count to check. She produced coloured graduated rods 'so that children could see before them a concrete picture of the number series'

(Stern 1953:21). (These were made by Bakelund, who invented Bakelite, which was the forerunner of plastic.) The rods were scored to show their value in individual cubes, but they were introduced through various matching and pairing activities, encouraging children to memorize bonds to 10 by colour and size, before naming numbers. Stern devised self-checking inset boards, including some showing the pattern of odd and even numbers arranged in two columns. She also produced base 10 materials, including a square representing a hundred and a cube for a thousand. An insert number track allowed children to check, for instance, the value of an array by making a 'train' of cubes. Through this range of apparatus, Stern was encouraging children to link different images of the number system.

In 1938 she went to the USA, where she worked with Wertheimer, the founder of Gestalt Psychology. Wertheimer's 'productive thinking' (1945) emphasized the importance of oscillating between the part and the whole sense of an object or situation, especially when trying to understand or solve problems. This is in contrast to considering thinking and learning as reproductive and associated with repetition, conditioning, habits or familiar intellectual territory. This approach links with seeing arithmetical understanding in terms of part-whole relationships. Stern described her approach as one in which

> 'counting is taught but not used in computation… Structural Arithmetic provides materials to be used in experiments that reveal the structural characteristics of number and number relationships" (Stern 1953:15).

Stern proposed that a given sequence of experiments with the materials led to the discovery of number facts and relations. These were remembered, not through repetition, but because of the strength of visual images that the pupil can reconstruct mentally. Children would memorise number facts in groups, for instance as doubles or 'neighbours of the doubles'. Her workbooks have illustrations which duplicate the mental pictures gained while working with the materials (Stern 1953:18). The actual work with the manipulative devices leads directly to the algorithms of arithmetic (1953:19) and children recording their findings by writing equations. Problem solving follows investigative number activities and develops mathematical reasoning: children learn how to analyse problems and 'how to decide to which group a problem belongs...the reasoning used at this level is that used in algebra' (1953:21).

Stern criticized 'progressive education' and ridiculed the theory that young children would gradually abstract numerical relations from manipulating everyday objects. She also rejected a problem-based approach, whereby children would engage with arithmetic in order to solve interesting problems, regarding this as inefficient. Instead, she proposed that children should learn pure mathematics first, then apply it:

> They discover with delight how their mastery of arithmetic helps them explore their surroundings.. In teaching Structural Arithmetic we do not study life situations filled with numbers. We fill numbers with life!
> (Stern, 1953:16)

In Europe after the Second World War, manipulatives were promoted by several mathematics educationalists including Gattegno and Dienes. Less famous others include Castelnuevo in Italy, whose focus, mainly on geometry, was described as using concrete materials to foster exploration, conjecturing and the creation of examples and counter examples (Furenghetti and Menghini, 2014). Like Montessori and Stern, these educationalists saw structured materials as embodying mathematical relations and stimulating learning in a process of guided discovery.

1952 saw the foundation of the Association for Teaching Aids in Mathematics (ATAM) by a group of enthusiasts led by Caleb Gattegno and Roland Collins (https://www.atm.org.uk/History). This organisation later became the Association of Teachers of Mathematics (ATM). Gattegno was then a lecturer at the London University Institute of Education. Members of ATAM experimented with ideas put forward by Cuisenaire and Goutard and promulgated teaching materials including films, coloured rods and geoboards.

'**New mathematics'**, of which Dienes was one of the advocates, challenged mathematics education in the 1960s. Walters (1963) argued that primary teachers needed to have more of an understanding of new topics in mathematics such as set theory and matrices. There was a move away from an emphasis on speed and accuracy in computation as goals for mathematics teaching, towards a belief that

> 'children should be vitally interested in their mathematics work and should be led to appreciate both the power of mathematics in application and the beauty and fascination of its own internal patterns. Throughout their work children should have a more creative part to play'
> (Walters, 1963: 51).

Discussion about mathematics was considered important as was the need for children to express their mathematical ideas in written form. Advocates of structured materials were therefore in line with the ethos of mathematics educationalists at the time, in emphasizing children discovering and being creative with mathematical patterns and relationships, as well as in expressing themselves mathematically through discussion and writing.

Such approaches link with Piaget's (1952) account of children learning through reflection on actions with objects and materials, involving 'successive restructurings of facts and relations' (Resnick & Ford, 1984:111). Piaget implied that children's understanding of number was dependent on their ability to visualize transformations: for instance understanding the invariance of number requires the mental reversal of actions such as spreading out beads. Understanding also depends on children's ability to reason logically, for instance, thinking, 'if I did this, then..' (Piaget, 1952:37). Piaget suggested that children could do this at a 'concrete operations stage' from about 7 years old. He also drew attention to young children's difficulty in processing complex information: for instance understanding the concept of inclusion, as with seeing 4 and 2 within 6, requires a child to see the whole number and its parts simultaneously, or understand part-whole relationships. Similarly, he found younger children unable to consider two adjustments at the same time, for instance, after changing 4 and 4 into 1 and 7, they said there were more altogether, because they only focused on one alteration. In order to understand the number system, children had to integrate number relations such as inclusion and order, ultimately performing 'an operational synthesis of classification and seriation' (Piaget, 1952: viii).

Piagetian theory therefore suggested a need for young children to manipulate objects, in order to develop visualization, reasoning and understanding. Because he saw children as constructing their own systems of ideas, the active discovery approaches of Montessori and Stern resonate with Piaget's ideas. Piaget also pointed out that number understanding requires young children to focus on the quantity rather than the properties of objects. His emphasis on young children's difficulties with complex information perhaps also endorsed structured manipulatives which present number relations simply, facilitating comprehension of part-whole relationships.

Gattegno recognized the potential of **Cuisenaire** rods which had been devised in the 1920s by Georges Cuisenaire at about the same time as Stern was developing her manipulative devices. Cuisenaire's intention was to make mathematics visible, and as readily comprehensible to children as music, which he also taught. The 10 coloured rods were very similar to Stern's, except they were smaller (based on one centimetre units) and they had no marks to indicate numbers, so were more likely to deter children from calculating by counting. Similar colours indicate related factors eg 3 and 6 are both green. There were no accompanying numerals or other materials. The rods were used mainly in the Belgian village school where Cuisenaire taught, until his success, particularly with 'weaker' students, prompted Gattegno to visit in 1953. He subsequently promoted the rods and wrote workbooks for them (Gattegno & Cuisenaire (1954). He believed that the rods both embodied the core relationships and structures of mathematics and stimulated 'intuition and enquiry' (Resnick and Ford, 1981:120).

Gattegno's (1954) approach with Cuisenaire rods taught children to visualise transformations: therefore his workbooks contained no illustrations. He addressed teachers through these, thereby also encouraging teachers to visualize and generalize, so bypassing the need for a teacher's manual which might be overlooked. His pedagogical approach began with free play, followed by closed questions and open challenges to stimulate investigation and reflection.

Language was used to express the same relation in different ways, eg *1 is what of 2? What is half of 2?* This emphasized principles such as equivalence and inverse. Children were instructed to answer questions or test statements by visualizing. They were taught to use standard signs for equations, first through matching, then visualizing and recording their own arrangements. Like Stern, Gattegno directed children to first describe relations between the rods by colour, before introducing numbers. This pre-numerical stage includes finding common differences and equivalent fractions, then recording equations in terms of colours. Gattegno therefore intended that children would generalize algebraic relationships, such as commutativity or inverse operations, before engaging with numbers or symbols. Children then investigated the numbers to five in terms of addition, subtraction, multiplication and fractions, so they became thoroughly familiar with the composition of each number through different operations and relationships.

Like Stern rods, Cuisenaire rods encourage discoveries about equivalence, since the same length or area can be made in many different ways. Because children are also encouraged to express number facts in different ways, as bonds, differences, factors and fractions, they can investigate connections between arithmetical relations through familiarity with a few numbers, using mathematical reasoning processes, such as generalizing and testing. The rods can also be used to stand for different numbers, so that orange might be 5 or 10, or 1 or 100 or any other number, which also helps children to derive number facts and generalise relationships.

One criticism of Gattegno's Cuisenaire approach is that it involves learning two codes, first recording relationships with colours, then with numbers. On the other hand, because the Cuisenaire approach requires children to think with only one image of the number system, there were doubts as to whether it would enable them to abstract arithmetical understanding and generalize to other situations. However Goutard's implementation of Gattegno's approach demonstrates that abstraction and understanding can result from this approach.

