Durham University

Developing the use of visual representations in the primary classroom

Patrick Barmby, David Bolden, Stephanie Raine & Lynn Thompson

Contents

Acknowledgements

The authors of the report would like to thank the Nuffield Foundation for their funding of this project. The Nuffield Foundation is an endowed charitable trust that aims to improve social well-being in the widest sense. It funds research and innovation in education and social policy and also works to build capacity in education, science and social science research. The Nuffield Foundation has funded this project, but the views expressed are those of the authors and not necessarily those of the Foundation. More information is available at www.nuffieldfoundation.org

Also many thanks to the schools who participated in the project and the mathematics coordinators involved for their enthusiasm and dedication.

1. Introduction

The Nuffield Foundation funded a research project that ran from September 2011 to July 2012. The aim of this project was to involve 8 maths coordinators from the Durham region in professional development sessions looking at developing teachers' use of visual representations for mathematics in the primary classroom. The sessions drew on the research on how we can use visual representations in the classroom, specifically looking at representations of multiplication and fractions.

Coordinators taking part in the project attended three one-day sessions, looking at how the research ideas on representations could be applied in the primary classroom. Coordinators were then asked to work with Year 3 and Year 5 teachers in their schools in order to try and incorporate these ideas into practice. The impact of the project was assessed through pre- and post-tests for pupils on multiplication and fractions, observations of Year 3 and Year 5 lessons, interviews with the Year 3 and Year 5 teachers, and interviews with the maths coordinators involved. In carrying out this project, there was an observed impact on the pedagogy of teachers involved and the mathematical understanding of pupils. The project provided an example for taking the research on representations and applying these ideas in schools, and therefore highlights improvements that we could make in the future.

The findings of the project were disseminated at a one-day conference held at Durham University in July 2012. Coordinators involved in the project were invited to present their experiences of being part of this project at this conference. In addition to the dissemination conference, an outcome of this project is this report detailing the project and its findings. The National Centre for Excellence in Teaching Mathematics (NCETM) has agreed to make this report available through their website and so disseminate the project's findings to teachers throughout the UK.

In this report, we begin by outlining the research on using visual representations of mathematical ideas in the classroom. We then provide the design of the project in terms of the training for teachers and also in terms of measuring the outcomes of the project. The results of the project are then presented, and then the conclusions that are drawn from the project. The final section of this report looks at the results of the external evaluation carried out for the project.

2. The research on visual representations

(a) The importance of visual representations

Research has highlighted the importance of visual representations both for teachers and pupils in their teaching and learning of mathematics. The use of multiple representations in general is an important part of teachers' knowledge of mathematics and they can play an important role in the explanation of mathematical ideas (Leinhardt *et al*., 1991).

> "Skilled teachers have a repertoire of such representations available for use when needed to elaborate their instruction in response to student comments or questions or to provide alternative explanations for students who were unable to follow the initial instruction" (Brophy, 1991, p. 352)

Also, external representations can highlight specific aspects of a mathematical concept (e.g. the array representation illustrating the commutative and distributive nature of multiplication – see below), therefore supporting this process of explanation (Kaput, 1991; Ainsworth, 1999). In addition, the ability to draw on multiple representations is an important aspect of pupils' mathematical understanding (Hiebert & Carpenter, 1992; Greeno & Hall, 1997). Visual representations enable pupils to make connections between their own experience and mathematical concepts (Post & Cramer, 1989), and therefore gain insight into these abstract mathematical ideas (Duval, 1999; Flevares & Perry, 2001).

(b) The possible drawbacks to visual representations

In addition to recognising the benefits of using visual representations, the possible difficulties involved in using these in the classroom must also be acknowledged. In particular, teachers cannot assume that students recognise these representations in the manner expected (Hall, 1998); the meaning that particular representations have for the teacher may be quite different to the meaning they have for the student (Cobb et al., 1992). Therefore, if particular representations are to be used in the classroom, then teachers need to support students in learning to interpret representations (Flevares & Perry, 2001), through providing "effective transitional experiences" (Boulton-Lewis, 1998, p. 222) to support students' progression onto using these different representations.

(c) The empirical evidence for using visual representations

Perhaps due to these possible drawbacks, despite the theoretical importance of visual representations in the teaching and learning of mathematics, the empirical evidence to support the use of these representations in the classroom is mixed and somewhat lacking. Sowell (1989) carried out a study looking at the effectiveness of external representations in mathematics classrooms and concluded that there were no significant benefits associated with using visual representations in comparison to more abstract representations. A more recent meta-analysis by Gersten *et al*. (2009) found more positive results. This study looked specifically at mathematics instruction for students with learning disabilities and found that the use of visual representations significantly benefitted students. Due to the mixed nature of these results and the lack of empirical studies specifically looking at the use of visual representations in the primary school context, the present study therefore contributes to the research on this pedagogical approach to teaching mathematics.

(d) A focus on multiplication and fractions

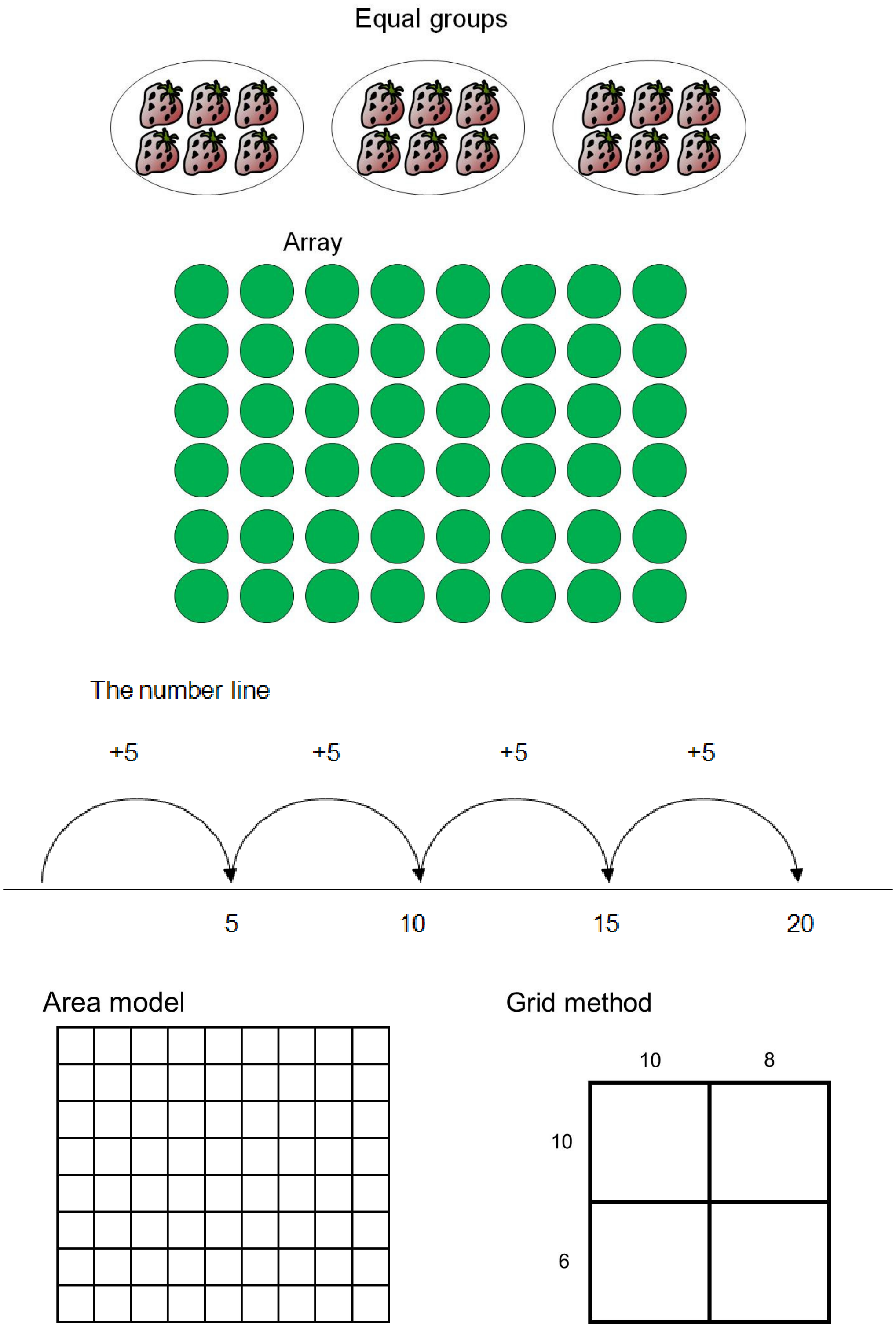
In trying to exemplify the theoretical issues highlighted above, we chose two areas of primary school mathematics to focus upon in the study, namely multiplication and fractions. In coming to understand multiplication, Greer (1992) highlighted the importance of a range of different 'classes of situations'. These included equal groups, equal measures, rate, multiplicative comparison, multiplicative change and Cartesian product situations. These different situations are exemplified below:

