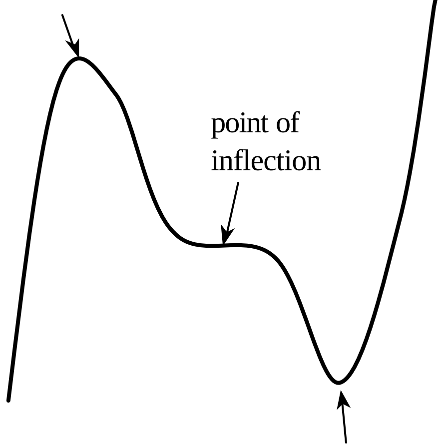
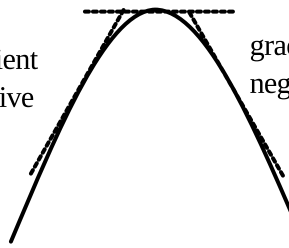
Stationary Points

There are 3 types of stationary points: maximum points, minimum points and points of inflection.



Maximum Points

Consider what happens to the gradient at a **maximum point.** It is positive just before the maximum point, zero at the maximum point, then negative just after the maximum point.

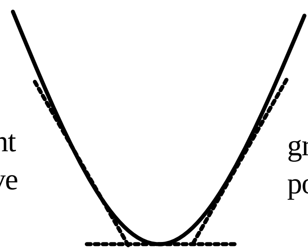


The value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is decreasing so the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x is negative

i.e. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is negative.

Minimum Points

Just before a **minimum point** the gradient is negative, at the minimum the gradient is zero and just after the minimum point it is positive.



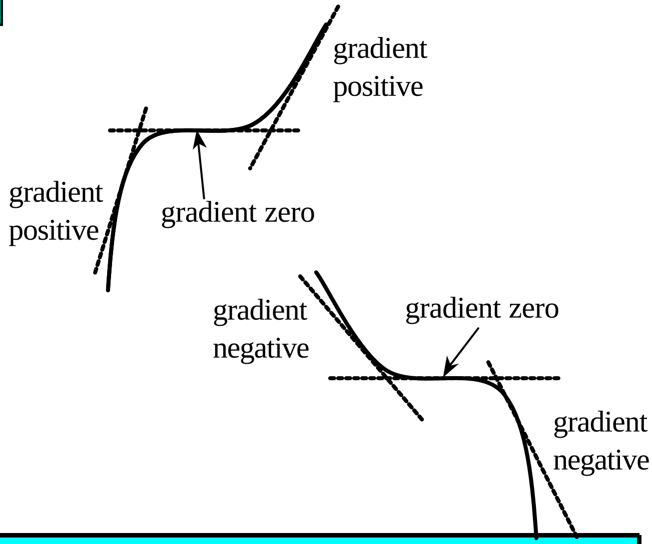
The value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ is increasing so the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x is positive

i.e. $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is positive.

Points of Inflection

At some points $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$.

Two such points are shown in the sketches.
They are called **points of inflection**.



Note that $\frac{d^2y}{dx^2}$ is also zero at some maximum and minimum points.
To find the type of stationary point, consider the gradient at each side of it.

Sketching Curves

Find the stationary point(s):

- Find an expression for $\frac{dy}{dx}$ and put it equal to 0, then solve the resulting equation to find the x co-ordinate(s) of the stationary point(s).
- Find $\frac{d^2y}{dx^2}$ and substitute each value of x to find the kind of stationary point(s).
  (+ suggests a minimum, – a maximum, 0 could be either or a point of inflection)
- Use the curve's equation to find the y co-ordinate(s) of the stationary point(s).

Find the point(s) where the curve meets the axes:

- Substitute $x = 0$ in the curve's equation to find the y co-ordinate of the point where the curve meets the y axis.
- Substitute $y = 0$ in the curve's equation. If possible, solve the equation to find the x co-ordinate(s) of the point(s) where the curve meets the x axis.

Sketch the curve, then use a graphic calculator to check.

