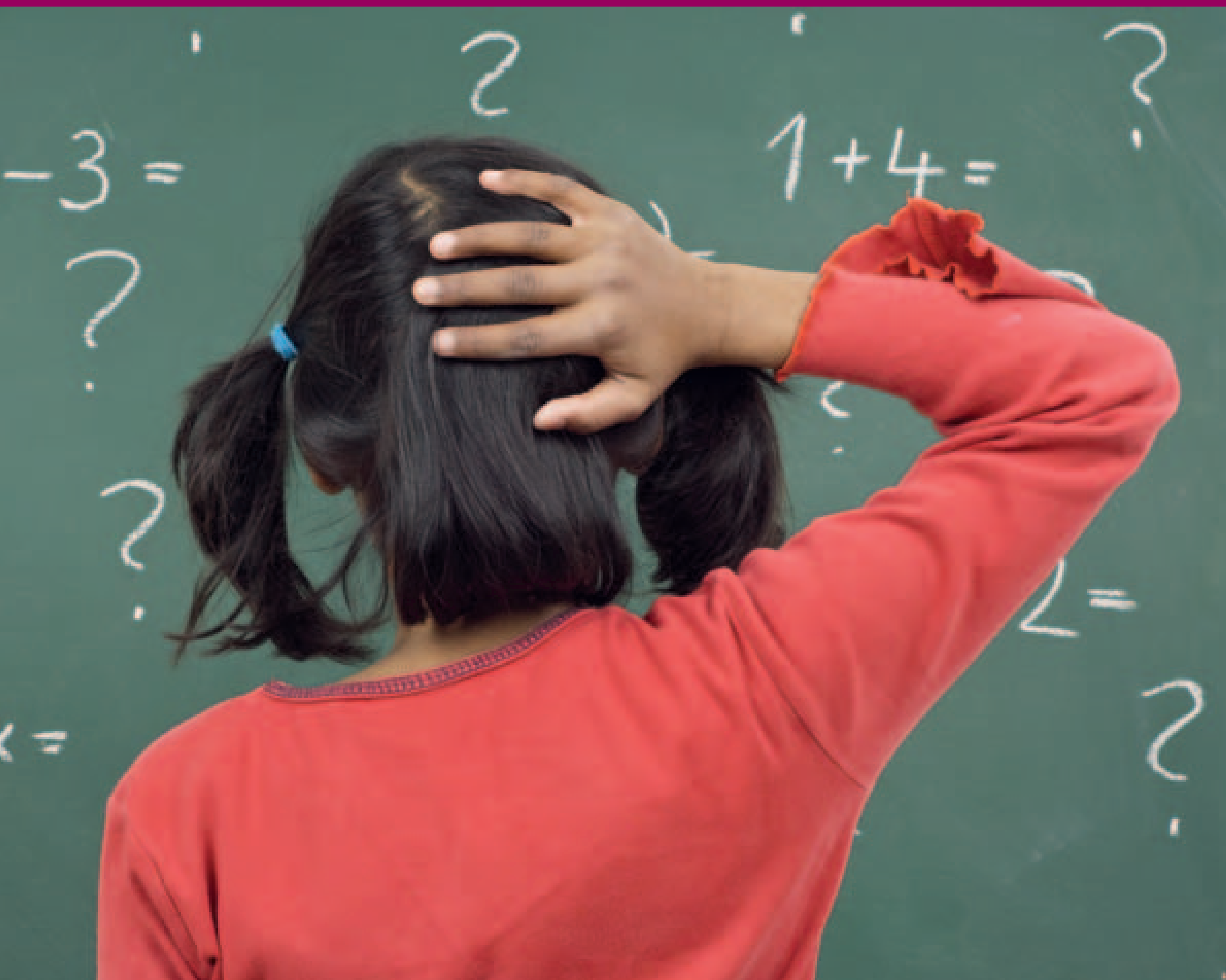
Nuffield Foundation

Key understandings in mathematics learning

Summary papers

By Terezinha Nunes, Peter Bryant and Anne Watson, University of Oxford



Introduction

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers, the first seven of which are summarised here.

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools. Paper 1 is the *Overview*, and Paper 8, not included here, is the *Methodological appendix*.

Full versions of the 8 papers, together with an *Introduction and summary of findings*, are available to download from our website, www.nuffieldfoundation.org

The review as a whole illuminates important aspects of mathematics learning from the perspectives of educational psychology and practice. It identifies important issues of significance to policy makers and practitioners as well as identifying significant gaps in our evidence base.

We are grateful to the authors for their commitment to this task and for producing such a comprehensive analysis of the extensive literature in this important field. We welcome the review and are confident it will usefully inform continuing debates about how best to improve curriculum design, teaching and learning for all students of elementary mathematics.

Anthony Tomei
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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

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Summaries

Full versions of these papers, together with *Paper 8: Methodological appendix*, are available to download from:

www.nuffieldfoundation.org

PAPER 1: Overview

By Terezinha Nunes, Peter Bryant and Anne Watson, University of Oxford

Aims

Our aim in the review is to present a synthesis of research on mathematics learning by children from the age of five to the age of sixteen years and to identify the issues that are fundamental to understanding children's mathematics learning. In doing so, we concentrated on three main questions regarding key understandings in mathematics.

- What insights must students have in order to understand basic mathematical concepts?
- What are the sources of these insights and how does informal mathematics knowledge relate to school learning of mathematics?
- What understandings must students have in order to build new mathematical ideas using basic concepts?

Theoretical framework

While writing the review, we concluded that there are two distinct types of theory about how children learn mathematics.

Explanatory theories set out to explain how children's mathematical thinking and knowledge change. These theories are based on empirical research on children's solutions to mathematical problems as well as on experimental and longitudinal studies. Successful theories of this sort should provide insight into the causes of children's mathematical development and worthwhile suggestions about teaching and learning mathematics.

Pragmatic theories set out to investigate what children ought to learn and understand and also identify obstacles to learning in formal educational settings. Pragmatic theories are usually not tested for their consistency with empirical evidence, nor examined for the parsimony of their explanations vis-à-vis other existing theories; instead they are assessed in multiple contexts for their descriptive power, their credibility and their effectiveness in practice.