> 4 packs of oranges, each with 3 oranges (equal groups)
>
> 3 children each have 4.2 litres of orange juice (equal measures)

> Peter walks at 2 miles per hour. In 1½ hours, he walks 3 miles (rate)
>
> Anne has 3 times more sweets than her younger brother Robert. If Robert has 5 sweets, how many sweets does Anne have? (multiplicative comparison)
>
> The apple tree in our garden has grown to 2.2 times its height that it was five years ago. If the tree was 1.5m tall five years ago, how tall is it now? (multiplicative change)
>
> A crisp manufacturer produces crisps with three different flavourings, and sold in two different sizes of bags. How many types of bags of crisps does the manufacturer produce? (Cartesian product)

The range of situations illustrates the range of possible representations that can constitute our understanding of multiplication (Hiebert & Carpenter, 1992). Closely linked to these different situations, Greer (1992) also highlighted the range of different external diagrammatic representations associated with different contexts of multiplication. For example, equal groups can be represented by diagrams of equal groups of objects and/or arrays respectively. Cartesian product situations can also be represented by arrays. The number line can be used to represent multiplication, particularly in relation to multiplication as repeated addition. Alongside the representations highlighted above, Outhred and Mitchelmore (2004) stated that the rectangular array model (or area representation) is an important model for multiplication. Also, Lampert (1986) highlighted the importance of 'computational knowledge', manipulating numerical symbols often according to procedural rules. This can be done through traditional methods of multiplication or through methods such as the grid method. These different visual representations of multiplication are shown in Figure 1 below.

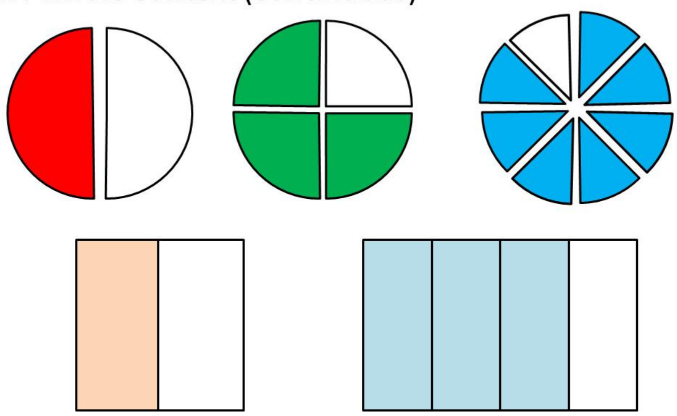
Figure 1: Visual representations of multiplication



Moving on to fractions, Behr et al. (1983) stated that the topic of fractions, or more broadly rational numbers, "involves a rich set of integrated subconstructs and processes" (p.92). Behr et al. (1983) detailed five sub-constructs for the rational number concept: the part-whole construct (both for continuous and discrete quantities) and the closely-associated measure construct; the ratio construct; rational numbers as indicated division or quotient; and rational number as an operator (e.g. as a transforming function). Related to these sub-constructs are the ways in which these constructs can be represented. For example, diagrammatically, the part-whole construct can be represented by a part of a continuous whole (e.g. a part of a shape) or by a group of discrete objects. The measure construct can be represented by the number line. Figure 2 illustrates some of the possible diagrammatic representations for fractions.

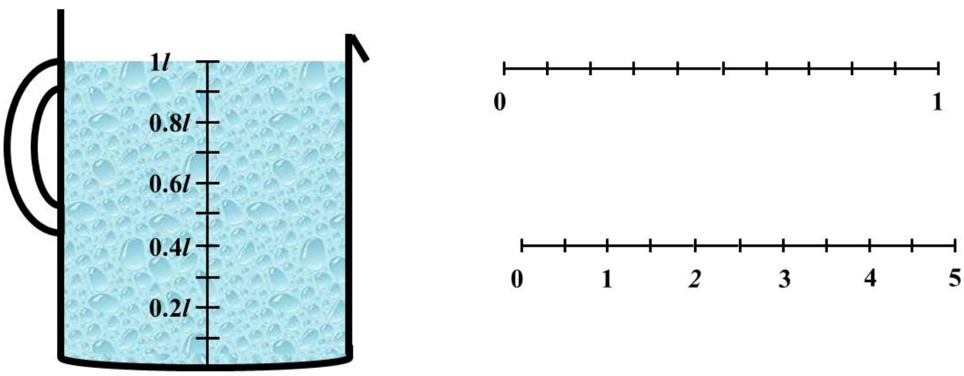
Figure 2: Visual representations of fractions

Part-whole context (continuous)



Part-whole context (discrete)

The measure context



(e) A professional development programme for teachers

Based on the above literature on the use of diagrammatic representations, the research project aimed to develop the use of these representations by primary teachers. This was carried out through a professional development programme for teachers, the design of which was informed by the research literature of teacher professional development, particularly with a focus on mathematics. This literature suggested a number of key characteristics for successful professional development programmes. Firstly, a programme should specify and highlight subject matter and knowledge required by the teachers (Garet et al., 2001; Borko, 2004; Hill & Ball, 2004). Secondly, a programme should highlight how children learn that subject matter (Wilson & Berne, 1999; Franke et al., 2001; Garet et al., 2001; Borko, 2004). Thirdly, during the programme, teachers should be actively engaged in their learning (Desimone et al., 2002), providing opportunities for teachers to work together (Hill & Ball, 2004) with meaningful discussion and planning (Garet et al., 2001), with a "privileging of teachers' interaction with one another" (Wilson & Berne, 1999, p.195). Fourthly, the programme should try and develop a 'community of practice' through professional development that is sustained over time (Garet et al., 2001) and teacher collaborations (Franke et al., 2001).

Therefore, based on the above literature, the professional development programme for the teachers involved in the project was structured as follows:

- Attendance at the first training day at the start of the autumn term, looking at the general research ideas on using diagrammatic representations, and also focussing these ideas on multiplication, with consideration of implications for practice.

- Attendance at the second training day at the start of the spring term, reflecting on the results and teachers' experiences from the first term, and subsequently focussing the research ideas on fractions, with consideration of implications for practice.
- Attendance at the third and final training day at the start of the summer term, reflecting on the results and teachers' experiences from the project in general, and looking at the overall implications for classroom practice and also for future research.

In addition to introducing the teachers to the research on the use of diagrammatic representations, an emphasis during the training days was to encourage discussions between teachers by providing necessary opportunities. This included pairing up teachers as partners for discussion during the training days themselves, and also requesting teachers to meet and discuss their progress in the project outside of the training sessions. In providing the training over three days in three different terms, the project tried to achieve the balance between providing sustained opportunities for professional development, and minimising the impact of taking the teachers out of schools. Also, the design of the training was wary of 'feed-forward' issues (Korthagen & Kessels, 1999), where barriers for teachers in implementing ideas may occur due to a lack of personal concerns from the teachers about the issues at hand. Therefore, the professional development was provided to the mathematics coordinators from the schools involved (not directly to the class teachers), building on their remit to improve the teaching of the subject in their establishments. This did have the drawback that the 'indirect' approach to the transfer of the research ideas may have led to difficulties, however, advantages were also identified through the hoped-for discussions and reflections taking place between coordinators and class teachers.

Having described the relevant research literature associated with the project, the next section will describe the methods used to measure the impact of the project on teachers and pupils.

3. Methods used in the study

The study sought to answer the following research questions:

- Did the programme of professional development for teachers result in greater understanding of multiplication and fractions amongst pupils?
- How did the professional development programme impact on teachers' practice in terms of using diagrammatic representations?

In answering the first research question, an experimental design was employed in order to observe the impact of the professional development programme on pupils. In answering the second research question, a qualitative approach was to be taken.

(a) The experimental design

The quantitative part of the research involved an experimental design using pre- and post-tests with children in Year 3 and Year 5 in the 8 schools of the maths coordinators involved in the project. Year 3 was chosen because of previous research (Barmby et al., 2009) which indicated these pupils' lack of understanding of multiplication representations. Year 5 children were chosen for fractions so that there were no constraints due to End of Key Stage 2 tests. Initially, Year 3 pupils were given a pre-test on multiplication involving multiplication questions requiring explanations for working out. Half of the maths coordinators were then be asked to implement in their Year 3 classes, over a period of up to one term (but determined by the teachers), a teaching programme developed in the first professional development session utilising visual representations of multiplication. At the end of the term, a post-test was carried out in all 8 schools with the Year 3 children. This process was repeated in the second term, with Year 5 pupils and the topic of fractions, with the other half of the maths coordinators implementing a teaching programme in their Year 5 classes. The experimental design can be summarised follows:

Table 1: A summary of the experimental design

Schools	Group	Autumn Term (focus on multiplication/Year 3s)		
A, B, C, D	Treatment 1	Pre-test	Implementation	Post-test
E, F, G, H	Treatment 2	Pre-test	No implementation	Post-test
I, J	Control	Pre-test		Post-test

Schools	Group	Spring Term (focus on fractions/Year 5s)		
A, B, C, D	Treatment 1	Pre-test	No implementation	Post-test
E, F, G, H	Treatment 2	Pre-test	Implementation	Post-test
I, J	Control	Pre-test		Post-test

The methodological design provided a quantitative measure of the impact of the teaching programmes on pupils, with a comparison at each stage of the project between the schools receiving the training, and also with the schools not receiving the training.