Example To sketch $y = 4x - x^2$

$$\frac{dy}{dx} = 4 - 2x \ldots\ldots\ldots(1)$$

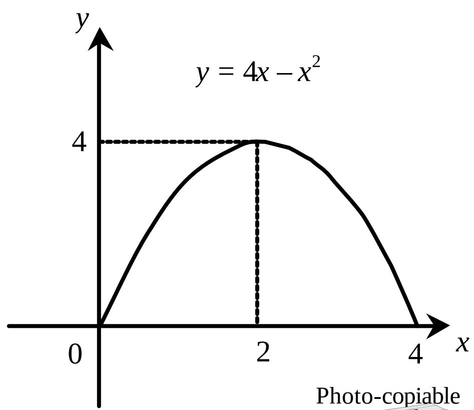
At stationary points $\frac{dy}{dx} = 0$

This gives $2x = 4$ so $x = 2$

From (1) $\frac{d^2y}{dx^2} = -2$ suggesting a maximum.

Substituting $x = 2$ into $y = 4x - x^2$ gives:
$y = 8 - 4 = 4$ so **(2, 4)** is the **maximum point**

When $x = 0$ $y = 0$
When $y = 0$ $4x - x^2 = 0$
$x(4 - x) = 0$
so $x = 0$ or 4
Curve **crosses the axes** at **(0, 0)** and **(4, 0)**



Example To sketch $y = 2 + 3x^2 - x^3$

$$\frac{dy}{dx} = 6x - 3x^2 \quad \ldots\ldots.(1)$$

At stationary points $\frac{dy}{dx} = 0$

This means $6x - 3x^2 = 0$

Factorising gives $3x(2 - x) = 0$
with solutions $x = 0$ or $x = 2$

From (1) $\frac{d^2y}{dx^2} = 6 - 6x$ which is positive when $x = 0$ and negative when $x = 2$.

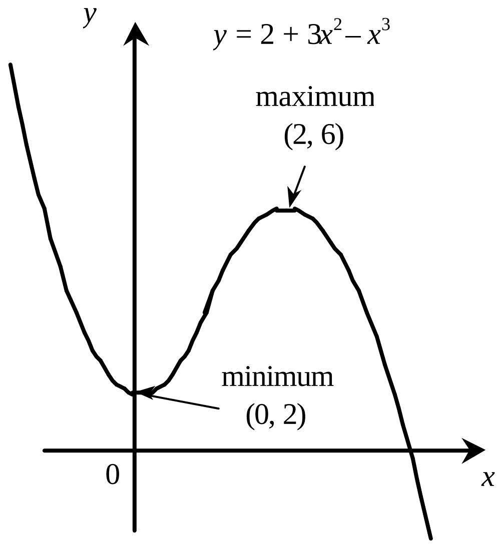
Substituting the values of x into $y = 2 + 3x^2 - x^3$:

$x = 0$ gives $y = 2$ and $x = 2$ gives $y = 6$

(0, 2) is a **minimum point**
and **(2, 6)** is a **maximum point**

When $x = 0$ $y = 2$
When $y = 0$ $2 + 3x^2 - x^3 = 0$

Solving such cubic equations is difficult and not necessary as it is possible to sketch the curve using just the stationary points and the fact that it **crosses the *y* axis** at **(0, 2).**



Note
Use a graphic calculator to check the sketch. If you wish, you can use the trace function to find the x co-ordinate of the point where the curve crosses the x axis. In this case the curve crosses the x axis at approximately (3.2, 0).

Example To sketch $y = x^4 - 4$

$$\frac{dy}{dx} = 4x^3 \ldots\ldots\ldots(1)$$

At stationary points $\frac{dy}{dx} = 0$

This gives $4x^3 = 0$ so $x = 0$ and $y = -4$

From (1) $\frac{d^2y}{dx^2} = 12x^2 = 0$ when $x = 0$

In this case the stationary point could be a maximum, minimum or point of inflection.
To find out which, consider the gradient before and after $x = 0$.