Our starting point in the review is that children need to learn about quantities and the relations between them and about mathematical symbols and their meanings. These meanings are based on sets of relations. Mathematics teaching should aim to ensure that students' understanding of quantities, relations and symbols go together.

Conclusions

This theoretical approach underlies the six main sections of the review. We now summarise the main conclusions of each of these sections.

Whole numbers

- Whole numbers represent both quantities and relations between quantities, such as differences and ratio. Primary school children must establish clear connections between numbers, quantities and relations.

- Children's initial understanding of quantitative relations is largely based on correspondence. One-to-one correspondence underlies their understanding of cardinality, and one-to-many correspondence gives them their first insights into multiplicative relations. Children should be encouraged to think of number in terms of these relations.

- Children start school with varying levels of ability in using different action schemes to solve arithmetic problems in the context of stories. They do not need to know arithmetic facts to solve these problems: they count in different ways depending on whether the problems they are solving involve the ideas of addition, subtraction, multiplication or division.

- Individual differences in the use of action schemes to solve problems predict children's progress in learning mathematics in school.

- Interventions that help children learn to use their action schemes to solve problems lead to better learning of mathematics in school.

- It is more difficult for children to use numbers to represent relations than to represent quantities.

Implications for the classroom

Teaching should make it possible for children to:

- connect their knowledge of counting with their knowledge of quantities
- understand additive composition and one-to-many correspondence
- understand the inverse relation between addition and subtraction
- solve problems that involve these key understandings
- develop their multiplicative understanding alongside additive reasoning.

Implications for further research

Long-term longitudinal and intervention studies with large samples are needed to support curriculum development and policy changes aimed at implementing these objectives. There is also a need for studies designed to promote children's competence in solving problems about relations.

Fractions

- Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. As with whole numbers, children need to make connections between quantities and their representations in fractions in order to be able to use fractions meaningfully.

- Two types of quantities that are taught in primary school must be represented by fractions. The first involves measurement: if you want to represent a quantity by means of a number and the quantity is smaller than the unit of measurement, you need a fraction; for example, a half cup or a quarter inch. The second involves division: if the dividend is smaller than the divisor, the result of the division is represented by a fraction; for example, three chocolates shared among four children.

- Children use different schemes of action in these two different situations. In division situations, they use correspondences between the units in the numerator and the units in the denominator. In measurement situations, they use partitioning.

- Children are more successful in understanding equivalence of fractions and in ordering fractions by magnitude in situations that involve division than in measurement situations.

- It is crucial for children's understanding of fractions that they learn about fractions in both types of situation: most do not spontaneously transfer what they learned in one situation to the other.

- When a fraction is used to represent a quantity, children need to learn to think about how the numerator and the denominator relate to the value represented by the fraction. They must think about direct and inverse relations: the larger the numerator, the larger the quantity, but the larger the denominator, the smaller the quantity.

- Like whole numbers, fractions can be used to represent quantities and relations between quantities, but they are rarely used to represent relations in primary school. Older students often find it difficult to use fractions to represent relations.

Implications for the classroom

Teaching should make it possible for children to:

- use their understanding of quantities in division situations to understand equivalence and order of fractions
- make links between different types of reasoning in division and measurement situations
- make links between understanding fractional quantities and procedures
- learn to use fractions to represent relations between quantities, as well as quantities.

Implications for further research

Evidence from experimental studies with larger samples and long-term interventions in the classroom are needed to establish how division situations relate to learning fractions. Investigations on how links between situations can be built are needed to support curriculum development and classroom teaching.

There is also a need for longitudinal studies designed to clarify whether separation between procedures and meaning in fractions has consequences for further mathematics learning.

Given the importance of understanding and representing relations numerically, studies that investigate under what circumstances primary school students can use fractions to represent relations between quantities, such as in proportional reasoning, are urgently needed.

Relations and their mathematical representation

- Children have greater difficulty in understanding relations than in understanding quantities. This is true in the context of both additive and multiplicative reasoning problems.

- Primary and secondary school students often apply additive procedures to solve multiplicative problems and multiplicative procedures to solve additive problems.

- Teaching designed to help students become aware of relations in the context of additive reasoning problems can lead to significant improvement.

- The use of diagrams, tables and graphs to represent relations in multiplicative reasoning problems facilitates children's thinking about the nature of the relations between quantities.

- Excellent curriculum development work has been carried out to design programmes that help students develop awareness of their implicit knowledge of multiplicative relations. This work has not been systematically assessed so far.

- An alternative view is that students' implicit knowledge should not be the starting point for students to learn about proportional relations; teaching should focus on formalisations rather than informal knowledge and only later seek to connect mathematical formalisations with applied situations. This alternative approach has also not been systematically assessed yet.

- There is no research that compares the results of these diametrically opposed ideas.

Implications for the classroom

Teaching should make it possible for children to:

- distinguish between quantities and relations
- become explicitly aware of the different types of relations in different situations
- use different mathematical representations to focus on the relevant relations in specific problems
- relate informal knowledge and formal learning.

Implications for further research

Evidence from experimental and long-term longitudinal studies is needed on which approaches to making students aware of relations in problem situations improve problem solving. A study comparing the alternative approaches – starting from informal knowledge versus starting from formalisations – would make a significant contribution to the literature.

Space and its mathematical representation

- Children come to school with a great deal of informal and often implicit knowledge about spatial relations. One challenge in mathematical education is how best to harness this knowledge in lessons about space.

- This pre-school knowledge of space is mainly relational. For example, children use a stable background to remember the position and orientation of objects and lines.

- Measuring length and area poses particular problems for children, even though they are able to understand the underlying logic of measurement. Their difficulties concern iteration of standard units and the need to apply multiplicative reasoning to the measurement of area.

- From an early age children are able to extrapolate imaginary straight lines, which allows them to learn how to use Cartesian co-ordinates to plot specific positions in space with little difficulty. However, they need help from teachers on how to use co-ordinates to work out the relation between different positions.

- Learning how to represent angle mathematically is a hard task for young children, even though angles are an important part of their everyday life. Initially children are more aware of angle in the context of movement (turns) than in other contexts. They need help from to teachers to be able to relate angles across different contexts.