The study involved 10 volunteer schools (A to J) in the North East of England, with schools being randomly allocated to the different groups. The sample sizes of children were 109 and 106 for treatment group 1, 121 and 106 for treatment group 2, and 44 both terms for the control group in each of the autumn and spring terms respectively.

The tests used in each term were constructed specifically for the study, with a focus on children's understanding of multiplication and fractions. In each term, the same test was used for the pre- and post-test. In the autumn term, a 19 item multiplication test was constructed and used, based on the research on multiplication previously highlighted. In the spring term, a 39 item fractions test was constructed and used, based on the research on fractions. Both these tests are included in the Appendices to this report

(b) Qualitative methods

In addition to the quantitative approaches used above, semi-structured interviews with coordinators and class teachers were carried out at the start, the middle and end of each implementation to assess their views on their practice, the teaching programmes and their impact on children's learning of mathematics. We incorporated three visits to schools over the duration of each term in order to gain insight into how teachers' practices developed over time. Carrying out three interviews with teachers provided opportunities to track teachers' practice over time. Prior to each interview, an unstructured observation of a Year 3 or Year 5 maths lesson was also carried out to observe how visual representations were being used in the classroom.

4. Results of the study

In order to find out whether the programme of professional development for teachers resulted in greater understanding of multiplication and fractions amongst pupils, the pre- and post-test results in each term was compared across the different groups of schools. Figures 3 and 4 compare the average test score achieved by each group of schools (treatment schools who attended the training and the other two control schools) at the beginning and end of each term. Figure 3 shows the multiplication test scores. Figure 4 shows the fractions test scores.

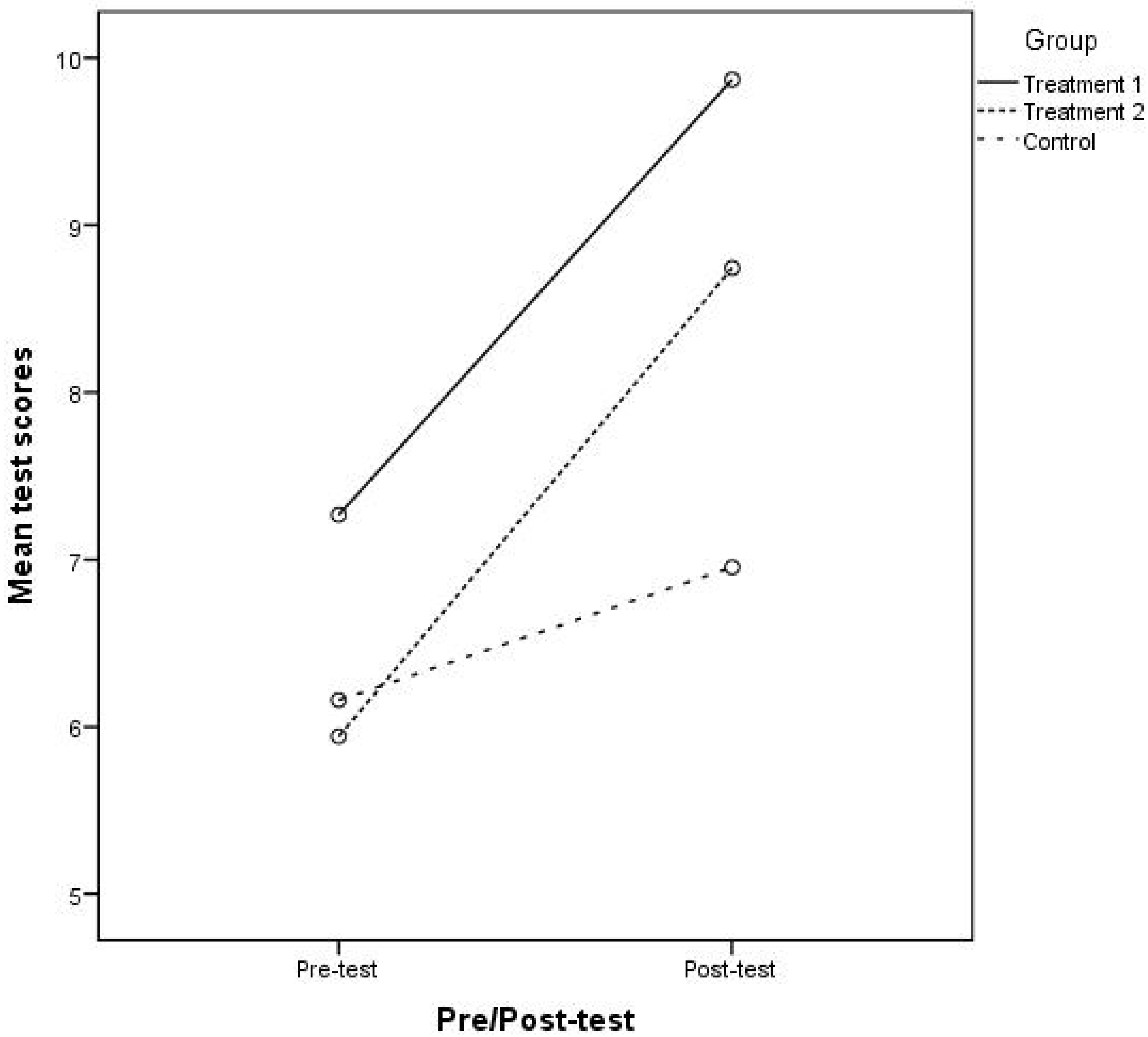


Figure 3: The average pre- and post-test scores for the multiplication test

In Figure 3, we can see that the two groups of schools who attended the training made the same progress in terms of multiplication test scores over the course of the term (shown by the steepness of each graph), even though treatment group 1 were the group of schools who were asked to implement the ideas. However, both treatment groups made greater progress than the control group schools, showing that the training programme had an impact in some way. The differences between the treatment groups and the control group were statistically significant.