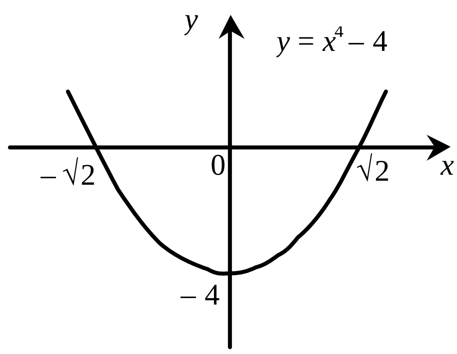
When x is negative $\frac{dy}{dx} = 4x^3$ is negative

When x is positive $\frac{dy}{dx}$ is positive

so **(0, – 4)** is a **minimum point**.

When $x = 0$ $y = -4$
When $y = 0$ $x^4 - 4 = 0$
$x^4 = 4$
so $x^2 = 2$ and $x = \pm\sqrt{2}$

Curve crosses the axes at **(0, – 4)** ,
(– Ö2, 0) and **(Ö2, 0)**



Try these:

For each of the curves whose equations are given below:

- ***find each stationary point and what type it is;***
- ***find the co-ordinates of the point(s) where the curve meets the x and y axes;***
- ***sketch the curve;***
- ***check by sketching the curve on your graphic calculator.***

1	$y = x^2 - 4x$	2	$y = x^2 - 6x + 5$
3	$y = x^2 + 2x - 8$	4	$y = 16 - x^2$
5	$y = 6x - x^2$	6	$y = 1 - x - 2x^2$
7	$y = x^3 - 3x^2$	8	$y = 16 - x^4$
9	$y = x^3 - 3x$	10	$y = x^3 + 1$

For each of the curves whose equations are given below:

- ***find each stationary point and what type it is;***
- ***find the co-ordinates of the point where the curve meets the y axis;***
- ***sketch the curve;***
- ***check by sketching the curve on your graphic calculator.***

11	$y = x^3 + 3x^2 - 9x + 6$	12	$y = 2x^3 - 3x^2 - 12x + 4$
13	$y = x^3 - 3x - 5$	14	$y = 60x + 3x^2 - 4x^3$
15	$y = x^4 - 2x^2 + 3$	16	$y = 3 + 4x - x^4$

Teacher Notes

Unit Advanced Level, Modelling with calculus

Skills used in this activity:
- finding mamimum and minimum points
- sketching curves.

Preparation
Students need to be able to:
- differentiate polynomials;
- solve linear and quadratic equations;
- sketch curves on a graphic calculator.

Notes on Activity
A Powerpoint presentation with the same name includes the same examples and can be used to introduce this topic.

Answers

1 Minimum at (2, – 4), meets axes at (0, 0), (4, 0)
2 Minimum at (3, – 4), meets axes at (0, 5), (1, 0), (5, 0)
3 Minimum at (– 1, – 9), meets axes at (0, – 8), (– 4, 0), (2, 0)
4 Maximum at (0, 16), meets axes at (0, 16), (– 4, 0), (4, 0)
5 Maximum at (3, 9), meets axes at (0, 0), (6, 0)
6 Maximum at (– 0.25, 1.125), meets axes at (0, 1), (– 1, 0), (0.5, 0)
7 Minimum (2, – 4), maximum at (0, 0), meets axes at (0, 0), (3, 0)
8 Maximum at (0, 16), meets axes at (0, 16), (– 2, 0), (2, 0)
9 Minimum (1, – 2), maximum at (– 1, 2), meets axes at (0, 0), ($-\sqrt{3}$, 0), ($\sqrt{3}$, 0)
10 Point of inflection at (0, 1), meets axes at (– 1, 0), (0, 1)
11 Minimum at (1, 1), maximum at (– 3, 33), meets axis at (0, 6)
12 Minimum at (2, – 16), maximum at (– 1, 11), meets axis at (0, 4)
13 Minimum at (1, – 7), maximum at (– 1, – 3), meets axis at (0, – 5)
14 Minimum at (– 2, – 76), maximum at (2.5, 106.25), meets axis at (0, 0)
15 Maximum at (0, 3), minima at (– 1, 2) and (1, 2), meets axis at (0, 3)
16 Maximum at (1, 6), meets axis at (0, 3)