Goutard (1964) developed Gattegno's Cuisenaire approach in Canada to teach 6 and 7 year olds, with impressive results. Goutard's approach pre-empted later concerns, not only about generalizability, symbolic abstraction and problem solving, but also about the role of teacher and the social environment of the classroom.

Like Gattegno, Goutard encouraged free play followed by directed exploration, discussing colours before introducing numbers. The children identified families of equivalent additions, fractions, differences and ratios. She advocated three phases for teaching each operation:

- empirical,
- systematization
- mastery of structures.

The first of these is essentially exploratory, for instance finding lots of examples of pairs of rods with the same difference. The second involves ordering specific examples systematically and looking for patterns: Goutard emphasises that this should only be encouraged by the teacher when children have started to organise their pairs and started this phase for themselves: then they will be asked to put the rods away and visualise pairs and patterns. In the third phase children can generalise, for instance saying that if the same number is added to each of a pair of rods, the difference will stay the same, and they produce 'free writing' with equivalent equations involving much bigger numbers than they have used physically. Goutard therefore similarly emphasises this process as the child actively constructing representations and understanding.

Her approach might be summarized as 'do, talk and record' (Open University, 1982) and comprised:

- manipulative games
- conscious analysis
- writing of mathematics facts obtained.

The children were taught how to record using mathematical symbols. At the end of each lesson, the rods were put away and children were encouraged to write freely, starting from what they had found out, creating new equations. This creative mathematical writing was a distinctive aspect of Goutard's approach. The free writing sessions seemed to encourage children to generalize and create patterns, with some original results.

Children not only were engaged in investigating and testing generalisations, but became very familiar with facts for individual numbers, including complements and factors. Goutard argued that free composition developed children's fluency in reading and writing mathematics: children were not intimidated by complex expressions because they were confident in producing them.

Her approach was also significant in fostering children's enjoyment of pure mathematics. She described children playing with and being 'charmed by' mathematical expressions and so writing 'arithmetical puns' (1964:52):

b = b + (p-p) = b – (d-d) = b + (t-t)
(9-9)+(9-9) = 0   (20-20) + (10-10) = 0

1x1 =1
1x1x2 =2
1x1x1x3 =3
1x1x1x1x4=4

In the examples above, children seem to be enjoying the paradoxes created by zero and 1, by the way inverse operations can produce 0, or how repeatedly multiplying by 1 has no effect. There seems to be a playfully humourous element in creating impressively elaborate and sophisticated equations which reduce to simple statements like 10 x 1 =10. These are also reminiscent of younger children's number jokes, as noted by Gifford (2004). Like Stern, Goutard was also convinced that children could enjoy mathematical investigations in pure mathematics.

The teacher's role in Goutard's approach was of paramount importance: she observed and analysed children's responses, then planned accordingly. Activities were usually challenges to find equivalences, alternatives and families, so tasks were open-ended, undifferentiated, inclusive and creative. Goutard also described classroom discussions where children investigated and justified each other's findings. While the structuring of Cuisenaire rods makes discovering numerical relationships possible, Goutard's carefully judged challenges and responses made this happen. She emphasized the importance of valuing all responses and not transmitting any negative messages. She was insistent about giving children freedom to explore, to own their investigations and to create examples for as long as they wanted. In this Goutard's work links with later research findings that effective mathematics teachers expect that all children can learn (Askew, 2002). However it also seems clear that she was creating communities of mathematical enquiry in her classes (Boaler 2009).

In summary, Goutard's approach to using Cuisenaire rods, like Gattegno's, focuses on algebraic relationships using 'do, talk and record', within an enquiry - based learning community and with the distinctive strategy of removing apparatus before writing freely, which stimulates generalizing and creative investigation. It seems to counter concerns about the limitations of one image, as children move on successfully to thinking with abstract symbols. However, it does seem to require commitment to using the rods consistently for extended periods of time, as well as requiring a highly skilled and mathematically expert teacher.

Dienes (1960) was another major proponent of structured materials in the 1950s and 1960s. From the end of the nineteenth century, teachers in many classrooms across Europe had commonly used home-made materials, such as bundles of ten spills tied with raffia, to help children understand place value and to carry out arithmetical procedures. Dienes took this further, developing arithmetic blocks reminiscent of Stern's base 10 blocks but covering other number bases as well, and hence called multibase arithmetic blocks (MAB). 'Because patterns and relationships are not obvious in children's everyday environments', he proposed creating embodiments 'to bring them within the realm of concrete experience' (Resnick and Ford, 1981: 116), echoing Piagetian terminology of 'concrete operations'. He argued that specially designed mathematics materials were free of distractors and enabled early learning about algebraic principles without numbers or symbols. He also believed 'children are by nature fundamentally constructivist rather than analytic', learning through 'active exploration'. Nevertheless, his materials were aimed at analyzing common features of mathematical relationships, between embodiments in several number bases. 'Multiple embodiments', provided 'a rich store of mental images' (Resnick and Ford, 1981:122) with 'perceptual variability' ie looking as different from each other as possible and 'mathematical variability' eg of context or number involved, including geometric, physical, social, arithmetical and algebraic forms. Dienes proposed a learning cycle, summarized by Resnick and Ford (1981): free play was followed by structured games directing children's attention to key properties of the 'multiple embodiments' of a concept. This process was accompanied by talk, drawing 'pictures, graphs or maps', then attaching symbols, which might be of children's own choosing or creation. These symbols were associated with images and became the 'tools for mental manipulation'. Learning became systematized by playing with symbols and rules. Throughout the whole process, materials and images were revisited in order to reconnect symbols with concrete experience.

Dienes' pedagogical approach is similar to that of Gattegno and others in advocating free play followed by guided discovery, visualisation and deferred symbolization. The symbols then become objects to play with (as exemplified by

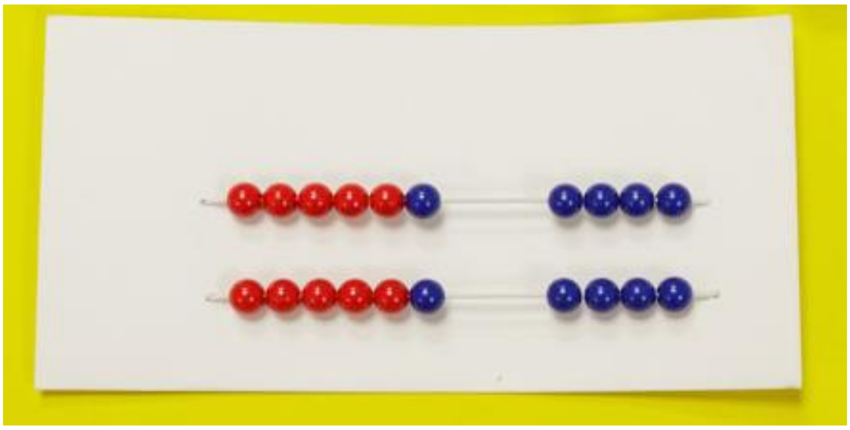
Goutard's children), in a cyclical or spiralling process of increasing abstraction. Dienes differed from Gattegno in emphasizing generalization from varied examples of materials, of number systems with different bases. Goutard also used Cuisenaire to represent different number bases, insisting that an understanding of place value should be based on understanding powers. What is distinctive in Dienes' cycle is his clear theory of how children develop abstract understandings of symbols, by linking them to diagrammatic images of the abstracted concepts, aided by talk, and his suggestion that children should record through drawing and choosing their own symbols.

However, Dienes' multibase approach was not widely adopted in primary schools. Later use of Dienes' apparatus commonly focused just on modelling addition and subtraction algorithms (Askew, 2012), rather than building children's understanding of how the number system works.

Realistic Mathematics Education (RME), which developed in the Netherlands in the 1970s, followed the ideas of Freudenthal (1975) and opposed the 'structuralist' approach (Treffers, 1991). The RME approach is based on providing story problems which lead children to 'mathematize', or develop mathematical ideas, models and strategies from these situations, rather than learning directly from models of mathematical structures. Van den Brink (1991) noted that 'realistic' referred to children 'realising' their own ideas and imaginings, because RME values children's understandings and the role of visualization. The RME approach emphasises making mathematical operations more meaningful and memorable, encouraging talk, and providing models which link the context and the arithmetical relations, also leading to abstract notation. For instance, the story of a shepherd trying to keep track of his sheep leads children to 'invent' ideas of place value, through the model of exchanging 10 pebbles for a coloured stone. Gravemeijer (1991) suggests that Freudenthal drew on ideas of reinvention from the history of mathematics.