- An important aspect of learning about geometry is to recognise the relation between transformed shapes (rotation, reflection, enlargement). This can be difficult, since children's preschool experiences lead them to recognise the same shapes as equivalent across such transformations, rather than to be aware of the nature of the transformation.

- Another aspect of the understanding of shape is the fact that one shape can be transformed into another by addition and subtraction of its subcomponents. For example, a parallelogram can be transformed into a rectangle of the same base and height by the addition and subtraction of equivalent triangles. Research demonstrates a danger that children learn these transformations as procedures without understanding their conceptual basis.

Implications for the classroom

Teaching should make it possible for children to:

- build on spatial relational knowledge from outside school
- relate their knowledge of relations and correspondence to the conceptual basis of measurement
- iterate with standard and non-standard units
- understand the difference between measurements which are/are not multiplicative
- relate co-ordinates to extrapolating imaginary straight lines
- distinguish between scale enlargements and area enlargements.

Implications for further research

There is a serious need for longitudinal research on the possible connections between children's pre-school spatial abilities and how well they learn about geometry at school.

Psychological research is needed on: children's ability to make and understand transformations and the additive relations in compound shapes; the exact cause of children's difficulties with iteration; how transitive inference, inversion and one-to-one correspondence relate to problems with geometry, such as measurement of length and area.

There is a need for intervention studies on methods of teaching children to work out the relation between different positions, using co-ordinates.

Algebra

- Algebra is the way we express generalisations about numbers, quantities, relations and functions. For this reason, good understanding of connections between numbers, quantities and relations is related to success in using algebra. In particular, understanding that addition and subtraction are inverses, and so are multiplication and division, helps students understand expressions and solve equations.

- To understand algebraic symbolisation, students have to (a) understand the underlying operations and (b) become fluent with the notational rules. These two kinds of learning, the meaning and the symbol, seem to be most successful when students know what is being expressed and have time to become fluent at using the notation.

- Students have to learn to recognise the different nature and roles of letters as: unknowns, variables, constants and parameters, and also the meanings of equality and equivalence. These meanings are not always distinct in algebra and do not relate unambiguously to arithmetical understandings.

- Students often get confused, misapply, or misremember rules for transforming expressions and solving equations. They often try to apply arithmetical meanings inappropriately to algebraic expressions. This is associated with over-emphasis on notational manipulation, or on 'generalised arithmetic', in which they may try to get concise answers.

Implications for the classroom

Teaching should make it possible for children to:

- read numerical and algebraic expressions relationally, rather than as instructions to calculate (as in substitution)
- describe generalisations based on properties (arithmetical rules, logical relations, structures) as well as inductive reasoning from sequences
- use symbolism to represent relations
- understand that letters and '=' have a range of meanings
- use hands-on ICT to relate representations
- use algebra purposefully in multiple experiences over time
- explore and use algebraic manipulation software.

Implications for further research

We need to know how explicit work on understanding relations between quantities enables students to move successfully between arithmetical to algebraic thinking.

Research on how expressing generality enables students to use algebra is mainly in small-scale teaching interventions, and the problems of large-scale implementation are not so well reported. We do not know the longer-term comparative effects of different teaching approaches to early algebra on students' later use of algebraic notation and thinking.

There is little research on higher algebra, except for teaching experiments involving functions. How learners synthesise their knowledge of elementary algebra to understand polynomial functions, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations is not known.

There is some research about the use of symbolic manipulators but more needs to be learned about the kinds of algebraic expertise that develops through their use.

Modelling, solving problems and learning new concepts in secondary mathematics

Students have to be fluent in understanding methods and confident about using them to know why and when to apply them, but such application does not automatically follow the learning of procedures. Students have to understand the situation as well as to be able to call on a familiar repertoire of facts, ideas and methods.

Students have to know some elementary concepts well enough to apply them and combine them to form new concepts in secondary mathematics. For example, knowing a range of functions and/or their representations seems to be necessary to understand the modelling process, and is certainly necessary to engage in modelling. Understanding relations is necessary to solve equations meaningfully.

Students have to learn when and how to use informal, experiential reasoning and when to use formal, conventional, mathematical reasoning. Without special attention to meanings, many students tend to apply visual reasoning, or be triggered by verbal cues, rather than analyse situations to identify variables and relations.

In many mathematical situations in secondary mathematics, students have to look for relations between numbers, and variables, and relations between relations, and properties of objects, and know how to represent them.

Implications for the classroom

Teaching should make it possible for children to:

- learn new abstract understandings, which is neither achieved through learning procedures, nor through problem-solving activities, without further intervention
- use their obvious reactions to perceptions and build on them, or understand conflicts with them
- adapt to new meanings and develop from earlier methods and conceptualizations over time
- understand the meaning of new concepts 'know about', 'know how to', and 'know how to use'
- control switching between, and comparing, representations of functions in order to understand them
- use spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

Implications for further research

Existing research suggests that where contextual and exploratory mathematics, integrated through the curriculum, *do* lead to further conceptual learning it is related to conceptual learning being a rigorous focus for curriculum and textbook design, and in teacher preparation, or in specifically designed projects based around such aims. There is therefore an urgent need for research to identify the key conceptual understandings for success in secondary mathematics. There is no evidence to convince us that the new U.K. curricula will necessarily lead to better *conceptual* understanding of mathematics, either at the elementary level which is necessary to learn higher mathematics, or at higher levels which provide the confidence and foundation for further mathematical study.

We need to understand the ways in which students learn new ideas in mathematics that depend on combinations of earlier concepts, in secondary school contexts, and the characteristics of mathematics teaching at higher secondary level which contribute both to successful conceptual learning and application of mathematics.

Common themes

We reviewed different areas of mathematical activity, and noted that many of them involve common themes, which are fundamental to learning mathematics: number, logical reasoning, reflection on knowledge and tools, understanding symbol systems and mathematical modes of enquiry.