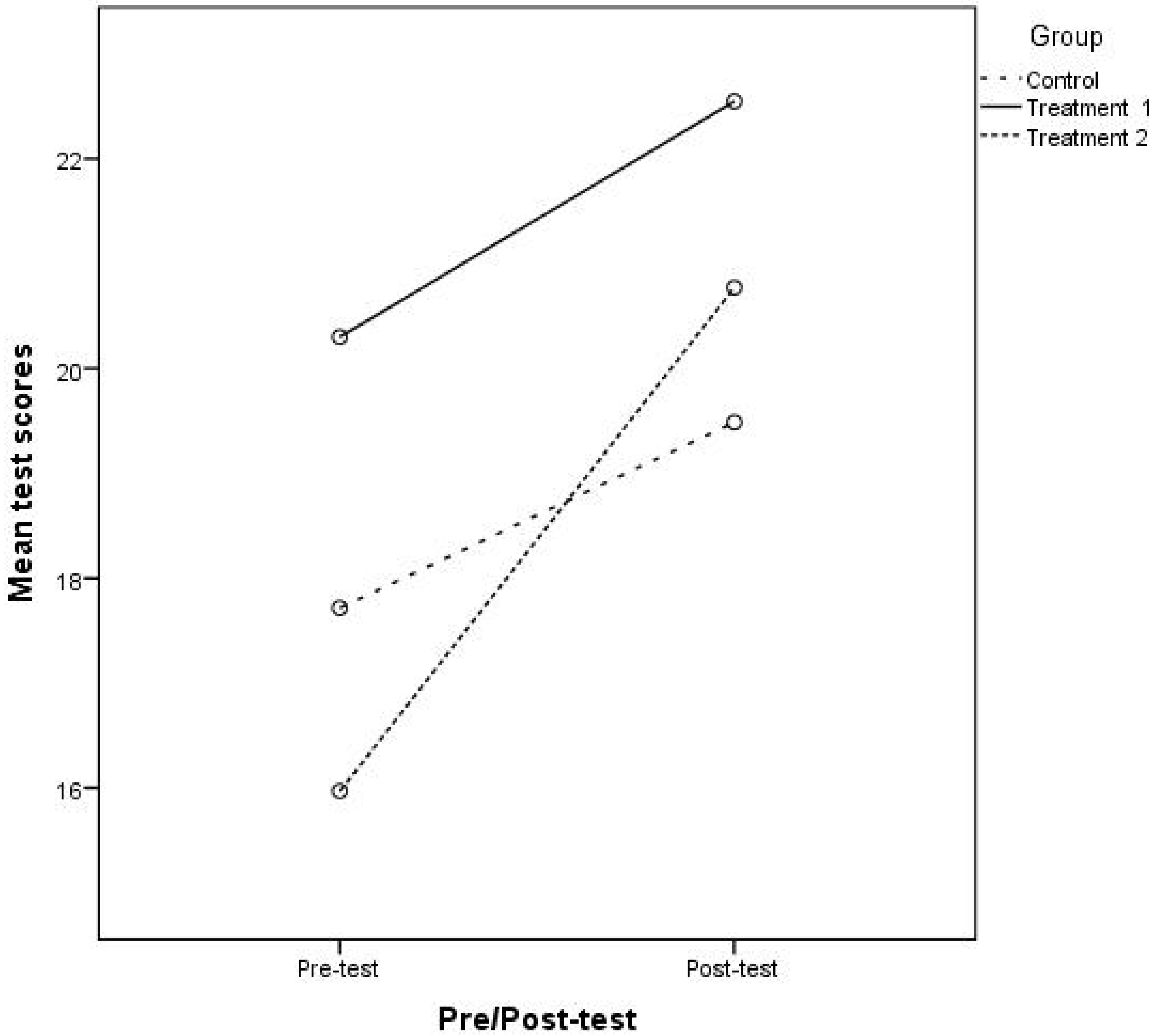


Figure 4: The average pre- and post-test scores for the fractions test

Figure 4 shows that this time, there was a difference in the progress made by the schools asked to implement the fractions ideas, compared to the other treatment group schools who attended the training but were asked not to implement the ideas, and the control group schools. This difference was statistically significant.

Moving on to look at the qualitative data obtained from the interviews with teachers and the lesson observations, the following issues emerged from this data:

- Impact on the pupils;
- Impact on the class teachers;
- Impact on the maths coordinators;
- Difficulties faced by teachers in the project.

Looking first of all at the impact of the project on the pupils, the quantitative data given above showed that there was an impact. This was confirmed by the teachers, who identified children's ability to explain and understand the mathematical ideas, and their increased confidence as a result of the project.

> *They've got a better understanding of what a fraction actually is. If we ask them what a fraction is they'll get the vocabulary and they'll say it has to be equal, its starts with a whole ... And they know the different ways of how to explain the numerator and the denominator and things like that.*

> *They could explain things. Some were still struggling but they could explain things and they could draw pictures for me and they were wanting to draw little diagrams for me.*

> *I think it's given them more confidence if I'm honest. I mean there's one little girl, I don't know if you saw her ... She's a lower ability child who came with no maths confidence at all, and I've tried to do everything very visually for her. Even addition, subtraction, not just*

> *multiplication. And because she sees it visually, it's easier for me to talk it through where she's gone wrong and correct it.*

With regards to the class teachers, some of the teachers particularly in the first term did identify changes to their practice.

> *I mean the way I've taught it has been definitely different. I've not taught it using arrays before so I think that's had an impact on the children but also on me.*
>
> *I'm more aware of the misconceptions children have of multiplication through doing it this way. Because you're not just saying to them "how have you got the answer?" They're actually showing you, "well, this is how I got the answer".*

In the second term however, teachers were more likely to state that "they were doing it anyway" in terms of using visual representations. However, teachers in both terms identified that the project had provided the opportunity to reflect on their practice.

> *It has made me think more about how I would teach it and the different ways to teach it.*
>
> *Seeing the simplest visual method because sometimes it's easy to maybe kind of jump a step and you won't perform the next step. So I suppose it's having to think "right, what's the first stage", and then working through in kind of relevant and logical steps, and progress the steps.*

Likewise for the maths coordinators, they identified that the project had developed their knowledge and practice. This was also enhanced through the discussions they were able to have with other teachers during the training days.

> *For me, yeah, knowledge about the teaching of multiplication … I think we've thought about it in a bit of a deeper kind of way, that perhaps some children do understand but they can't tell you what they're doing.*
>
> *To talk about what works and it was good to get ideas and hear about the research and some of the academic research side of it. But then there was plenty of chance for us to talk about how it's really going to work in the classroom, what's going to fit together ... It's always a nice chance to get together with another group of teachers and have a talk about how things work.*

Therefore, the qualitative data did highlight the impact of the project on pupils and teachers. However, difficulties faced by the teachers were highlighted as well. This included external pressures such as OfSTED, whether the coordinator had time to discuss the project with the class teacher, and the timing of the project in the second term.

> *So she's going to use some arrays and some visualisations to help them with the division side of that. The only thing she said is sometimes the work is a bit messy in the books and thinking about the school side of it, especially because we're due OFSTED this year, it's kind of make sure we've got lots of evidence in books and putting it in. Sometimes it's not the neatest work …*
>
> *Not sitting down and discussing it all, just in little bits ... But that's just the reality as it was. You know (the coordinator) went to the thing, came back and said this is what we're going to do. Gave us the power point and information, the different resources that we got. Some of the online stuff and then get on with it ... I mean to be fair it was probably the timing that its happened as well in that he's just*

had his first baby and everything that's happened with that, and it's just in a primary school it's a timing issue. When do you have time to get together and sit down and talk when you're not running off to football matches and various different things?

I mean part of the problem that I've had would be really that this would have been better if we'd had this in the autumn, because it was in the autumn term that Year 5 did this kind of fractions work.

5. Conclusions drawn from the study

Based on the results of the study highlighted in the previous section, we drew the following conclusions with regards the impact of the project on pupils and teachers, and also the implications for teaching and research that it provided.

(a) Impact on pupils

It was concluded that the training programme for the mathematics coordinators, introducing them to the research on using diagrammatic representations, did have a positive impact on both pupils in the respective schools. Although the nature of the impact was complex (which will be discussed below), we found that the in general, the pupils in the schools taking part in the project progressed significantly in their learning of multiplication and fractions as measured by the quantitative tests. Qualitatively as well, teachers highlighted that diagrammatic representations benefited pupils' understanding, and also the confidence of pupils in mathematics.

In reporting the quantitative findings of the study so far, we acknowledged that the pupils in both the treatment groups of the study in term 1 made the same average test gains. However, in term 2, only treatment group 2 made the greater gains. Therefore, although the study has claimed that the training programme did have a significant impact on pupils' test scores, this further complexity in the impact needs to be explained. The study identified two possible reasons for this complexity. Firstly, in reflecting on the results with the maths coordinators during the second and third training days of the programme, the coordinators identified that after the first training day, some teachers in treatment group 2 returned to school and talked to colleagues about the training and the research project. Furthermore, it would be unusual for a teacher to return to school after any professional development training and not talk to colleagues about the training. Therefore, the design of the research and asking the half of the treatment group teachers to not do anything in the first term was unrealistic, and it was difficult for the treatment group 2 teachers to adhere to their role as control group schools. This problem was not reported by the treatment group 1 schools in the second term which may explain some of the difference.

A second possible reason for this, but also related to the first explanation, was a possible Hawthorne effect which was identified during discussions with the mathematics coordinators. This is a problem that can arise in field experiments and occurs when "… subjects' knowledge that they are in an experiment modifies their behaviour from what it would have been without the knowledge" (Adair 1984, cited in Diaper 1990, 261-262). As has been identified, mathematics coordinators in the non-implementing group in the autumn term acknowledged that they returned to school and talked to colleagues about the research project and the training they had received because it was a 'new' project. This did not seem to occur in the second term because the project was no longer 'new' for the teachers involved.

(b) Impact on teachers

In addition to the impact of the use of diagrammatic representations on pupils, the project also identified the impact of the project on teachers' knowledge and practice. Although not all of the teachers identified changes in their actual classroom practice, there was some evidence both from interviews and the lesson observations that a greater range of diagrammatic representations were being used by teachers. More importantly, most of the classroom teachers interviewed identified a development in awareness and knowledge about using diagrammatic representations in their mathematics teaching. Looking at this issue of awareness and knowledge further, both the interviews with teachers and the observations of lessons identified a degree of progression in the sophistication of this knowledge of use of diagrammatic representations:

Little or no use of diagrammatic representations → Using a variety of diagrammatic representations → Consideration of the progression for pupils between different representations

The qualitative findings of the project suggest that in order to maximise the benefits for pupils, it is not a simple matter of using 'more' representations, but also to consider how the teachers and the pupils themselves can make the connections between different representations.

(c) Barriers for teachers

The aspects where the training programme was less successful, as identified by the interviews with coordinators, were that that little or no discussion with one another took place outside the training days, and also highlighted the short-term nature of the project. In terms of encouraging collaborative working between the coordinators and the class teachers as well, the results of the project were rather mixed. In some cases, the context of the school and the class teachers (e.g. NQTs) positively supported collaborative working. In other cases however, the influence of external pressures, time and unforeseen circumstance impacted negatively on these collaborations. These factors can be drawn upon to make future improvements to the design of the project, and will be further highlighted in the next section.