Models are often pictorial or diagrammatic, rather than involving manipulatives: they included number lines and arrays, as well as signs like arrows (for instance used to record people getting on or off a bus): thereby 'models of ' a situation become 'models for' abstract mathematics. The contexts are carefully chosen to realistically present mathematical models of place value, division or fractions, with later teaching of the abstract symbols. In more recent examples from the USA, Fosnot and Dolk (2001, 2002) use models and images which are integral to their story problems: for instance grouping in tens was used to count classroom resources, a 'soda' machine provided an array image of stacked cans and rectangular sandwiches could be drawn as fraction bars. Contexts thereby provided both meaning and images: according to Gravemeijer (1991: 75-76) 'understanding and insight are supported by the context, which can serve as a situation model. 'Strong', 'fitting models' helped children 'bridge the gap between informal, context bound work and the formal, standardized manner of operation, through the constructive contribution of the children themselves' (Treffers, 1991: 33).

In the later RME approach, children's contributions became more central and models were based on observations of children's intuitive strategies. Originally, according to Gravemeijer (1991), discussion was elicited to follow a previously mapped out route of 'discovery'. Subsequently, possible learning routes were planned, based on children's own solution methods or 'shortcut' strategies. These included the use of structured apparatus. For instance, close observation of children using the abacus for addition revealed that it prompted them to count, instead of using number facts. Researchers also noticed that some children used doubles, fives and tens as points of reference. An abacus was then devised with beads grouped in fives by colour, to encourage 'reading off ' the numbers of beads by subitising, thereby prompting more children to use these shortcuts. The arithmetic rack (rekenrek) was subsequently designed (two rows of ten beads grouped in 5s) so that 6 +6 could be recognized as double 5 +2 or double 6.



These developments also reflected the RME focus on mental arithmetic, whereas mechanistic and stucturalist approaches were seen as prioritizing written algorithms (Gravemeijer, 1991).

The RME group wanted structural apparatus to be used to develop understanding of mathematical relationships rather than as a calculating device: the abacus should be used as a thinking tool, not just a working tool. This is similar to the concern expressed by Goutard (1964:25): the role of material aids in mathematics teaching is not as a calculating machine.

The RME group was significant in using structured materials together with close observation, creating manipulatives that reflected children's effective ways of seeing numbers. 'The material is used to elicit (mental) arithmetic actions which other children have previously developed themselves' *(*Gravemeijer, 1991*:76).* This was a significant departure from previous apparatus based on adults' analysis of mathematical structures. Following the influence of Vygotsky (1978), constructivist theories of learning were diversifying, with more emphasis on social learning. For the RME group, constructivism meant giving the initiative to the child to build their own understanding. Children would create idiosyncratic constructions, which were attuned through consultation and negotiation in social interaction. There was also considerable problematisation of the different meanings of representation, either as materials embodying number relationships and or as mental pictures of activity.

One concern was the discrepancy between the manipulative action and the intended mental activity, as with subtracting using the abacus. There remained the problem of how children learned to think abstractly:

> The danger exists that working with manipulative materials does not prepare for working without manipulatives … The problem of the transition from thinking about material to thinking in terms of mathematical relationships and concepts. (Gravemeijer, 1991:75)

Gravemeijer (1991, citing Van Parreren) suggested that the process of internalizing was not about imagining using the blocks, so physical actions did not have to mirror mental ones: learning was about creating shortcuts, including imagining actions, automatizing actions or restructuring tasks. The process of learning was seen as the gradual creation of schemas or mental organized knowledge structures, through abbreviation and generalization, supported by the use of realistic contexts. According to van den Brink (1991:83), Freudenthal regarded children's mathematisation as activities in which they move back and forth between the real world and the world of symbols: as with Dienes, abstraction was not seen as a linear process.

RME's overall contribution to the development of mathematics manipulatives was to observe how children used apparatus and, through seeing learning as a process of organizing, abbreviating and creating shortcuts, then to create structured material that would encourage the use of children's shortcuts or points of reference. Since the 1990s, the focus on children's mental strategies has developed, alongside the use of manipulatives such as the rekenrek and beadstrings, as well as the empty number line (van den Heuvel-Panhuizen, 2008). RME also posed a challenge to the use of structured apparatus, offering criticisms which will be considered in more depth in the next section.

Postscript

The development of new materials leads to new possibilities. For example, advances in the manufacture of plastics led to the invention of interlocking cubes allowed for the flexible construction of number rods, but with the possibly distracting opportunity to count individual cubes.

In summary, ideological approaches, technological advances and government policies, as well as psychological analyses of children's learning, have resulted in the changing use and cycles of rediscovery of structured manipulatives to teach arithmetic. For example, the endorsement of Dienes' apparatus in an Ofsted report has led to a resurgence in interest in base ten equipment (2012: 27).

D. Critiques of the use of manipulatives

Empirical studies of the effective use of manipulatives

This section considers empirical research into the general use of manipulatives. Uttall, Scudder and Deloache (1997) noted that reviews of research into the effectiveness of manipulatives have generally been inconclusive. We particularly found that empirical studies using control groups produced unclear results since such studies are unable to identify and control for the many variables involved. Qualitative studies reveal insights into the complexity of the processes involved in teaching, including teachers' views.

Quantitative studies

Carbonneau, Marley & Selig (2013) conducted a recent meta–study of research which used control groups and compared manipulatives or images of manipulatives with just abstract symbols (eg approaches using only text books). They identified four aspects of measuring efficient learning: retention, problem solving, transfer and justification. It was not clear how long after the intervention retention was assessed. They defined 'problem solving' as not being told a method, 'transfer' as applying to a new situation eg from addition to multiplication, and 'justification' as explaining a chosen method. Arguably the latter might seem most indicative of understanding. They considered a range of mathematics topics.

Their findings may be summarised as follows:

- Instruction: overall there was a small to medium effect of instruction using manipulatives, with a larger effect on retention than problem solving, transfer and justification. Higher levels of instructional guidance improved retention and problem solving, while lower levels improved transfer. It was not clear why this might be, indicating that more research was needed about the topics involved.
- Age of children: manipulatives were more effective with 7 to 11 year olds than 3 to 6 year olds (or over 11s). They suggested that younger children did not understand 'that an object can stand for the item while simultaneously representing a larger mathematical concept'.
- Perceptual richness: realistic manipulatives suppressed learning in terms of retention and problem solving but seemed to improve transfer, which contradicted the findings of previous studies, indicating the need for further investigation.
- Length of time: medium length studies (under 45 days) were generally more effective, with shorter studies (under 15 days) more effective for retention and medium length worse for problem solving. These findings seemed insecure (especially considering that Utall et al. (2013) reported that longer use was more effective).
- Topics: manipulatives were more effective with fractions and algebra than arithmetic (including the four operations).

There were several limitations of the meta-review:

- There was a lack of distinction in studies between manipulatives and images of manipulatives.
- Timing was unclear: presumably the post–test was administered immediately after the study, in order to assess retention.
- These are studies with an empiricist experimental methodology, and so do not evaluate the effects of long term uses of manipulatives, such as Goutard's (1964) or Ainsworth's (2013) several years of teaching using Cuisenaire.

The oddity of some results, such as the effectiveness of less instructional guidance and more perceptually rich manipulatives improving transfer rather than retention, cast doubt on the validity and reliability of the review's findings. It seems likely that the studies which used tests of transfer to measure effectiveness, had other features which affected findings, since no reasonable explanation for these results was offered. Similarly, the finding that shorter interventions were more effective seemed insecure, with no explanation relating to manipulative use (for instance, it may have been due to a short-term Hawthorne effect and the difficulties in sustaining momentum in longer studies). The need for further investigation and analysis is clear.

Regarding topics, it is interesting that fractions and algebra were more successful than arithmetic, as this might be deduced from the focus of the main proponents of structured apparatus, Cuisenaire and Dienes, as discussed above. The main findings, that perceptually rich manipulatives are less effective, that more instruction is more effective and that very young children may have difficulties with recognising the symbolization of the manipulatives, are supported by the literature. The latter point is interesting because it challenges Piagetian assumptions that concrete materials are more appropriate with younger children and likely to be less effective with older children.