Number

Number is not a unitary idea, which children learn in a linear fashion. Number develops in complementary strands, sometimes with discontinuities and changes of meaning. Emphasis on procedures and manipulation with numbers, rather than on understanding the underlying relations and mathematical meanings, can lead to over-reliance and misapplication of methods in arithmetic, algebra, and problem-solving. For example, if children form the idea that quantities are only equal if they are represented by the same number, a principle that they could deduce from learning to count, they will have difficulty understanding the equivalence of fractions. Learning to count and to understand quantities are separate strands of development. Teaching can play a major role in helping children co-ordinate these two forms of knowledge without making counting the only procedure that can be used to think about quantities.

Successful learning of mathematics includes understanding that number describes quantity; being able to make and use distinctions between different, but related, meanings of number; being able to use relations and meanings to inform application and calculation; being able to use number relations to move away from images of quantity and use number as a structured, abstract, concept.

Logical reasoning

The evidence demonstrates beyond doubt that children must rely on logic to learn mathematics and that many of their difficulties are due to failures to make the correct logical move that would have led them to the correct solution. Four different aspects of logic have a crucial role in learning about mathematics.

The logic of correspondence (one-to-one and one-to-many correspondence) The extension of the use of one-to-one correspondence from sharing to working out the numerical equivalence or non-equivalence of two or more spatial arrays is a vastly important step in early mathematical learning. Teaching multiplication in terms of one-to-many correspondence is more effective than teaching children about multiplication as repeated addition.

The logic of inversion Longitudinal evidence shows that understanding the inverse relation between addition and subtraction is a strong predictor of children's mathematical progress. A flexible understanding of inversion is an essential element in children's geometrical reasoning as well. The concept of inversion needs a great deal more prominence than it has now in the school curriculum.

The logic of class inclusion and additive composition Class inclusion is the basis of the understanding of ordinal number and the number system. Children's ability to use this form of inclusion in learning about number and in solving mathematical problems is at first rather weak, and needs some support.

The logic of transitivity All ordered series, including number, and also forms of measurement involve transitivity ($a > c$ if $a > b$ and $b > c$: $a = c$ if $a = b$ and $b = c$). Learning how to use transitive relations in numerical measurements (for example, of area) is difficult. One reason is that children often do not grasp the importance of iteration (repeated units of measurement).

The results of longitudinal research (although there is not an exhaustive body of such work) support the idea that children's logic plays a critical part in their mathematical learning.

Reflection on knowledge and tools

Children need to re-conceptualise their intuitive models about the world in order to access the mathematical models that have been developed in the discipline. Some of the intuitive models used by children lead them to appropriate mathematical problem solving, and yet they may not know why they succeeded. Implicit models can interfere with problem solving when students rely on assumptions that lead them astray.

The fact that students use intuitive models when learning mathematics, whether the teacher recognises the models or not, is a reason for helping them to develop an awareness of their models. Students can explore their intuitive models and extend them to concepts that are less intuitive, more abstract. This pragmatic theory has been shown to have an impact in practice.

Understanding symbol systems

Systems of symbols are human inventions and thus are cultural tools that have to be taught. Mathematical symbols are human-made tools that improve our ability to control and adapt to the environment. Each system makes specific cognitive demands on the learner, who has to understand the systems of representation and relations that are being represented; for example place-value notation is based on additive composition, functions depict covariance. Students can behave as if they understand how the symbols work while they do not understand them completely: they can learn routines for symbol manipulation that remain disconnected from meaning. This is true of rational numbers, for example.

Students acquire informal knowledge in their everyday lives, which can be used to give meaning to mathematical symbols learned in the classroom. Curriculum development work that takes this knowledge into account is not as widespread as one would expect given discoveries from past research.

Mathematical modes of enquiry

Some important mathematical modes of enquiry arise in the topics covered in this synthesis.

Comparison helps us make new distinctions and create new objects and relations Comparisons are related to making distinctions, sorting and classifying; students need to learn to make these distinctions based on mathematical relations and properties, rather than perceptual similarities.

Reasoning about properties and relations rather than perceptions Throughout mathematics, students have to learn to interpret representations before they think about how to respond. They need to think about the relations between different objects in the systems and schemes that are being represented.

Making and using representations Conventional number symbols, algebraic syntax, coordinate geometry, and graphing methods, all afford manipulations which might otherwise be impossible. Coordinating different representations to explore and extend meaning is a fundamental mathematical skill.

Action and reflection-on-action In mathematics, actions may be physical manipulation, or symbolic rearrangement, or our observations of a dynamic image, or use of a tool. In all these contexts, we observe what changes and what stays the same as a result of actions, and make inferences about the connections between action and effect.

Direct and inverse relations It is important in all aspects of mathematics to be able to construct and use inverse reasoning. As well as enabling more understanding of relations between quantities, this also establishes the importance of reverse chains of reasoning throughout mathematical problem-solving, algebraic and geometrical reasoning.

Informal and formal reasoning At first young children bring everyday understandings into school and mathematics can allow them to formalise these and make them more precise. Mathematics also provides formal tools, which do not describe everyday experience, but enable students to solve problems in mathematics and in the world which would be unnoticed without a mathematical perspective.

Epilogue

We have made recommendations about teaching and learning, and hope to have made the reasoning behind these recommendations clear to educationalists (in the extended review). We have also recognised that there are weaknesses in research and gaps in current knowledge, some of which can be easily solved by research enabled by significant contributions of past research. Other gaps may not be so easily solved, and we have described some pragmatic theories that are or can be used by teachers when they plan their teaching. Classroom research stemming from the exploration of these theories can provide new insights for further research in the future, alongside longitudinal studies which focus on learning mathematics from a psychological perspective.

PAPER 2: Understanding extensive quantities and whole numbers

By Terezinha Nunes and Peter Bryant, University of Oxford

Headlines

- Whole numbers are used in primary school to represent quantities and relations. It is crucial for children's success in learning mathematics in primary school to establish clear connections between numbers, quantities and relations.

- Using different schemes of action, such as setting objects in correspondence, children can judge whether two quantities are equivalent, and if they are not, make judgements about their order of magnitude. These insights are used in understanding the number system beyond simply producing a string of number words in a fixed order. It takes children some time to make links between their understanding of quantities and their knowledge of number.