6. Evaluation of the study and recommendations for developments

In addition to drawing on the results of the study, an external evaluation of the project was carried out by Dr Stephen Atherton from Aberystwyth University. This external evaluation involved separate interviews with the mathematics coordinators involved in the project, looking specifically at the benefits and drawbacks of the project and ways of improving the design of the project in the future. The sections below summarise the main themes emerging from the evaluation interviews.

(a) Benefits of the project

The benefits of the project highlighted in the evaluation were in line with the findings from the project presented previously. There was a clear consensus from the interviews that the mathematics coordinators had enjoyed being part of the research project.

> *Actually when I first came along I was a bit daunted by being involved with something at the university … I thought it would be all, I*

> *know this sounds terrible, but "highbrow" and this kind of thing, but the people that have been working with us have been really nice.*
>
> *I sort of had the letter put on my desk and told "you're doing this". So it wasn't a choice thing, so it was a bit daunting at first. And then when we came to the first day, I'd sort of had a different remit from the head as to what it was about; and so when I came I thought "ooh I don't think I really understand this", but after that day and then putting it into practice it's been really enjoyable. And then coming to days like this where you get a chance to share ideas and discuss things is really valuable to your future teaching. I think it's been really good.*

One of the main benefits identified with the project was the opportunity to reflect on the teaching of mathematics.

> *Personally speaking I've enjoyed taking part in something where for the first time in quite a long time. I've had to focus on some aspect of the curriculum, and completely rethink the way it's taught. As I said before there isn't a lot of time for reflection in the profession at all, I don't think. And just the opportunity to be out of school at all, as school is inevitable busy, just to come here, it's nice to have time.*
>
> *I think it's definitely made me be more reflective on my practice.*

In addition, another positive aspect of the project has been the opportunity for teachers to discuss and share ideas with other colleagues.

> *Sort of sharing with other schools as well all the different sorts of visual representations that you can use that we maybe might not have thought about in the past, so that has been really worthwhile and does have a big impact on the whole school.*
>
> *I think it's been really valuable to reflect on your practice and we all bring very different things, we're from different schools and we're a*

> *very different group, but it's been really interesting listening to everyone else, beginning to share ideas from their schools and you put them into practice.*

As a result of the project, the teachers identified that there had been an impact on their knowledge of teaching mathematics and actual changes to their teaching.

> *It's highlighted quite a lot of issues or things we need to improve on in maths even though we've only focused on two areas of it.*
>
> *We've thought a lot about the different areas. I mean my school was in on the fractions and with fractions being a notoriously poorly taught area, it's made us think how we do it, how we progress the children and it has had a big impact on their learning so it's been really worthwhile.*
>
> *There was a particular member of staff … who quite surprisingly had never used arrays to teach multiplication. She's taught that year group for a very long time, she just didn't like (arrays), but because of this she had to and she found she loved them and it really worked surprisingly enough. That's a teacher who had been teaching for twenty/thirty years who has then changed their practice, so it definitely benefits the school. It's just like I say, the opportunity to think is so invaluable.*

(b) Drawbacks to the project

Despite the positive impact of the project, there were of course some drawbacks identified by the teachers. These were again in line with the findings of the project. The main issue identified by the coordinators was the issue of time.

> *But then finding the time to go back and sit with someone for an hour is very difficult in schools, so you say it in passing in the hope that that person will think yeah I'll do that. But there's so many other things going on all the time.*
>
> *I think it's always good practice to share but I think that time is always an issue too you know, unless you've got staff meetings set aside or you're linking up with other schools. Your ideas you share with your link group partner but unless there's a whole staff meeting set aside for it, it's always a timing issue.*

The issue of allocating time to the particular maths topics focused upon in the project was also identified as a possible issue.

> *The one major issue that we found was that the time that was spent extra on one area was then taken away from another area of maths, so it's then trying to find time slots to fit in so we're not leaving another area sort of as a weakness. So that's one of our areas that we struggled with but not a great deal.*

(c) Improvements to the project

Following on from the possible drawbacks, the coordinators suggested improvements to the project. The main suggestion was including the class teachers in the training days for the project.

> *Some greater involvement of the teachers who are actually going to be teaching, somehow. How that would happen I'm not really sure, some more direct input for them would be useful.*

> *I think it would have been beneficial if the actual teacher who was implementing it was here as well.*

In addition, putting into place further opportunities to enhance the discussion between the coordinators was suggested.

> *And I think maybe one of the other issues was speaking to other schools as well, because once you get back to school it's just so difficult to do that, other things take precedent, you can't help that. So sometimes until you come back for the next training day it's in the back of your mind a little bit so we would like to be able to go to other schools to see their practice.*
>
> *I know there was a lot of keep in touch with your partner, you know see what they're doing in their schools. And there's no time set aside for that. I would have liked sort of a midterm sort of meeting. Out of school again, rather than just saying 'oh give them a ring, email them', what have you.*

All these suggestions will be incorporated into further work that we carry out in the future.

7. Next steps

To conclude this report, we highlight particular implications that can be taken from this study in order to inform the pedagogical use of visual representations by teachers (an in turn the professional development of teachers), and perhaps more generally the primary mathematics curriculum.

Firstly, as has been highlighted, the study has identified the benefits that visual representations can bring to the learning of mathematics for primary children. The quantitative findings suggested that children progressed significantly in their learning of multiplication and fractions through the use of visual representations by their teachers. Qualitatively as well, teachers highlighted the fact that diagrammatic representations benefit pupils' understanding and also their confidence in mathematics. It is important that we emphasise these benefits to teachers in their training and professional development. Within the profession, OfSTED in their report and *Mathematics: understanding the score* (OfSTED, 2008) emphasised the importance of developing pupils' understanding in the subject:

> *The fundamental issue for teachers is how better to develop pupils' mathematical understanding. Too often, pupils are expected to remember methods, rules and facts without grasping the underpinning concepts, making connections with earlier learning and other topics, and making sense of the mathematics so that they can use it independently.* (p.5)

In another more recent report, *Good practice in primary mathematics: evidence from 20 successful schools* (OfSTED, 2011), OfSTED highlighted the beneficial use of visual images, drawing on an example where "One school has recently adopted the Singapore curriculum, which emphasises the consistent use of visual representation to aid conceptual understanding" (p.29). Therefore, the awareness of the beneficial nature of using visual representations is there, but this report provides additional detail on why and how those benefits may arise. The next step for the authors of this study, in their roles of working with trainee teachers and on professional development programmes, is to emphasise on these courses the potential of visual representations in primary mathematics.

The study stresses however that it is not just a straightforward matter of using more visual representations in our teaching of mathematics. The study identified the progression in teachers' use of representations from little or no use of representations, to using a variety of representations, and then importantly on to the next step of considering the progression for pupils in using different representations for mathematical concepts. This is an additional aspect that needs to be stressed in the training and professional development of teachers, and once again, we have incorporated these ideas in the work that we do with trainee and more experienced teachers. Also in our research work, we have identified possible examples of progression in the use of visual representations. For example, from Barmby *et al*. (2009), we have this suggestion for progressing through different visual representations for multiplication:

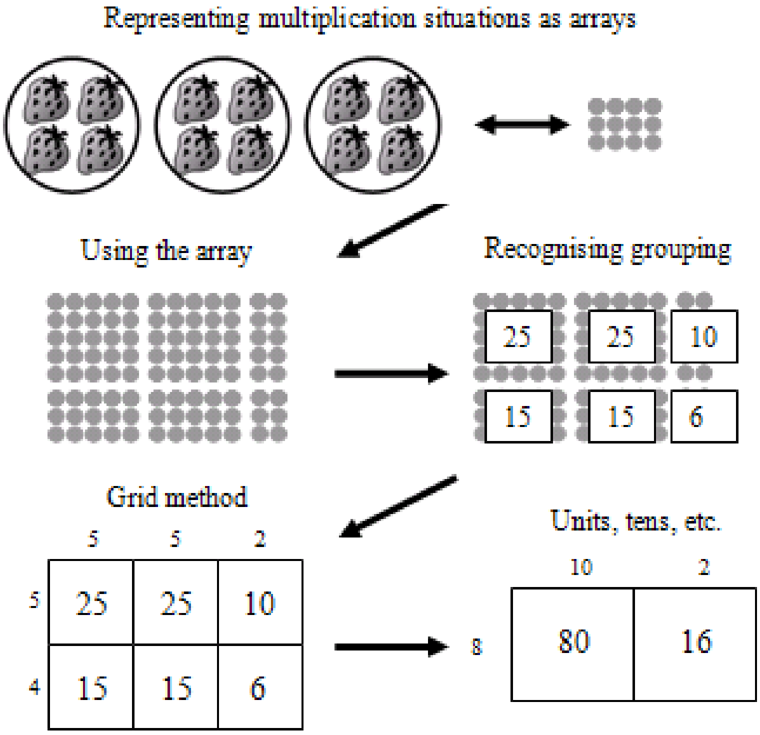


Figure 5: A suggested progression in using visual representations for multiplication.