What is lacking from these studies is consideration of the kind of activities that the children were engaged in and the social context- whether they were about performing calculations or pattern seeking, solitary or collaborative. It also seems likely, as Goutard argues, that the success of manipulative use may be affected by the skill of the teacher in assessing children and challenging them appropriately.

In summary, it may not be possible for empirical research to assess the effectiveness of the use of manipulatives, if it focuses on short term use and does not control for other key factors, such as kinds of activities, pedagogy, teacher expertise and social context.

Qualitative studies of effective use

Qualitative studies of teachers using manipulatives reveal more about the complex variables involved in using manipulatives, as Goutard's account of her own practice shows. Frequently they reveal the inappropriate use of manipulatives, so that they may even hinder learning. Qualitative studies have analysed the use of manipulatives from both the children's and teachers' perspectives.

Some researchers, such as Lesh, Behr and Post (1987), found that children had more difficulty with concrete than abstract problems. Boulton- Lewis (1998) used information processing theory to explain children's difficulties using base ten materials to support column addition and subtraction algorithms. She had found that, given the choice, children used 'verbal and mental methods' or fingers, rather than apparatus or written methods to solve two or three digit addition and subtraction problems (although they sometimes used apparatus to please the interviewer). She argued that difficulties arose because of the complexities involved in children mapping between the symbolic problem and the materials, which were used as analogies for the numerical relationships:

> When the complexity of the mappings for three-digit subtraction is analyzed … It seems that perhaps the most difficult way to perform two or three digit addition, in terms of the load on processing capacity, is to use analogs to support limited knowledge of place value, while trying to learn to use an algorithm. (Boulton- Lewis,1998: 221)

One issue here seems to be that Dienes' materials were being used to teach understanding of the written algorithm to children who also lacked understanding of the place value system. Gravemeijer (1991) cited Resnick and Cobb, who both described children, who did not yet understand the relations between the tens and ones blocks, attempting to use base 10 blocks to solve addition and subtraction problems. , Taylor, A., O'Donnell, B. & Fick,

Moyer (2001), studying sixth to eighth grade teachers, found that manipulatives were used as 'little more than a diversion in classrooms where teachers were not able to represent mathematics concepts themselves', thereby pointing to the lack of pedagogical subject knowledge of the teachers (2001:175). With grade 9 teachers, Golafshani (2013) also identified that teachers saw manipulatives mainly as affecting students' engagement and that they had concerns about their competency in using them. The teachers identified several factors affecting the use of manipulatives, including classroom control and noise level, management of tidying up, availability of resources and time factors.

Puchner, Taylor, O'Donnell and Fick (2008) described teachers mis-using manipulatives in lesson study: they cited Schram and Ball asserting that teachers assume children will automatically internalize understanding from manipulatives. They concluded that teachers designing lessons were not clear how the manipulatives would help learning and were using them because they assumed they were 'a good thing', 'as an end in themselves'. The teachers failed to assess that the children already understood what they intended to teach. In one lesson the manipulatives hindered learning, because cubes were used for counting, rather than helping the students to think algebraically as intended. Generally the teachers had '…a practice of using the manipulatives without carefully thinking how the content will actually be learned using the manipulative in question' (2008:322). Teachers had not identified what students' possible responses might be and so were frustrated when they were not as expected. Instead they needed the exercise of analyzing the intersection between the content goal, the specific type of manipulative, the way the manipulative would be used and the way students make sense of the representation. They suggested analyzing scenarios in terms of linkages between pedagogy, mathematical content and learning, including some with inappropriate manipulatives and some without any. Teachers needed to know their advantages and disadvantages and the theory behind their use. They suggested that a better focus would be children's 'drawings and self- invented strategies', because that would problematise children's thinking for teachers, rather than their assuming that understanding would be automatic. Puchner et al. (2008) concluded that because 'hands-on' learning was appealing to elementary teachers, a general recommendation to use apparatus was not helpful.

Similar comments about 'hands on' learning have been made by other researchers, linked with doubts about how well Piagetian theory has been understood by educationalists and whether the 'concrete operations stage' was later seen as requiring activity with objects, rather than visualization of transformations and reasoning about these (Clements, 1999). Swan and Marshall (2010) found that some Australian teachers justified using manipulatives as 'concrete visualisation' and 'hands on learning', terms which echo Piagetian theory. Swan and Marshall concluded that 'teachers could not identify exactly what it is about manipulatives that assists in the learning of mathematics' (2010:14). They found that teachers also associated manipulatives with concept formation, perhaps considering them unnecessary for more complex mathematics with older children. This view may also derive from Piagetian stage theory, that older children at the formal operations stage are able to reason abstractly, without reference to objects.

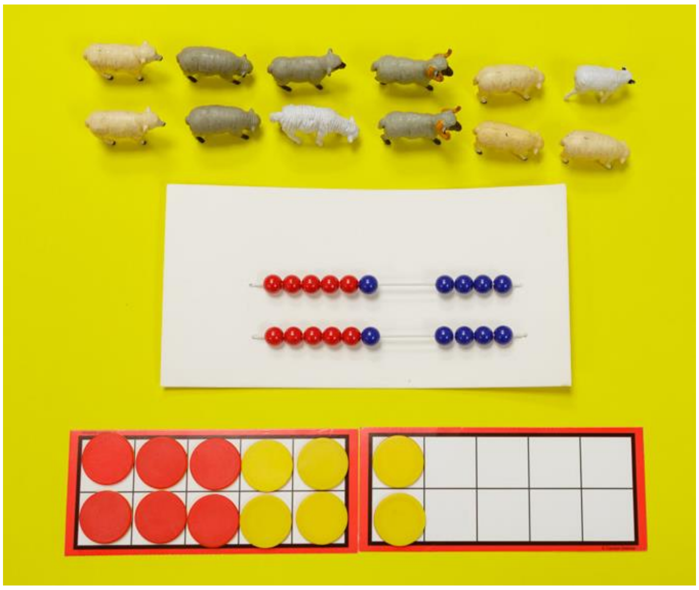
Moscardini (2009) identified teachers either using manipulatives as 'tools' to help children make sense of problems or as 'crutches' to enable them to complete a procedure (often poorly understood). This implies that the use of manipulatives as a 'crutch' is associated with aims of successful calculation, rather than relational understanding. Manipulatives as 'crutches' may also be seen as serving aims of inclusion, by allowing some children to achieve success at class tasks they would otherwise find too difficult, by low level counting strategies. This points to the issues of the kinds of activities and mathematical aims for which manipulatives are used, as well as teachers' analysis of learning.

This research therefore underlines the importance of teachers considering the processes of learning involved with manipulatives, the links between them and the mathematics, as well as assessment of children's understanding before and during teaching. These studies raise the following questions about manipulative use:

- How does the manipulative represent the mathematics?
- How do children make sense of the manipulatives? Are some representations too opaque or complex for some children?
- How do children move from understanding mathematical relationships in concrete to abstract form?

How does the manipulative represent the mathematics?

The use of manipulatives to help children learn maths requires them to see physical materials as symbols for mathematical relationships: however these symbols necessarily represent some aspects at the expense of others. Whereas for Stern (1953:15) the rods provided children with an unproblematic 'picture of the number series' for them to reconstruct mentally, it is one of many possible pictures. Where Stern rods emphasize numbers increasing by one, base 10 blocks illustrate numbers increasing by powers of 10. The number 12, represented using an array of toys, may emphasise 12 as double six and six pairs, demonstrating commutativity; on the rekenrek 12 may be seen as the double of 5 plus one (illustrating the distributive law) and as 10 and 2, whereas ten-frames may be used to show 12 as 6 plus 4 plus 2, or the result of 'bridging through ten'.



All of these present a very different image of the number system to the number line, which uses a continuous rather than discrete model of number. An approach using this model was advocated by Russians such as Davydov (Schmittau & Morris, 2004).

Kaput (1987:23) identifies five elements in representation:

- the represented world,
- the representing world,
- what aspects of the represented world are being represented,
- what aspects of the representing world are doing the representing and
- the correspondence between them.

For instance, with Cuisenaire, numbers are represented as wooden rods, emphasizing increasing in ones, shown by 1cm lengths and colours. The increasing lengths correspond quite closely to the numerical values, but the colours are arbitrarily linked to factor families. The lack of marked divisions on the rods, in contrast to Stern's rods, helps to focus attention on the proportional relationships between the rods. However, children may not focus on the numerical relations intended by representations: for instance, they may associate numbers with colours and not register the relations between the lengths or shapes. Or they may see numbers as collections of ones, not as whole entities, raising the issue of whether children need to have already formed abstract number concepts, in order to see rods as symbols for them.