- Children start school with varying levels of ability in using different action schemes to solve arithmetic problems in the context of stories. They do not need to know arithmetic facts to solve these problems: they count in different ways depending on whether the problems they are solving involve the ideas of addition, subtraction, multiplication or division.

- Individual differences in the use of action schemes to solve problems predict children's progress in learning mathematics in school.

- Interventions that help children learn to use their action schemes to solve problems lead to better learning of mathematics in school.

- It is considerably more difficult for children to use numbers to represent relations than to represent quantities. Understanding relations is crucial for their further development in mathematics in school.

Understanding extensive quantities and whole numbers

In children's everyday lives and before they start school, they have experiences of manipulating and comparing quantities. For example, even at age four, many children can share sweets fairly between two recipients by using correspondences: they share giving one-for-you, one-for-me, until there are no sweets left. They do sometimes make mistakes but they know that, when the sharing is done fairly, the two people will have the same amount of sweets at the end. Even younger children know some things about quantities: they know that if you add sweets to a group of sweets, there will be more sweets there, and if you take some away, there will be fewer. However, they might not know that if you add a certain number and take away the same number, there will be just as many sweets as there were before.

At the same time that young children are developing these ideas about quantities, they are often learning to count. They learn to say the sequence of number words in the right order, they know that each object that they are counting must be counted once and only once, and that it does not matter if you count a row of sweets from left to right or from right to left, you should get to the same number.

Four-year-olds are thus amazing learners of mathematics. But they lack one thing which is crucially important: they do not at first make connections between their understanding of quantities and their knowledge of numbers. So if you ask a four-year-old, who just shared some sweets fairly between two dolls, to count the sweets that one doll has and then tell you, without counting, how many sweets the other doll has, the majority (about 60%) will tell you that they do not know. Knowing that the dolls have the same quantity is not sufficient to know that if one has 8 sweets, the other one has 8 sweets also, i.e. has the same number.

Quantities and numbers are not the same thing. We can use numbers as measures of quantities, but we can think about quantities without actually having a measure for them. Until children can understand the connections between numbers and quantities, they cannot use their knowledge of quantities to support their understanding of numbers and *vice versa*. Because the connections between quantities and numbers are many and varied, learning about these connections could take three to four years in primary school.

An important link that children must make between number and quantity is the link between the order of number words in the counting sequence and the magnitude of the quantity represented. How do children come to understand that the any number in the counting sequence is equal to the preceding number plus 1?

Different explanations have been proposed in the literature. One is that they simply see that magnitude increases as they count. But this explanation does not work well: our perception of magnitude is approximate and knowing that any number is equal to its predecessor plus 1 is a very precise piece of knowledge. A second explanation is that children use perception, language and inferences together to reach this understanding. Young children discriminate well, for example, one puppet from two puppets and two puppets from three puppets. Because they know these differences precisely, they put these two pieces of information together, and learn that two is one more than one, and three is one more than two. They then make the inference that all numbers in the counting sequence are equal to the predecessor plus one. But this sort of generalisation could not be stretched into helping children understand that any number is also equal to the last-but-one in the sequence plus 2. This process of putting together perception with language and then generalising is an explanation for only the $n + 1$ idea; it would be much better if we could have a more general explanation of how children understand the connection between quantities and the number sequence.

The third explanation for how children connect their knowledge of quantities with the magnitude of numbers in the counting sequence is that children's schemes of action play the most important part in this development. The actions of adding and taking away help them understand part–whole relations. When they can link their understanding of part–whole relations with counting, they will understand many things about relations between numbers. A critical change in young children's behaviour when they add two sets is from 'count all' to 'count on'. If they know that they have 5 sweets, and you add 4 to the 5, they could either start from 1 and count all the sweets (count all) or they could point to the 5, and count on from there. 'Count on' is a sign that the children have linked their knowledge of part–whole relations with the counting sequence: they have understood the additive composition number. This explanation works for the relation between a number and its immediate predecessor and any of its predecessors. It is supported by much research that shows that counting on is a sign of abstraction in part–whole relations, which opens the way for children to solve many other problems: they can add a quantity to an invisible set, count coins of different denominations to form a single total, and are ready to learn to use place value to represent numbers in writing.

Adding and subtracting elements to sets also give children the opportunity to understand the inverse relation between addition and subtraction. This insight is not gained in an all-or-nothing fashion: children first apply it only to quantities and later on to number also. The majority of five-year-olds realises that if you add 3 sweets to a set of sweets and then take the *same* sweets away, the number of sweets in the set remains the same. However, many of these children will not realise that if you add 3 sweets to the set and then take 3 *other* sweets away, the number of sweets is still the same. They see that adding and taking away the *same quantity* leaves the original quantity the same but this does not immediately mean to them that adding and taking away the *same number* also leaves the original number the same. Research shows that the step from understanding the inverse relation between addition and subtraction of quantities is a useful start if one wants to teach children about the inverse relation between addition and subtraction of number.

Adding, taking away and understanding part–whole relations form one part of the story of what children know about quantities and numbers in the early years of primary school. They relate to how additive reasoning develops. The other part of the story is surprising to many people: children also know quite a lot about multiplicative reasoning when they start school.

Children use two different schemes of action to solve multiplication and division problems before they are taught about these operations in school: they use one-to-many correspondence and sharing. If five- and six-year-olds are shown, for example, four little houses in a row, told that they should imagine that in each live three dogs, and asked how many dogs live in the street, the majority can say the correct number. Many children will point three times to each house and count in this way until they complete the counting at the fourth house. They are not multiplying: they are solving the problem using one-to-many correspondence. Children can also share objects to recipients and answer problems about division. They do not know the arithmetic operations, but they can use their reasoning to count in different ways and solve the problem. So children manipulate quantities using multiplicative reasoning and solve problems before they learn about multiplication and division in school.

If children are assessed in their understanding of the inverse relation between addition and subtraction, of additive composition, and of one-to-many correspondence in their first year of school, this provides us with a good way of anticipating whether they will have difficulties in learning mathematics in school. Children who do well in these assessments go on to attain better results in mathematics assessments in school. Those who do not do well can improve their prospects through early intervention. Children who received specific instruction on these relations between quantities and how to use them to solve problems did significantly better than a similar group who did not receive such instruction.