The progression begins with a ‘groups of’ representation for multiplication but then moves to the array because of its usefulness in showing different properties of multiplication. However, by linking back to the ‘groups of’ representation, we can try and avoid any difficulty for pupils in identifying the array as representation for multiplication. In turn, modifying the array can help explain the grid representation for multiplication calculations, which in turn can be linked to formal column multiplication approaches. The important issue here is considering how we progress pupils to representations that we want them to use whilst at the same time supporting their understanding of the representations. In terms of next steps, this is another area which the authors will be examining, considering possible progression pathways for representations of different mathematical concepts (e.g. addition involving number lines, place value etc.) and we hope to develop materials for teachers based on these considerations.

A final 'next step' that we would highlight from this study is with respect to the proposed new curriculum for primary mathematics that will be introduced in 2014. The programme of study for Key Stages 1 to 2 (Department for Education, 2013) stress as one of its aims that pupils "become fluent in the fundamentals of mathematics" (p.3). In relation to this, the idea of developing conceptual understanding is highlighted by the programmes of study, although the aim of developing this understanding is stated as "pupils are able to recall and apply their knowledge rapidly and accurately to problems" (p.3). The concern is that this emphasis on fluency and speed could in fact detract from teachers aiming for conceptual understanding. Looking further at the programme of study, although some visual representations are highlighted (e.g. number line, array), their use in developing understanding is not emphasised and no sense of progression involving these representations is made clear. Therefore, the findings of this study have implications for how teachers enact the new programme of study. Once again, through awareness of why and how to use visual representations in the teaching and learning of primary mathematics, it is hoped that any possible drawback to the new primary mathematics curriculum can be avoided.

References

Ainsworth, S. (1999). The functions of multiple representations. *Computers & Education*, *33*, 131-152.

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2009). The array representation and primary children's understanding and reasoning in multiplication. *Educational Studies in Mathematics*, *70*, 217–241.

Behr, M. J., Lesh, R., Post, T. P., & Silver, E. A. (1983). Rational-number concepts. In R. Lesh, & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 91-126). New York: Academic Press.

Borko, H. (2004). Professional development and teacher learning: Mapping the terrain. *Educational Researcher*, *33*(8), 3-15.

Boulton-Lewis, G. M. (1998). Children's strategy use and interpretations of mathematical representations. *Journal of Mathematical Behavior*, *17*(2), 219-237.

Brophy, J. (1991). Conclusion. In J. Brophy (Ed.), *Advances in research on teaching* (Vol. 2, pp. 349-364). Greenwich, CT: JAI Press.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view of mind in mathematics education. *Journal for Research in Mathematics Education*, *23*(1), 2-33.

Desimone, L. M., Porter, A. C., Garet, M. S., Yoon, K. S., & Birman, B. F. (2002). Effects of Professional Development on Teachers' Instruction: Results from a Three-Year Longitudinal Study. *Educational Evaluation and Policy Analysis*, *24*(2), 81-112.

Diaper, G. (1990). The Hawthorne effect: A fresh examination. *Educational Studies*, 16(3), 261-267.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. In F. Hitt, & M. Santos (Eds.), *Proceedings of the Twenty-first Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 3-26). Columbus, Ohio: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Flevares, L. M., & Perry, M. (2001). How many do you see? The use of nonspoken representations in first-grade mathematics lessons. *Journal of Educational Psychology*, *93*(2), 330-345.

Franke, M. L., Carpenter, T. P., Levi, L., & Fennema, E. (2001). Capturing Teachers' Generative Change: A Follow-up Study of Professional Development in Mathematics. *American Educational Research Journal*, *38*(3), 653-689.

Garet, M. S., Porter, A. C., Desimone, L., Birman, B. F., & Yoon, K. S. (2001). What Makes Professional Development Effective? Results from a National Sample of Teachers. *American Educational Research Journal*, *38*(4), 915-945.

Gersten, R., Chard, D. J., Jayanthi, M., Baker, S. K., Morphy, P. & Flojo, J. (2009). Mathematics instruction for students with learning disabilities: A meta-analysis of instructional components. *Review of Educational Research*, *79*(3), 1202–1242.

Greeno, J. G., & Hall, R. P. (1997). *Practicing representation: Learning with and about representational forms.* The Phi Delta Kappan, *78*(5), 361-367.

Greer, B. (1992). Multiplication and division as models of situations. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 276-295). New York: Macmillan.

Hall, N. (1998). Concrete representations and procedural analogy theory. *Journal of Mathematical Behavior*, *17*(1), 33-51.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65–97). New York: Macmillan.

Hill, H. C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's mathematics professional development institutes. *Journal for Research in Mathematics Education*, *35*(5), 330-351.

Kaput, J. J. (1991). Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.), *Radical constructivism in mathematics education* (pp. 53-74). Dordrecht: Kluwer.

Korthagen, F. A. J., & Kessels, J. P. A. M. (1999). Linking Theory and Practice: Changing the Pedagogy of Teacher Education. *Educational Researcher*, *28*(4), 4-17.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction*, *3*, 305-342.

Leinhardt, G., Putnam, R. T., Stein, M. K., & Baxter, J. (1991). Where subject knowledge matters. In J. Brophy (Ed.), *Advances in research on teaching:*

Vol. 2. Teachers' knowledge of subject matter as it relates to their teaching practice (pp. 87-113). Greenwich, CT: JAI Press.

OfSTED (2009). *Mathematics: understanding the score*. London: OfSTED.

OfSTED (2011). *Good practice in primary mathematics: evidence from 20 successful schools*. London: OfSTED.

Outhred, L., & Mitchelmore. M. (2004). Student's structuring of rectangular arrays. In M. J. Høines, & A. Fuglestad (Eds.), *Proceedings 28th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, 465-472). Bergen, Norway: PME.

Post, T. R., & Cramer, K. A. (1989). Knowledge, representation, and quantitative thinking. In M. C. Reynolds (Ed.), *Knowledge base for the beginning teacher* (pp. 221-232). New York: Pergamon.

Sowell, E. J. (1989). Effects of manipulative materials in mathematics education. *Journal for Research in Mathematics Education*, *20*(5), 498-505.

Wilson, S. M., & Berne, J. (1999). Learning and the Acquisition of Professional Knowledge: An Examination of Research on Contemporary Professional Development. *Review of Research in Education*, *24*, 173-209.

Appendix A: Multiplication test used

MATHS TEST

NAME: ……………………..

CLASS: ……………………..

For the first part of the test, the teacher will read out the question, and you will need to tick the correct box to show which calculation is being shown in the question.

Question 1

George has 15 sweets but he gives 4 sweets to his sister. How do you work out how many sweets George has left? (Tick one box)

15 + 4	15 ÷ 4
15 - 4	15 × 4

Question 2

Six children are holding two books each. How do you work out how many books they have altogether? (Tick one box)

6 + 2	6 - 2
6 × 2	6 ÷ 2

Question 3

Eight sweets are shared between four children? How do you work out how many sweets each child gets? (Tick one box)

8 + 4	8 - 4
8 × 4	8 ÷ 4

Question 4

It takes Chris 4 minutes to wash a window. He wants to know how many minutes it will take him to wash 8 windows at this rate. Which of these should he do? (Tick one box)

4 × 8	8 ÷ 4
8 - 4	8 + 4

For the second part of the test, the teacher will read out all the questions. Sometimes you will need to tick the correct box, and sometimes you will need to work out and write down your answer.

Question 5

What is 5 times 2?

7	3
10	52

Question 6

Work out what 8 times 5 is. (Write down your answer)

Question 7

A class full of children lined up in four rows. There were seven children in each row. How many children were there in the class? (Tick one box)

28	18	12	11

Question 8

What is 3 times 23? (Tick one box)

323	233	69	26

Question 9

Maurice the monkey eats one bunch of bananas every day. If there are six bananas in a bunch, how many bananas does Maurice eat in a week? (Tick one box)

7	30	36	42

Question 10

25 × 18 is more that 24 × 18. How much more? (Tick one box)

1	18	24	25

Question 11

A boat moves at 2 metres every second. How far does it move in 7 seconds?

9	14	5	7

Question 12

If you know that 18 × 4 = 72, how would you work out 18 × 3? (Tick one box)

18 + 3	72 + 18
72 - 18	72 - 3

Question 13

Emily swims four times further than James. If James swims four lengths of the pool, how many lengths does Emily swim? (Tick one box)

16	8	2	12

Question 14

When Sarah's guinea pig was born, it was only six centimetres long. It grew up to be three times this size. How big did it grow to?