How do children make sense of the manipulatives?

We do not know how children make sense of manipulatives and Boulton–Lewis' (1998) analysis of mapping representations highlights a dominant theme of cognitive theories which problematised manipulatives. It seems that by the 1980s structured manipulatives were being used in a directed way to teach calculations, or as self explanatory, rather than with the original approach of guided discovery of mathematical relations. The pedagogy of structured materials was apparently not disseminated in the latter years of the 20th century and Gattegno's (1954) and Dienes' (1960) focus on algebraic relations was no longer a priority. Ma (2015) suggests that US elementary school mathematics focuses on learning to calculate, in contrast to Chinese mathematics education which encourages exploration of relationships between operations.

Some theorists identified several levels of mapping or translation that needed to take place in order for structured materials to help children understand symbolic arithmetic. Hiebert and Carpenter (1992) identified the difference between internal and external representation. Thinking requires internal representations, including images and language: 'To think about mathematical ideas, we need to represent them internally, in a way that allows the mind to operate on them' (1992:66). Communication requires external representations, including spoken language, symbols, pictures and objects.

The demand of understanding manipulatives used to represent dynamic operations is another issue. Dufour-Janvier et al. (1987: 119) pointed out that when children manipulate rods, we cannot be sure how they remember these actions and whether, as Piaget proposed, they subsequently imagine hypothetical transformations or reversals. It has been noted that criticisms of manipulatives often focus on Dienes apparatus being used to model two or three digit subtraction involving decomposition, which requires several fleeting transformations. It is also notable that subtraction using Cuisenaire is introduced as a static 'difference' relationship, rather than dynamic 'taking away', with the advantage that the numbers in the problem remain visible for comparison (Gattegno, 1954). Similarly, other activities suggested by Gattegno and Goutard (1964) involve grouping equivalent fractions or making number patterns which can be analysed at leisure, rather than disappearing in a transformation. Criticisms of manipulatives do not seem to include these more contemplative algebraic activities. Goutard's (1964) practice implies that fluency in calculation followed activities focusing on equivalence, and would be done mentally or symbolically. It may be that manipulatives are easier to understand when used to represent static patterns, enabling them to be discussed in a more contemplative way.

Another issue with symbolic representations is that they have operating conventions or 'syntax' (Vergnaud, 1987). Usually children have to learn the teacher's ways of manipulating and describing equipment. According to Dufour-Janvier et al (1987:118), the child is forced to 'learn' the representation that is submitted to him: the rules of usage, the conventions, the symbols and the language linked to the representation. Similarly, teachers, schools and advisors may also develop their own cultures of procedures for using manipulatives, with associated language and recording. All of this points to the complexities of an additional learning load which manipulatives can present for children. Teachers therefore need to be aware of the problematic relationship between manipulatives and mathematical ideas, and the conventions involved in using them.

Are some representations too opaque or complex for some children?

There may be developmental constraints with some particular models in terms of their complexity. There is neuroscientific evidence that working memory, which processes complex information, expands from the age of six (Siegel and Ryan, 1989). Boulton-Lewis (1998) pointed out that children could only deal with system mapping (ie relations between 3 elements) from the age of five. Resnick (1983) proposed that children first constructed a number line image of number and that a part-whole model was a later development. This view may be supported by the argument of some neuroscientists that a number line is an intuitive image (e.g. Dehaene, 2001).

Fuson (2009) argues that the continuous number model of the number line is too difficult for young children who want to count discrete items. If children do not have concepts of numbers as whole entities, then holding in mind the whole and parts of a number simultaneously would seem impossible for young children, preventing learning number combinations in this way.

Barmby, Bolden, Raine, and Thompson (2011) found that less than half of the eight and nine–year olds in their study could interpret representations of multiplication, including equal groups, the array and the number line. This may be because multiplicative thinking involves more complex part-whole relationships, such as envisaging numbers of numbers. Barmby et al. (2011) also found that some older primary children could not access the array, despite considerable exposure: this might be due to the cognitive demands of the array as simultaneously presenting numbers as groups in two directions. It therefore seems that some models may be too complex for some, not necessarily younger, children to process and understand.

Some children may lack the necessary concepts to understand a model, according to Cobb (1995). Despite a social constructivist view of learning, he argued that learning in a socially supportive situation, with one child explaining to another, may not work if a child has not constructed concepts to give meaning to the words used, despite being supported by demonstrations. He gives the example of John who could not add 'a ten' on a hundred square using the interval between numbers in a vertical column, because 'a ten' as an abstract composite unit simply 'did not exist for him' (1995:377). However, in some examples (eg Boulton Lewis, 1998) where children are described as having difficulty, it seems not just that they lack prerequisite understanding for the activity, but that they lack much more basic number concepts. So for instance, Cobb's John was being asked to add using place value: not only did he not understand place value, as he did not see 'ten' as a unit he seemed unlikely to understand addition as a part-whole relationship. This points to the need not only to analyse the complexity of models, but also the relationships and operations they are modeling, in order to identify prerequisite concepts and assess whether children understand these.

How do children move from understanding mathematical relationships in concrete to abstract form?

Another major issue about mathematical representations is even more problematic: according to Harries and Barmby (2006), we still 'do not know how children construct mental representations' (2006:27). It may be added, 'or whether they do at all': Bills (2000) found that children seldom used any mental visual images when calculating. Holmes and Adams (2006) found that younger children relied on visuo-spatial memory for mathematics tasks, with a shift to verbal recall between the ages of 7 and 9, suggesting the importance of manipulatives and images for younger children. While two year olds can symbolise with language, images and pretend play (Goswami and Bryant, 2007), understanding the symbolic function of manipulatives can be problematic for younger children. Uttal et al. (1997) found that very young children had difficulty interpreting models as representing something else, for instance in relating a scale model to a room. They needed to identify which features were being represented by which objects, which was difficult if these were complex. They therefore suggested that:

> 'simpler, less inherently interesting objects would be more useful as symbols than more complex, interesting objects'.(Uttal et al., 1997:52)

Symbolic understanding also depends on the child's interpretation of the situation: 'young children want to find the characteristics in a representation that they perceive as essential to the situation studied …' according to Dufour-Janvier et al. (1987:118). The complexity of both mathematical relationships and representations may make the identification of similarities more difficult. Uttal et al. (1997) concluded that manipulatives needed to be carefully selected, with strong similarity to the mathematics concepts being taught. They recommended that teachers explain the symbolic meanings and ways of using materials, rather than assume they were obvious to children:

> 'Children's use of manipulatives should be improved by explicit instructions and reminders of the representational nature of those objects.'
> (Uttall et al., 1997:16)

They also speculated that manipulatives which were only used for mathematics had the advantage of creating an expectation that they were being used to introduce a mathematics concept or symbol.

If children see mathematical relationships in materials, they may represent these in several different modes, in more complex ways than suggested by Bruner's

(1964) enactive, ikonic and symbolic sequence. Goldin (2002:211) identified different internal systems of representation, including:

- language or verbal-syntactic
- imagistic, including visual-spatial, tactile-kinaesthetic and auditory–rhythmic
- written symbols or formal notational
- affective, including stable beliefs and attitudes and changing states during learning and problem solving.

It seems that children may use none, one or all of these when thinking: for instance, Bills (2000) found that children commonly either imagined nothing or rehearsed verbal phrases mentally when calculating. Children may use several modes simultaneously: for instance with sharing problems, they might mentally rehearse dealing one-to-one as a rhythmic action, which they may associate with as pleasurable action, but not necessarily connect with division language or signs. Affective representations could include warm memories of successfully using certain materials, thus creating positive expectations. Goldin suggests that familiar contexts may produce *comfortable affect:* hence using RME stories or familiar manipulatives might mitigate the challenging emotions of problem solving. However, Goldin (2002*)* pointed out that while familiar contexts may be more memorable because they are *multiply encoded,* this may also create confusion for organization and retrieval from memory. It seems likely that cognitive analyses of mathematical challenges may not have sufficiently considered the complexities of different modes of representation, including the role of the affective.

How individual children create mental representations is difficult to ascertain, particularly if they are not able to articulate them. Asking children to draw or record in their own way, as Dienes (1960) suggested, is an alternative way of accessing children's mental representations. Dufour-Janvier et al. (1987: 119), proposed that children should construct their own representations, instead of using those 'imposed from the outside', arguing from a constructivist viewpoint. Hiebert and Carpenter (1992) suggested that children's own representations were important for assessing understanding: 'the way in which a student deals with or generates an external representation reveals something of how the student has represented that information internally' (Hiebert & Carpenter, 1992:66).