Finally, many studies have used story problems to investigate which uses of additive reasoning are easier and which are more difficult for children of primary school age. Two sorts of difficulties have been identified.

The first relates to the need to understand that addition and subtraction are the inverse of each other. One story that requires this understanding is: Ali had some Chinese stamps in his collection and his grandfather gave him 2; now he has 8; how many stamps did he have before his grandfather gave him the 2 stamps? This problem exemplifies a situation in which a quantity increases (the grandfather gave him 2 stamps) but, because the information about the original number in his collection is missing, the problem is not solved by an addition but rather by a subtraction. The problem would also be an inverse problem if Ali had some Chinese stamps in his collection and gave 2 to his grandfather, leaving his collection with 6. In this second problem, there is a decrease in the quantity but the problem has to be solved by an increase in the number, in order to get us back to Ali's collection before he gave 2 stamps away. There is no controversy in the literature: inverse problems are more difficult than direct problems, irrespective of whether the arithmetic operation that is used to solve it is addition or subtraction.

The second difficulty depends on whether the numbers in the problem are all about quantities or whether there is a need to consider a relation between quantities. In the two problems about Ali's stamps, all the numbers refer to quantities. An example of a problem involving relations would be: In Ali's class there are 8 boys and 6 girls; how many more boys than girls in Ali's class? (Or how many fewer girls than boys in Ali's class?). The number 2 here refers neither to the number of boys nor to the number of girls: it refers to the relation (the difference) between number of boys and girls. A difference is not a quantity: it is a relation. Problems that involve relations are more difficult than those that involve quantities. It should not be surprising that relations are more difficult to deal with in numerical contexts than quantities: the majority, if not all, the experiences that children have with counting have to do with finding a number to represent a quantity, because we count things and not relations between things. We can re-phrase problems that involve relations so that all the numbers refer to quantities. For example, we could say that the boys and girls need to find a partner for a dance; how many boys won't be able to find a girl to dance with? There are no relations in this latter problem, all the numbers refer to quantities. This type of problem is significantly easier. So it is difficult for children to use numbers to represent relations. This could be one step that teachers in primary school want to help their children take, because it is a difficult move for every child.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's pre-school knowledge of quantities and counting develops separately.	**Teaching** Teachers should be aware of the importance of helping children make connections between their understanding of quantities and their knowledge of counting.
When children start school, they can solve many different problems using schemes of action in coordination with counting, including multiplication and division problems.	**Teaching** The linear view of development, according to which understanding addition precedes multiplication, is not supported by research. Teachers should be aware of children's mathematical reasoning, including their ability to solve multiplication and division problems, and use their abilities for further learning.
Three logical-mathematical reasoning principles have been identified in research, which seem to be causally related to children's later attainment in mathematics in primary school. Individual differences in knowledge of these principles predict later achievement and interventions reduce learning difficulties.	**Teaching** A greater emphasis should be given in the curriculum to promoting children's understanding of the inverse relation between addition and subtraction, additive composition, and one-to-many correspondence. This would help children who start school at risk for difficulties in learning mathematics to make good progress in the first years. **Research** Long-term longitudinal and intervention studies with large samples are needed before curriculum and policy changes can be proposed. The move from the laboratory to the classroom must be based on research that identifies potential difficulties in scaling up successful interventions.
Children's ability to solve word problems shows that two types of problem cause difficulties for children: those that involve the inverse relation between addition and subtraction and those that involve thinking about relations.	**Teaching** Systematic use of problems involving these difficulties followed by discussions in the classroom would give children more opportunities for making progress in using mathematics in contexts with which they have difficulty. **Research** There is a need for intervention studies designed to promote children's competence in solving problems about relations. Brief experimental interventions have paved the way for classroom-based research but large-scale studies are needed.

PAPER 3: Understanding rational numbers and intensive quantities

By Terezinha Nunes and Peter Bryant, University of Oxford

Headlines

- Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. As with whole numbers, children need to make connections between quantities and their representations in fractions in order to be able to use fractions meaningfully.

- There are two types of situation in which fractions are used in primary school. The first involves measurement: if you want to represent a quantity by means of a number and the quantity is smaller than the unit of measurement, you need a fraction – for example, a half cup or a quarter inch. The second involves division: if the dividend is smaller than the divisor, the result of the division is represented by a fraction. For example, when you share 3 cakes among 4 children, each child receives $\frac{3}{4}$ of a cake.

- Children use different schemes of action in these two different situations. In division situations, they use correspondences between the units in the numerator and the units in the denominator. In measurement situations, they use partitioning.

- Children are more successful in understanding equivalence of fractions and in ordering fractions by magnitude in situations that involve division than in measurement situations.

- It is crucial for children's understanding of fractions that they learn about fractions in both types of situation: most do not spontaneously transfer what they learned in one situation to the other.

- When a fraction is used to represent a quantity, children need to learn to think about how the numerator and the denominator relate to the value represented by the fraction. They must think about direct and inverse relations: the larger the numerator, the larger the quantity but the larger the denominator, the smaller the quantity.

- Like whole numbers, fractions can be used to represent quantities and relations between quantities, but in primary school they are rarely used to represent relations. Older students often find it difficult to use fractions to represent relations.

Understanding rational numbers and intensive quantities

There is little doubt that students find fractions a challenge in mathematics. Teachers often say that it is difficult to teach fractions and some think that it would be better for everyone if children were not taught about fractions in primary school. In order to understand fractions as numbers, students must be able to know whether two fractions are equivalent or not, and if they are not, which one is the bigger number. This is similar to understanding that 8 sweets is the same number as 8 marbles and that 8 is more than 7 and less than 9, for example. These are undoubtedly key understandings about whole numbers and fractions. But even after the age of 11 many students have difficulty in knowing whether two fractions are equivalent and do not know how to order some fractions. For example, in a study carried out in London, students were asked to paint 2/3 of figures divided in 3, 6 and 9 equal parts. The majority solved the task correctly when the figure was divided into 3 parts but 40% of the 11- to 12-year-old students could not solve it when the figure was divided into 6 or 9 parts, which meant painting an equivalent fraction (4/6 and 6/9, respectively).

Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. If the teaching of fractions were to be omitted from the primary school curriculum, children would not have the support of school learning to represent these quantities. We do not believe that it would be best to just forget about teaching fractions in primary school because research shows that children have some informal knowledge that could be used as a basis for learning fractions. Thus the question is not whether to teach fractions in primary school but what do we know about their informal knowledge and how can teachers draw on this knowledge.

There are two types of situation in which fractions are used in primary school: measurement and division situations.

When we measure anything, we use a unit of measurement. Often the object we are measuring cannot be described only with whole units, and we need fractions to represent a part of the unit. In the kitchen we might need to use a ½ cup of milk and when setting the margins for a page in a document we often need to be precise and define the margin as, for example, as 3.17 cm. These two examples show that, when it comes to measurement, we use two types of notation, ordinary and decimal notation. But regardless of the notation used, we could not accurately describe the quantities in these situations without using fractions. When we speak of ¾ of a chocolate bar, we are using fractions in a measurement situation: we have less than one unit, so we need to describe the quantity using a fraction.

In division situations, we need a fraction to represent a quantity when the dividend is smaller than the divisor. For example, if 3 cakes are shared among 4 children, it is not possible for each one to have a whole cake, but it is still possible to carry out the division and to represent the amount that each child receives using a number, ¾. It would be possible to use decimal notation in division situations too, but this is rarely the case. The reason for preferring ordinary fractions in these situations is that there are two quantities in division situations: in the example, the number of cakes and the number of children. An ordinary fraction represents each of these quantities by a whole number: the dividend is represented by the numerator, the divisor by the denominator, and the operation of division by the dash between the two numbers.

Although these situations are so similar for adults, we could conclude that it is not necessary to distinguish between them, however, research shows that children think about the situations differently. Children use different schemes of action in each of these situations.

In measurement situations, they use partitioning. If a child is asked to show ¾ of a chocolate, the child will try to cut the chocolate in 4 equal parts and mark 3 parts. If a child is asked to compare

¾ and 6/8, for example, the child will partition one unit in 4 parts, the other in 8 parts, and try to compare the two. This is a difficult task because the partitioning scheme develops over a long period of time and children have to solve many problems to succeed in obtaining equal parts when partitioning. Although partitioning and comparing the parts is not the only way to solve this problem, this is the most likely solution path tried out by children, because they draw on their relevant scheme of action.

In division situations, children use a different scheme of action, correspondences. A problem analogous to the one above in a division situation is: there are 4 children sharing 3 cakes and 8 children sharing 6 identical cakes; if the two groups share the cakes fairly, will the children in one group get the same amount to eat as the children in the other group? Primary school pupils often approach this problem by establishing correspondences between cakes and children. In this way they soon realise that in both groups 3 cakes will be shared by 4 children; the difference is that in the second group there are two lots of 3 cakes and two lots of 4 children, but this difference does not affect how much each child gets.

From the beginning of primary school, many children have some informal knowledge about division that could be used to understand fractional quantities. Between the ages of five and seven years, they are very bad at partitioning wholes into equal parts but can be relatively good at thinking about the consequences of sharing. For example, in one study in London 31% of the five-year-olds, 50% of the six-year-olds and 81% of the seven-year-olds understood the inverse relation between the divisor and the shares resulting from the division: they knew that the more recipients are sharing a cake, the less each one will receive. They were even able to articulate this inverse relation when asked to justify their answers. It is unlikely that they had at this time made a connection between their understanding of quantities and fractional representation; actually, it is unlikely that they would know how to represent the quantities using fractions.

The lack of connection between students' understanding of quantities in division situations and their knowledge about the magnitude of fractions is very clearly documented in research. Students who have no doubt that recipients of a cake shared between 3 people will fare better than those of a cake shared between 5 people may, nevertheless, say that 1/5 is a bigger fraction than 1/3 because 5 is a bigger number than 3. Although they understand the inverse relation in the magnitude of quantities in a division situation, they do not seem to connect this with the magnitude of fractions. The link between their understanding of fractional quantities and fractions as numbers has to be developed through teaching in school.

There is only one well-controlled experiment which compared directly young children's understanding of quantities in measurement and division situations. In this study, carried out in Portugal, the children were six- to seven-years-old. The context of the problems in both situations was very similar: it was about children eating cakes, chocolates or pizzas. In the measurement problems, there was no sharing, only partitioning. For example, in one of the measurement problems, one girl had a chocolate bar which was too large to eat in one go. So she cut her chocolate in 3 equal pieces and ate 1. A boy had an identical bar of chocolate and decided to cut his into 6 equal parts, and eat 2. The children were asked whether the boy and the girl ate the same amount of chocolate. The analogous division problem was about 3 girls sharing one chocolate bar and 6 boys sharing 2 identical chocolate bars. The rate of correct responses in the partitioning situation was 10% for both six- and seven-year-olds and 35% and 49%, respectively, for six- and seven-year-olds in the division situation.

These results are relevant to the assessment of variations in mathematics curricula. Different countries use different approaches in the initial teaching of fraction, some starting from division and others from measurement situations. There is no direct evidence from classroom studies to show whether one starting point results in higher achievement in fractions than the other. The scarce evidence from controlled studies supports the idea that division situations provide children with more insight into the equivalence and order of quantities represented by fractions and that they can learn how to connect these insights about quantities with fractional representation. The studies also indicate that there is little transfer across situations: children who succeed in comparing fractional quantities and fractions after instruction in division situations do no better in a post-test when the questions are about measurement situations than other children in a control group who received no teaching. The converse is also true: children taught in measurement situations do no better than a control group in division situations.

A major debate in mathematics teaching is the relative weight to be given to conceptual understanding and procedural knowledge in teaching. The difference between conceptual understanding and procedural knowledge in the teaching of fractions has been explored in many studies. These studies show that students can learn procedures without understanding their conceptual significance. Studies with adults show that knowledge of procedures can remain isolated from understanding for a long time: some adults who are able to implement the procedure they learned for dividing one fraction by another admit that they have no idea why the numerator and the denominator exchange places in this procedure. Learners who are able to co-ordinate their knowledge of procedures with their conceptual understanding are better at solving problems that involve fractions than other learners who seem to be good at procedures but show less understanding than expected from their knowledge of procedures. These results reinforce the idea that it is very important to try to make links between children's knowledge of fractions and their understanding of fractional quantities.