9	12	36	18

Question 15

Calculate 14 × 6 (write down your answer)

Question 16

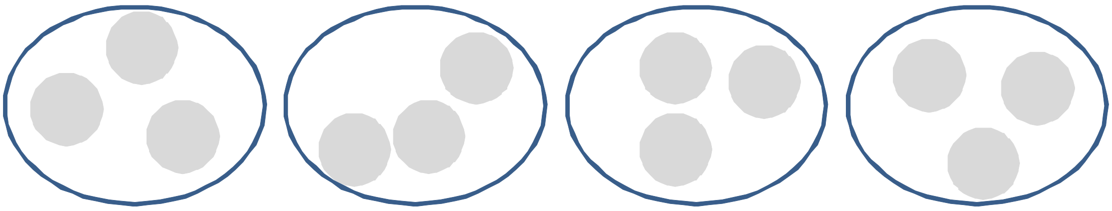
A shop makes sandwiches. You can choose from 3 sorts of bread, and from 6 sorts of filling. How do you work out how many different sandwiches you could choose?

6 + 3	6 × 3
6 - 3	6 ÷ 3

For the last part of the test, the teacher will read out the question, and you will need to look at the pictures in the questions so that you can answer the question.

Question 17

Look at the picture below:



Fill in the boxes below to show the calculation that is in the picture:

$$\square \times \square = \square$$

Question 18

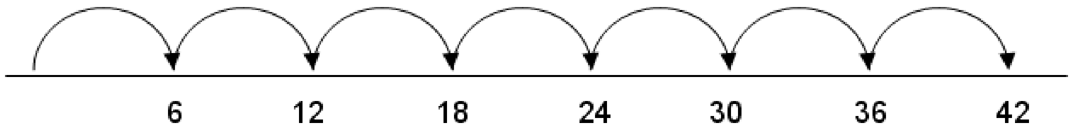
Look at the picture below:

Fill in the boxes below to show the calculation that is in the picture:

$$\square \times \square = \square$$

Question 19

Look at the picture below:

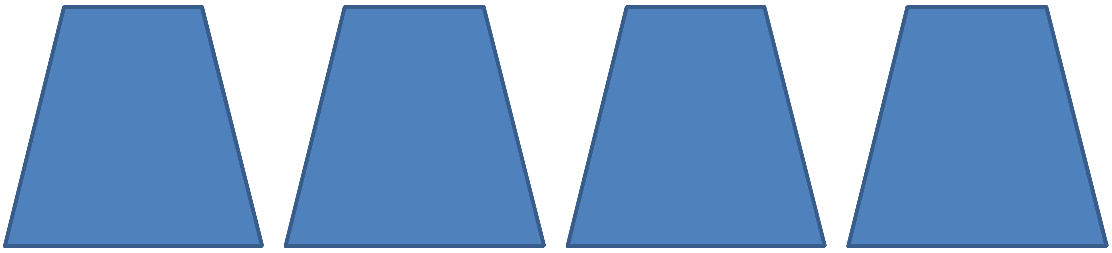


Fill in the boxes below to show the calculation that is in the picture:

$\square \times \square = \square$

Question 20

The picture below shows four cups. Each cup is hiding six sweets.

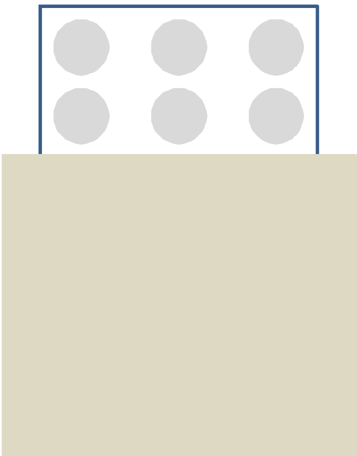


Fill in the boxes below to show the calculation of the number of sweets in the picture:

$\square \times \square = \square$

Question 21

The picture below shows a piece of card with rows of circles. The bottom of the piece of card is hidden behind a screen. Four rows of circles are hidden by the screen.



Fill in the boxes below to show the calculation that is in the picture on the card <u>when the screen is taken away</u>:

$\square \times \square = \square$

<u>WELL DONE FOR DOING THE TEST!</u>

Appendix B: Fractions test used

FRACTIONS TEST

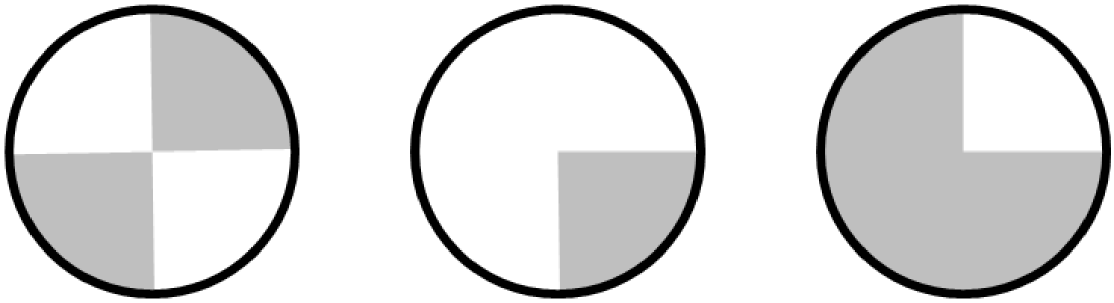
FIRST NAME: ……………..

SURNAME: ………………..

CLASS: …………………….

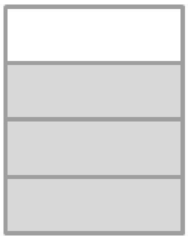
Question 1

Tick the circle which is three quarters coloured in with grey shading:



Question 2

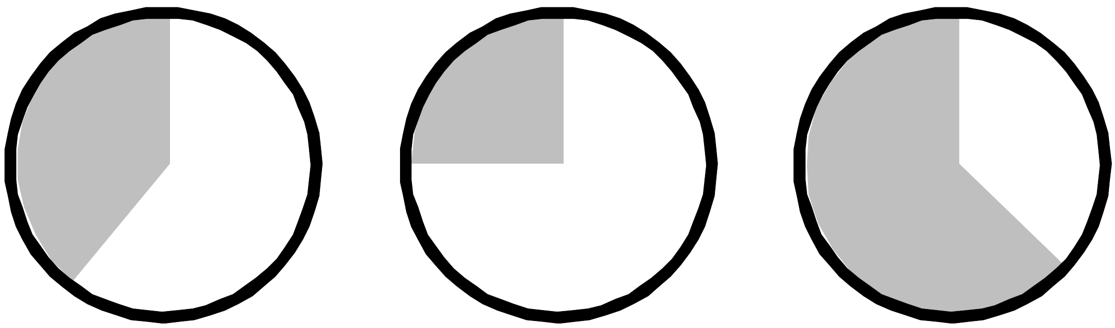
Write the fraction that is shaded grey in the shape below:



....

Question 3

Tick the circle which is two thirds coloured in with grey shading:



Question 4

What is half of 6? (Tick one box)

9	2	5	3

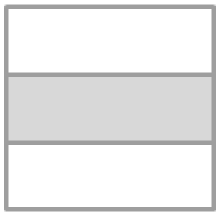
Question 5

What is a quarter of 8? (Tick one box)

2	4	6	8

Question 6

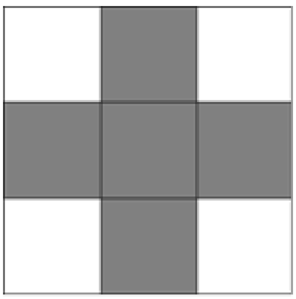
Write the fraction that is shaded grey in the shape below:



....

Question 7

Part of the figure is shaded in grey:



What fraction of the figure is shaded? (Tick one box)

$\frac{5}{4}$	$\frac{4}{5}$	$\frac{6}{9}$	$\frac{5}{9}$

<u>Question 8</u>

In a whole or 1, there are how many thirds?,

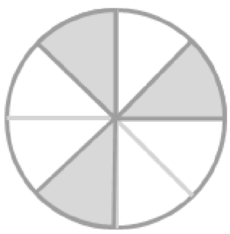
<u>Question 9</u>

A cake was cut into 8 pieces of equal size. John ate 3 pieces of the cake. What fraction of the cake did John eat? (Tick one box)

$\boxed{\frac{1}{8}}$ $\boxed{\frac{3}{8}}$ $\boxed{\frac{3}{5}}$ $\boxed{\frac{8}{3}}$

<u>Question 10</u>

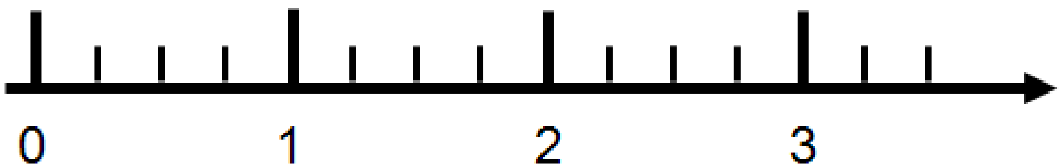
Write the fraction that is shaded grey in the shape below:



....