However, representing transformative operations is particularly difficult, as Dufour-Janvier et al. (1987) also pointed out. They found that young children tended to use written language instead of drawing or diagrams for operations, even though they found writing difficult, because they lacked knowledge of the 'graphic codes and symbols' needed to 'capture actions, transformations, and relations' (1987:121). It is sometimes assumed that an iconic mode of representation involves drawing apparatus: however, a static drawing is usually inadequate to capture the actions and changes involved, for instance, in 'taking away'. Young children's representations of operations sometimes reveal creativity in their use of devices such as hands and arrows to indicate movement (Hughes, 1986; Womack, 2000; Worthington and Carruthers, 2003; Davenall, 2015): it seems some children reject conventional plus and minus signs as not dynamic enough. This suggests that children's representations on paper will depend on their graphical repertoire and only provide partial information about their thinking. One advantage of manipulatives is that they can be moved to represent active operations, and children can use them to support the articulation of their reasoning, for instance about inverse operations and reversals.

Analysing children's representations can give insights into individuals' recognition of mathematical relationships. Thomas, Mulligan and Goldin (2002) found that children varied in the degree of structure inherent in mental images they reported as representing100. These were classified into 3 stages, according to the degree of structure and understanding revealed.

1. inventive/semiotic stage, characterised by meaning making, often pictorial and idiosyncratic, eg a dinosaur with 100 (ie a big number) on its back, a truck that can 'do a 100'
2. extended stage of structural development, connected to a known system, such as counting words, eg pictures of groups of 10s, numbers arranged in different ways, in lines or as spirals, not necessarily regularly, but attempting some pattern or structure
3. autonomous stage, where the new system of representation can function flexibly in new contexts, independently, such as inventive ways of grouping numerals, or showing the ten tens structure.

Some children's images seemed very inventive eg involving flashing lights or spirals. Thomas et al. (2002) proposed that children should acquire a repertoire of images for flexible use:

> 'A child will benefit from having available a variety of images for use in mathematical representation, so that salient features of particular imagistic representations can be drawn on in a variety of situations, and flexibility of thought developed.' (2002:130)

They also argued that teachers should encourage children's own representations: when the child's imagery is valued, positive affect also develops in connection with mathematics.

Mulligan and Mitchelmore (2009) subsequently proposed that children vary in their Awareness of Mathematical Pattern and Structure (AMPS), a general cognitive skill of recognizing key relationships in mathematical patterns. Assessment tasks required children of 5 to 6 years to visualize and draw patterns or structured images, for instance asking children to draw a triangular arrangement of six dots from memory, complete an incomplete 3 x 4 grid or draw 8 o'clock on an empty clock face. Children's levels of awareness were consistent across different tasks and correlated with their 'maths ability', according to their teachers and the test used. Mulligan and Mitchelmore (2009) found that low achievers produced poorly organized, pictorial, idiosyncratic drawings of images, focusing on non-mathematical surface features. Following Thomas et al. (2002), they devised a range of tests and categorized children's responses into four AMPS stages:

(i) Pre-structural

(ii) Emergent (inventive-semiotic)
(iii) Partial-structural
(iv) Structural.

These stages cover pictorial, ikonic and symbolic characteristics of children's representations, applicable to a range of concepts, thus emphasizing the connections between spatial and symbolic pattern recognition. In the following year, the Structural group made faster progress (to a fifth Advanced Structural stage) whereas many of the Pre-structural group continued to focus on irrelevant features. They suggest that this is because recognising structures then make mathematical properties easier to learn, especially with equal groups and unitising. They found that AMPS could be taught, even pre-school, thereby increasing achievement and supporting progress. Papic, Mulligan and Mitchelmore (2011) found that under fives also showed consistent AMPs levels which increased as the result of intervention. According to Mulligan & Woolcott, (2015), characteristics of AMPs include awareness of different structural groupings, namely sequences, structured counting, shape and alignment, equal spacing and partitioning and their recently developed teaching assessments and programmes focus on these. This approach supports a more spatial approach to number, which seems appropriate for young children, building on their visual memory skills. It involves manipulatives arranged in different spatial patterns, fostering key arithmetical skills such as subitising and unitizing. It also links to pre-algebraic thinking and to experiences of early measuring, suggesting a more connected approach to early mathematics. While younger children are encouraged to recreate patterns with manipulatives, the focus on children's drawings supports visualization and abstraction processes, as recommended by the literature.

This research therefore suggests that we still have much to learn from children about how they make sense of manipulatives, that we need to observe and listen to their discussions, as well as giving them opportunities to record in their own way from an extensive graphical repertoire. This stage can illuminate the process of how children come to understand written symbols and work with mathematics expressed symbolically.

This raises the issue of how children interpret written symbols and develop the abstract concepts they represent, or how children can progress from image-supported to image-independent representations (Goldin, 1998:301). Gravemeijer (1991) describes this as the problem of the transition from thinking about material to thinking in terms of mathematical relationships and concepts. However, according to Hiebert and Carpenter (1992), this is a process of connecting up images, rather than replacing them: 'It is useful to think of students' knowledge of mathematics as internal networks of representations' (Hiebert and Carpenter,1992:69). This view is supported by Goswami and Bryant (2007) in their summary of neuroscientific research: conceptual understanding in the brain consists of the networking of multiple representations from different sources, verbal, visual, auditory and kinaesthetic. They refute Piagetian theory (1952) that thinking develops in stages of increasing rationality and that 'sensory-motor' representations are replaced by symbolic ones. While agreeing that 'physical interaction with the world is a critical part of knowledge construction, 'enactive' representations are augmented by knowledge gained through action, language, pretend play and teaching' (Goswami and Bryant, 2007:7).

According to Hiebert and Carpenter (1992: 69) 'understanding grows as networks become larger and more organized'. In this view, understanding is defined as a representational network and rather than talking about a child acquiring an abstract concept, we should consider whether a symbol 'calls up' a meaningful network of associations. For instance, we may expect the symbol '9' to be quickly associated with the 'nineness of nine', but also to trigger a network of meanings, such as one less than 10, a square number or a factor of 360, all of which can be accessed flexibly according to the context.

The main issue is how the various inputs are integrated and organized, or 'the way an individual's internal representations are structured' according to Hiebert and Carpenter *(*1992:66). Vergnaud (1987) emphasised symbolic representation as involving generalizing concepts through several cognitive processes. Drawing on the semiotic terminology of 'referent', 'signified' and 'signifier', he proposed different mental interactions between real world experience (the referent) and the generalized concept (the signified). The latter was constructed by a process whereby 'invariants are recognized, inferences drawn, actions generated and predictions made' (Vergnaud, 1987:229). However, he emphasized that generalizing concepts in this way is a continuous process of refinement, of 'theorems-in-action'*:* for instance initial concepts of addition and subtraction are different from later concepts which include directed numbers. Networks are therefore dynamic not static, with more sophisticated and complex conceptualisations resulting from a greater diversity of linked examples.

However, some theorists have questioned whether all children learn by intuitively abstracting similarities from different examples. For instance, Dufour-Janvier et al. (1987) were dubious about children identifying commonalities from multiple representations. They found that some children happily accepted different answers for the same calculation using written methods and an abacus, even though they knew it was the same problem. Similarly Nunes, Carraher and Schliemann (1993) found that children solving real life problems did not relate these to the same problems presented as written algorithms. This is the issue of situated cognition, that people do not readily relate understanding in one situation to another, and may construct more sophisticated concepts in one context. For instance, Cobb (1995) found that John understood the abstract concept of 'a ten' as money, but did not relate this to 'a ten' on a hundred square. Similarly with calculating strategies, 'young children solve addition and subtraction word problems using counting strategies that mirror the semantic structure of the problems' (Hiebert and Carpenter, 1992:68). This limited understanding may prevent children from generalizing strategies: for instance, if they have difficulty seeing models of difference and take away as both being subtraction, they may not relate 'counting up' to a 'take away' problem.