<u>Question 11</u>

Draw an arrow to show where ½ is on the number line below:

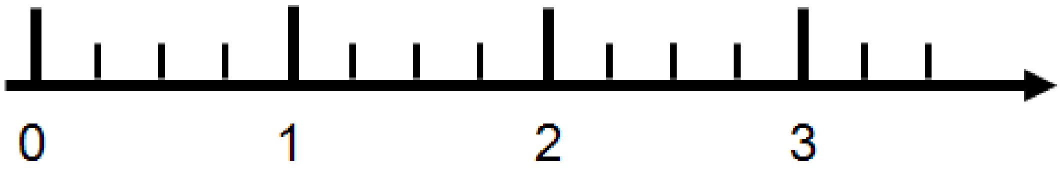


Question 12

Give a fraction that is equivalent to ½.

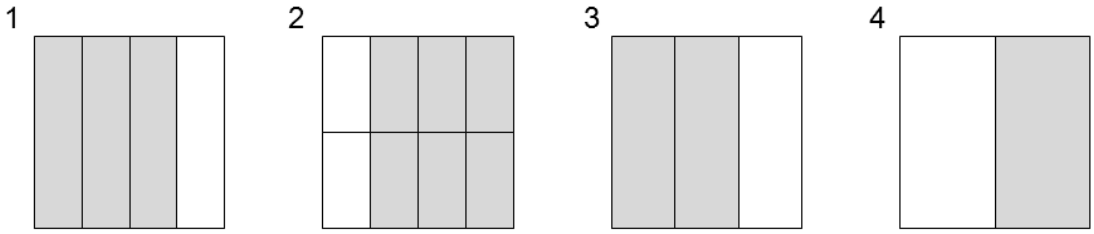
Question 13

Draw an arrow to show where 2¼ is on the number line below:



Question 14

Each figure represents a fraction.



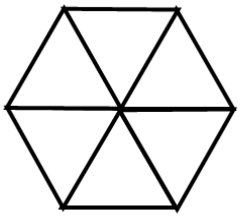
Which two figures represent the same fraction? (Tick one box)

3 and 4	1 and 3
2 and 3	1 and 2

Question 15

Colour the picture to match the given fraction:

$\frac{1}{3}$ of the shape



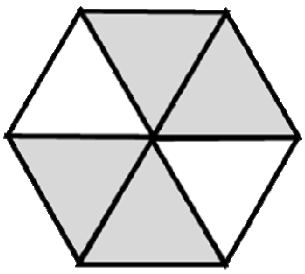
Question 16

Calculate $\frac{1}{8}$ of 24.

Question 17

Three quarters is how many times bigger than ¼ ?

Question 18



What fraction of the figure is shaded grey? (Tick one box)

$\frac{1}{2}$	$\frac{2}{3}$	$\frac{2}{4}$	$\frac{2}{6}$

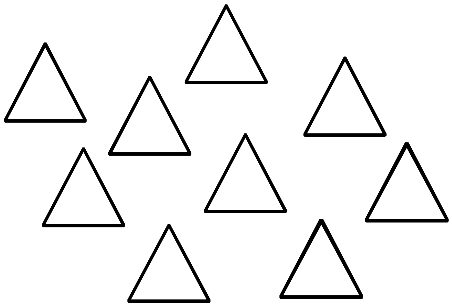
Question 19

Which is greater, $\frac{1}{2}$ or $\frac{3}{8}$?

Question 20

Colour the picture to match the given fraction:

$\frac{1}{3}$ of the triangles



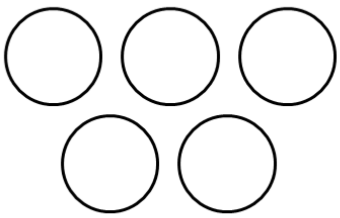
Question 21

Three pizzas are evenly divided among four children. How much pizza will each child get?

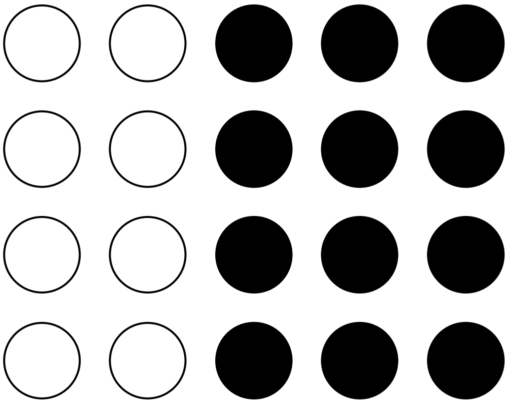
Question 22

Colour each picture to match the given fraction:

$2\frac{3}{4}$ of the pies below



Question 23



What fraction of the counters are white? (Tick one box)

$\frac{15}{25}$ $\frac{10}{15}$ $\frac{4}{5}$ $\frac{2}{5}$

Question 24

In which list of fractions are all the fractions equivalent? (Tick one box)

$\frac{3}{4}, \frac{6}{8}, \frac{12}{14}$ $\frac{3}{5}, \frac{5}{7}, \frac{9}{15}$

$\frac{3}{8}, \frac{6}{16}, \frac{12}{32}$ $\frac{5}{10}, \frac{10}{15}, \frac{1}{2}$

Question 25

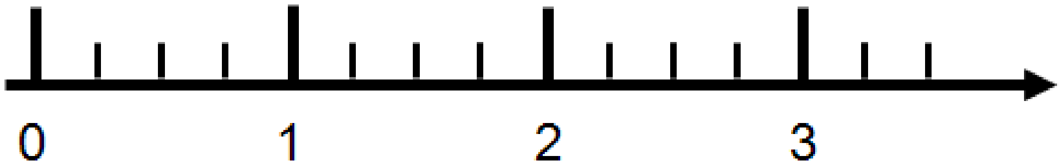
Eight chocolate bars are shared between twelve children. How much of a chocolate bar will each child get?

Question 26

Give two different fractions that add up to a total of 1.

Question 27

Draw an arrow to show where $\frac{8}{4}$ is on the number line below:



Question 28

Calculate $\frac{1}{5}+\frac{2}{5}$

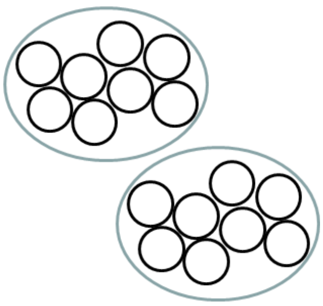
Question 29

Three pizzas were evenly shared among some friends. If each of them gets $\frac{3}{5}$ of the pizza, how many friends are there altogether?

Question 30

Colour the picture to match the given fraction:

$1\frac{3}{4}$ of the sets of marbles below



Question 31

Draw an arrow to show where $2\frac{6}{8}$ is on the number line below:

Question 32

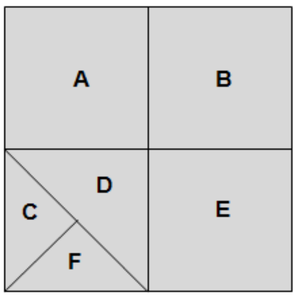
Twelve biscuits were evenly shared among some friends. If each of them gets 1½ biscuits, how many friends are there altogether?

Question 33

Calculate $5+\frac{5}{8}$

Question 34

Look at this diagram:



What fraction of the whole square is area D? (Tick one box)

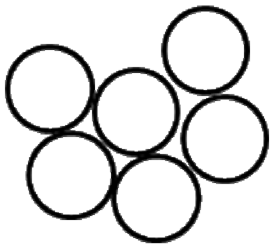
$\frac{1}{8}$ $\frac{1}{4}$ $\frac{1}{6}$ $\frac{1}{16}$

Question 35

Calculate $2+2\frac{3}{4}$

Question 36

This is $\frac{3}{5}$ of a set of marbles.



Draw the full set of marbles in the box below.

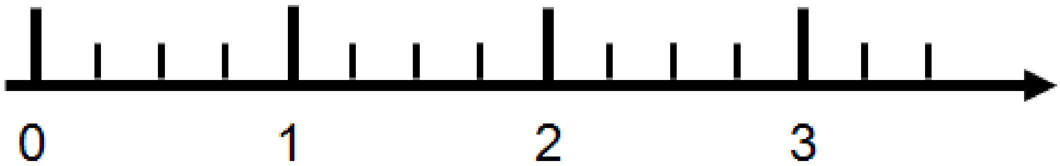
Question 37

Which number is largest? (Tick one box)

$\frac{4}{5}$ $\frac{3}{4}$ $\frac{5}{8}$ $\frac{7}{10}$

Question 38

Draw an arrow to show where $\frac{13}{8}$ is on the number line below:



Question 39

Which number is largest? (Tick one box)

$\frac{3}{4}$ $\frac{2}{3}$ $\frac{3}{5}$ $\frac{7}{9}$ $\frac{7}{10}$

WELL DONE FOR DOING THE TEST!