Dufour-Janvier et al. (1987) also found that children representing a problem did not select the most appropriate diagram (set or number line), but the most familiar, suggesting that the process of analysing a problem and identifying the key relationships, then selecting an appropriate representation, is very sophisticated and requires complex organisation and networking of representations. Threlfall (2000) questioned whether any such process of conscious selection is involved with mental calculation strategies, suggesting that a 'clicking' happens as connections are made subconsciously: the awareness of flexible possibilities from a bank of number facts results in connections being made. The implication is that it may take children considerable time to develop and explicitly use processes of analysis: developing a bank of representations and a flexible approach to equivalences may be more productive.

Utall et al. (1997), like proponents of Cuisenaire, suggested that using one kind of manipulative might be more effective than multiple embodiments, citing meta-reviews endorsing consistency of use, including Japanese practice. They argued that surface features were likely to be less distracting, with the children's focus clearly on mathematical relationships. Goutard's practice suggests that using one kind of manipulative may indeed assist abstract thinking, because children generalise relationships from many numerical examples made with the rods, through talking about these. Then putting the rods away and writing freely encourages them to generalise both to using larger or more complex number forms and also to using abstract written symbols. This suggests that an explicit move to abstraction without manipulatives may help this process, especially if children are free to make their own connections.

Instead of abstracting concepts from multiple embodiments, Vergnaud argued for the importance of translating between different symbol systems:

> 'It is because symbolic representation is transparent in different ways (also opaque for some properties.. that are not represented) that it is fruitful to use different symbol systems.' (Vergnaud, 1987:232)

Hiebert & Carpenter agreed that understanding is helped by making links between different representations, including spoken language, symbols, pictures and objects: 'Connections between internal representations can be stimulated by building connections between corresponding external representations' (Hiebert & Carpenter, 1992: 66). They argue that because 'each representation of a quantity or relationship captures some of its features, but not others', teachers need to focus attention on key aspects through discussion (Hiebert & Carpenter, 1992:71).

The consideration of communication, public symbol systems and discussion raises the issue of social aspects of learning. Cognitive theoretical perspectives tended to see representation and understanding as individual internal processes of knowledge construction, whereas socio-cultural perspectives, following Vygotsky (1978) saw learning as a social process. As Cobb (1990:212) pointed out, '..opportunities for individual children to construct mathematical knowledge arise in the course of their classroom social interactions'. The implication is that the nature of these opportunities and interactions may be determined by the relationships in the classroom created by the teacher. 'The social context in which materials are used may account in part for their effectiveness (or ineffectiveness) in helping students understand' (Hiebert & Carpenter (1992:72). From a socio-cultural perspective, Yackel and Cobb (1996) argued that learning might not be just about internal individual cognitive development, but about participation in social practices with 'taken for granted' conventions and about being accepted within the social group.

From a social-constructivist viewpoint, for instance, children construct their own non-standard algorithms, constrained by activities, manipulatives available, teachers' interventions, requirements to explain and attempts to understand interpretations and solutions of others. According to Cobb (1995) social situations, such as the example of John with another child who explained a representation, only facilitates learning within limits: the child still has to construct meaningful concepts for themselves. He found that some children would learn by copying another's actions, or like, John, despite understanding 'a ten ' as a coin, would later revert to counting by ones. John's learning to add 10 is seen as a process of cognitive construction which may be partly socially mediated by peer interaction and linked to the familiar representation of coins, but would be more helpfully supported by the image of a composite unit of 10 fingers or by a Dienes' ten rod, than the 100 square. This points to the role of the teacher in selecting images and manipulatives to suit children's levels of understanding as well as in organizing peer discussion.

From a social constructivist perspective, Hiebert and Carpenter (1992) also proposed a role for concrete materials in providing a shared focus for discussion:

> 'By interacting with the materials, and with others about those materials, students are more likely to construct the relationships that the teachers intend... This is non-trivial because students' varied backgrounds and goals often make it difficult to share a discussion about a common event or idea...'(Hiebert and Carpenter, 1992: 72).

Vergnaud (1998) expressed this difficulty in terms of the interaction between private meanings (the signified) and public symbol systems with their own syntax (the signifier). Vergnaud emphasized the gaps between thought and language:

> 'This partial mapping or even mismapping makes communication a kind of miracle, at least when individuals produce new ideas.' (Vergnaud,1998: 176)

Boulton-Lewis (1998) recommended that children make mental procedures explicit and 'that children and teachers discuss what they are doing and what they understand by it'. She suggested that 'concrete analogs' help understanding because they 'provide a means of verifying the truth', implying that children might use them to test ideas. For instance, as Goutard (1964) demonstrated, children can argue that different relationships are equivalent by rearranging or exchanging rods or blocks. In this way manipulatives can support mathematical reasoning within social groupings in the classroom, and help children develop organized networks of connected representations which constitute the understanding of abstract concepts.

Mason and Johnston-Wilder's (2006) characterization of the process of abstraction as MGA, (manipulating, getting a sense of and articulating) offers a pedagogy which identifies how learners may progress from manipulating objects to articulating generalisations, using formal and symbolic expressions. However, this assumes that the aim of learning is to understand relationships and to generalise, not just to perform calculations, and the role of the teacher in supporting this progression is key.

It therefore seems that the literature suggests that a process of carefully matching materials and activities to concepts, assessment of children's prerequisite understanding, with explicit reasoned discussion of mathematics ideas and representations should be effective in assisting learning through the multiple encodings that manipulatives have to offer.

E. Implications for teaching: a pedagogy

This literature review suggests that multiple embodiments using manipulatives can help learning, but also points to a key role for the teacher in assessing children's understanding, selecting manipulatives, mitigating difficulties with the conventions of their use and supporting translation between different representations, including verbal and written symbols.

According to recent neurological evidence (Goswami and Bryant, 2007), generalizing is strengthened by repeated experience, and probably also by direct teaching (echoing Uttal et al., 1997):

> ' ..cumulative learning is crucial. There will be stronger representation of what is common across experience ('prototypical') and weaker representation of what differs. It may be that direct teaching of what is intended to be prototypical (for example reminding of the general principles being taught via specific examples) will strengthen learning'. (2007: 4)

According to Hiebert and Carpenter (1992) important ways of connecting representations include identifying similarities and differences, 'Only by thinking and talking about the similarities and differences...can students construct relationships' (Hiebert and Carpenter, 1992:68). Another important kind of connection is 'inclusion and subsumption' (where situations are recognized as belonging to a group). It therefore seems that explicitly identifying these kinds of connections is likely to help children to interpret and organize representations.

Harries, Barmby and Suggate (2008: 172)**,** drawing on Hiebert and Carpenter (1992), propose a 'pedagogy of understanding through representation'. They suggest that connecting different representations also defines mathematical reasoning:

> 'To reason is to make connections between different representations (internal or external) of a mathematical concept, through formal processes (eg logic, proof) or informal process (eg examples).' (2008:164)

In turn this promotes understanding*:*

> …by encouraging children to explore and reason between representations, what we are doing is providing the opportunity for understanding to develop.

They therefore suggest that interrogating representations plays an important role in helping children develop understanding of mathematical concepts.

All of this has important implications for the role of the teacher. Boulton Lewis' (1998) suggestion, that children need to be able to use manipulatives with some automacity before using them to learn new procedures or concepts, implied that teachers need to monitor children's fluency with different manipulatives. Vergnaud (1998) argued that a major role of the teacher is as mediator to help students develop their repertory of schemes and representations. However Baumert et al. (2010) reported that**:**

> 'One of the major findings of qualitative studies on mathematics instruction is that the repertoire of teaching strategies and the pool of alternative mathematical representations and explanations available to teachers in the classroom are largely dependent on the breadth and depth of their conceptual understanding of the subject.' (2010:138).

This means that teachers may need professional development regarding subject knowledge in order to use manipulatives and other representations effectively.

However, Vergnaud (1998) further argued that the selection of activities is the most important factor in teaching:

> 'One must never forget that the most important mediation act of the teacher is to provide students with fruitful situations.'(1998:180)

This implies that teachers' repertoire of problems, statements and questions for children to investigate may be more important than the role of the manipulatives in supporting these enquiries. However, translating between alternative representations may provide such 'fruitful situations'.

It is interesting to consider Goutard's account of her classroom in the light of the issues raised by the literature: her teaching seems to exemplify many of the key elements identified. Children developed familiarity and automacity in using the rods before using them to represent numbers or more advanced concepts. In requiring children to articulate their findings and then record them symbolically, they translated between the different symbol systems of rods, language and mathematical notation. In requiring them to put away the rods before writing freely, she encouraged them to think abstractly and to generalize relationships, via language and symbols, to the number system beyond that shown by the rods, thereby overcoming the key problem of symbolic abstraction identified by other theorists. However, as with RME, children were also encouraged to move between the concrete and abstract, since they dealt with both in each lesson. From a socio-cultural perspective, Goutard also prioritized supporting children by communicating high expectations and creating identities for the children where they had positive self–images as mathematics learners, as well as in creating low-risk class discussions where all children's contributions were welcomed, taken seriously and explored. However, her choice of challenges, which required children to find equivalences and to use the rods to justify these, provided open-ended and inclusive investigations which nevertheless led to the identification and generalisation of key mathematical principles and relationships. These produced an impressive degree of flexibility and fluency in arithmetic with children as young as seven years old, thereby exemplifying Vergnaud's point about the importance of the choice of 'fruitful situations'.

The idea of equivalence as a focus for investigation and open-ended challenges therefore seems important (suggesting a re-consideration of classic resources, such as Quadling, who advocated this approach in 1969). Similarly, the current focus on variation theory offers an approach where activities provide examples for considered analysis, rather than quick computation or practising a modeled method (Marton & Tsui, 2004). Similar approaches are exemplified by Foster's (2014) mathematical etudes at secondary level and in Ainsworth's (2013) primary practice.

Ainsworth (2013) is a rare proponent of Goutard's approach in England. She has employed the same focus on equivalence, the phases of exploration, 'do, talk and record' and free writing, achieving outstanding success (although perhaps not to the same startling level with seven year olds, and in a school with a stable and not-deprived intake). It may be that with this approach, the teacher, rather than the materials, makes the key difference in children's attainment, since Ainsworth is also highly knowledgeable mathematically and immersed in teaching this approach. Furthermore, the children have been consistently taught using this approach throughout primary school, which presents challenges with children joining the school. Nevertheless, using Cuisenaire rods with this approach seems to offer potential strengths besides arithmetic, such as creative mathematical reasoning, abstract thinking and notation, combined with positive attitudes, including humour and delight. The children's creativity and willingness to take an idea to the limits, or test it out, is also similar to that of the children in the CAN project in the 1980s (Shuard, Walsh, Goodwin and Worcester, 1991). They would also try out ideas with large and negative numbers, enjoying patterns, and were encouraged to record in their own ways, which they did creatively. It may be that the pedagogy of enquiry, for instance by using 'how many ways?' activities, is the crucial element: calculators may be similar to colour rods as an engaging resource for generating impressive patterns. What this current practice does demonstrate is the possibility, that, as with Goutard's pupils, children can 'develop ...the expectation that numbers can amuse, delight, illuminate, inform, and excite'. (*Te Whariki,* New Zealand Ministry of Education, 2009)

F. Summary

Resnick and Ford (1981:126) identified the key issue with structure-oriented materials as depending on our ability to 'define psychologically the mathematical structures'. It seems that this is still problematic: we do not know how mathematical relationships are best represented mentally. Structured materials inevitably partially represent partial views of mathematical structures. Manipulatives also seem historically arbitrary and dependent on cultural practices, even for fingers. This raises the intriguing question of whether there are other manipulatives which could represent and help children understand key mathematics ideas, which have not yet been invented or identified as lacking. The use of manipulatives and images has been subject to fashion and marketing as well as national policies: some have been forgotten and rediscovered like ten-frames, while others which seem potentially useful are not common, such as number staircases or double number lines for ratio and multiplication.
We also still do not know:

- how children mentally represent their experiences with the materials
- how (and whether) all children develop abstract representations of the relationships, and come to understand written symbols.

We do know, however, that there are networks of linked representations in the brain in different modes, including muscle memory, words, rhythms, images and emotions: understanding may be defined in terms of the strength of these networks. There are still questions about:

- what influences the construction of images in different modes e.g.. how arranging blocks is remembered kinaesthetically, visually, verbally and emotionally
- how processes vary for individual children and at different ages
- whether and how some images are too complex or links too obscure for some children.

There are interesting questions about the role of children's own strategies and representations:

- Are there other manipulatives which would build on children's intuitive strategies and ways of children's thinking, if we studied this more closely?
- What do children's own representations tell us, and how can these be supported and developed?

Analysing what children focus on, in their own recording and discussions, might therefore provide clues to the sense they are making of activities with manipulatives.

These uncertainties provide useful avenues for investigation and discussion. In particular, interrogating the partiality of manipulatives seems a potentially powerful pedagogical strategy. For instance, comparing representations of a number using Dienes blocks or a beadstring, focuses attention on the tens structure of the number. Using rods or cubes to demonstrate reasoning about equivalences, by showing how 2:3 is equivalent to 4:6, or how adding the same number to each of a pair of numbers preserves the difference between them, provides a shared focus for discussion and explanation. Whereas it is not clear if multiple embodiments help or hinder abstraction, it seems that comparing what is the same or different about them is an important way of organizing thinking.

Harries et al's (2008) suggestion that connecting representations defines reasoning and understanding is persuasive. This idea is also supported by variation theory, which proposes that presenting a range of images and contexts, as well as varied mathematical aspects of a concept, increases depth of understanding (Marton and Tsui, 2004). It seems therefore that the English national curriculum's requirement for children to 'move fluently between representations '(DfE, 2013) is an important aim for developing flexible understanding.

However, teachers need to be aware that the symbolic meaning of manipulatives and their use may not be obvious to young children, unless they have the appropriate prerequisite understanding as well as explicit explanations. Particularly with younger children, there may be issues with complexity of the objects themselves and the expected focus, with potential overload of developing working memory space. Children will find it easier to learn new ideas if they are already familiar with structured manipulatives and can use them to represent numbers with automaticity. It also seems that teachers need deep understanding of concepts in order to work with a repertoire of alternative representations. Teachers also need to identify prerequisite understandings and skills before introducing an activity: for instance children will not understand two digit subtraction by using blocks if they do not first understand place value and single digit subtraction. The other main issue for teachers is to create the kind of community of learners where fruitful discussion can take place, with positive expectations for all – and where Vergnaud's (1998) communication of individual ideas as 'a kind of miracle' can take place.

To summarise: theories of representation indicate that manipulatives can support mathematics learning:

- Understanding can be defined in terms of mentally linking representations in different modes, including kinaesthetically, visually, verbally and emotionally, and of different levels of abstraction
- Linking external representations can stimulate articulated reasoning which links internal representations
- Manipulatives can be used by children to test ideas and focus explanations and justifications, leading to generalising

There are key benefits to using manipulatives:

- being multimodal, they help build stronger memory networks, including for instance, rhythmic action and positive affect
- they can show transformations, operations and actions, including reversibility, unlike static images
- they are impermanent, so support risk-taking, but can be photographed, videoed or drawn.

Concrete manipulatives therefore offer children a vehicle with which they can make sense of a complex, abstract and symbolic mathematical world.

The teaching implications are significant: the effective use of manipulatives depends on many aspects of pedagogy. Because invented representations are partial, and we know so little about what goes on mentally, the implications for teaching are that we need to allow time for observing and learning, including discussion and explanation by teachers and children. Pedagogically we need to:

- assess children's understanding and developmental constraints
- be aware of the advantages, limitations and conventions associated with manipulatives, and select appropriately
- allow time to develop familiarity through play and instruction
- explain features, purposes and conventions of manipulatives
- allow time to develop automaticity of use through activities
- encourage discussion, reasoning and explanation
- encourage visualization
- encourage generalisation
- encourage symbolization with and without materials
- encourage children's own representations.

Fruitful situations are needed: we need tasks which foster these conditions for learning. It seems that activities which invite children to provide alternatives, to explain equivalences and to compare and contrast can stimulate discussion and linking of representations. These might arise from modeling contextualized problems or involve challenges with materials, such as:

- *Can you find others in this family?* - providing examples which fit criteria or show a relationship (such as fingers showing 6, or pairs of rods where one is double another)
- *How many ways can you do this?*- examining alternative methods for calculating (such as adding 6 and 7 as double 5 and 2 and 1 or double 6 plus 1, shown on the rekenrek)
- *Show me*- different representations (such as adding 10 with Dienes, a beadstring and a calculator)

Class communities are needed which foster exploratory behaviour and talk, within an environment of trust and respect as advocated by Varol and Farran (2006). As they concluded, because research shows that children do not readily understand the connections between manipulatives and the mathematics they represent, discussion is needed to clarify this, which presupposes a classroom environment with safe risk-taking and questions 'that have no incorrect answers'. This may be the greatest pedagogical challenge.

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