Finally, there is little, if any, use of fractions to represent relations between quantities in primary school. Secondary school students do not easily quantify relations that involve fractions. Perhaps this difficulty could be attenuated if some teaching about fractions in primary school involved quantifying relations that cannot be described by a single whole number.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's knowledge of fractional quantities starts to develop before they are taught about fractions in school.	**Teaching** Teachers should be aware of children's insights regarding quantities that are represented by fractions and make connections between their understanding of these quantities and fractions.
There are two types of situation relevant to primary school teaching in which quantities cannot be represented by a single whole number: measurement and division.	**Teaching** The primary school curriculum should include the study of both types of situation in the teaching of fractions. Teachers should be aware of the different types of reasoning used by children in each of these situations. **Research** Evidence from experimental studies with larger samples and long-term interventions in the classroom are needed to establish whether division situations are indeed a better starting point for teaching fractions.
Children do not easily transfer their understanding of fractions from division to measurement situations and *vice-versa*.	**Teaching** Teachers should consider how to establish links between children's understanding of fractions in division and measurement situations. **Research** Investigations on how links between situations can be built are needed to support curriculum development and classroom teaching.
Many students do no make links between their conceptual understanding of fractions and the procedures that they are taught to compare and operate on fractions in school.	**Teaching** Greater attention may be required in the teaching of fractions to creating links between procedures and conceptual understanding. **Research** There is a need for longitudinal studies designed to clarify whether this separation between procedural and conceptual knowledge does have important consequences for further mathematics learning.
Fractions are taught in primary school only as representations of quantities.	**Teaching** Consideration should be given to the inclusion of situations in which fractions are used to represent relations. **Research** Given the importance of understanding and representing relations numerically, studies that investigate under what circumstances primary school students can use fractions to represent relations between quantities are urgently needed.

PAPER 4: Understanding relations and their graphical representation

By Terezinha Nunes and Peter Bryant, University of Oxford

Headlines

- Children have greater difficulty in understanding relations than in understanding quantities. This is true in the context of both additive and multiplicative reasoning problems.
- Primary and secondary school students often apply additive procedures to solve multiplicative reasoning problems and also apply multiplicative procedures to solve additive reasoning problems.
- Explicit instruction to help students become aware of relations in the context of additive reasoning problems can lead to significant improvement in children's performance.
- The use of diagrams, tables and graphs to represent relations facilitates children's thinking about and discussing the nature of the relations between quantities in problems.
- Excellent curriculum development work has been carried out to design instruction to help students develop awareness of their implicit knowledge of multiplicative relations. This programme has not been systematically assessed so far.
- An alternative view is that students' implicit knowledge should not be the starting point for students to learn about proportional relations; teaching should focus on formalisations rather than informal knowledge and seek to connect mathematical formalisations with applied situations only later.
- There is no research comparing the results of these diametrically opposed ideas.

Understanding relations and their graphical representation

Children need to learn to co-ordinate their knowledge of numbers with their understanding of quantities. This is critical for mathematics learning in primary school so that they can use their understanding of quantities to support their knowledge of numbers and vice versa. But this is not all that students need to learn to be able to use mathematics sensibly. Using mathematics also involves thinking about relations between quantities. Research shows quite unambiguously that it is more difficult for children to solve problems that involve relations than to solve problems that involve only quantities.

A simple problem about quantities is: Paul had 5 marbles. He played two games with his friend. In the first game, he won 6 marbles. In the second game he lost 4 marbles. How many marbles does he have now? The same numerical information can be used differently, making the problem into one which is all about relations: Paul played three games of marbles. In the first game, he won 5 marbles. In the second game, he won 6. In the third game, he lost 4. Did he end up winning or losing marbles? How many?

The arithmetic that children need to use to solve is the same in both problems: add 5 and 6 and subtract 4. But the second problem is significantly more difficult for children because it is all about relations. They don't know how many marbles Paul actually had at any time, they only know that he had 5 more after the first game than before, and 6 more after the second game, and 4 fewer after the third game. Some children say that this problem cannot be solved because we don't know how many marbles Paul had to begin with: they recognise that it is possible to operate on quantities, but do not recognise that it is possible to operate on relations. Why should this be so?

One possible explanation is the way in which we express relations. When we speak about quantities, we say that Paul won marbles or lost marbles; these are two opposite statements. When we speak about relations, statements that use opposite words may mean the same thing: after winning 5 marbles, we can say that Paul now has 5 more marbles or that before he had 5 fewer. In order to grasp the concept of relations fully, students must be able to view these two different statements as meaning the same thing. Research shows that some students are able to treat these different statements as having the same meaning but others find this difficult. Students who realise that the two statements mean the same thing are more successful in solving problems about relations.

A second plausible explanation is that many children do not distinguish clearly between quantities and relations when they use numbers. When they are given a problem about relations, they interpret the relations as quantities. If they are given a problem like 'Tom, Fred, and Rhoda put their apples into a bag. Tom and Fred together had 17 more apples than Rhoda. Tom had 7 apples. Rhoda had 5 apples. How many apples did Fred have?', they write down that Tom and Fred had 17 apples together (instead of 17 more than Rhoda). When they make this interpretation error, the problem seems very easy: if Tom had 7, Fred had 10. The information about Rhoda seems irrelevant. But of course this is not the solution. It is possible to teach children to represent quantities and relations differently, and thus to distinguish the two: for example, they can be taught to write 'plus 17' to show that this is not a quantity but a relation. Children aged seven to nine years can adopt this notation and at the same time improve their ability to solve relational problems. However, even after this teaching, they still seem to be tempted to interpret relations as quantities. So, learning to represent relations helps children take a step towards distinguishing relations and quantities but they need plenty of opportunity to think about this distinction.

A third difficulty is that relational thinking involves building a model of a problem situation in order to treat the relations in the problem mathematically. In primary school, children have little opportunity to explore situations in their mathematics lessons before solving a problem. If they make a mistake in solving a problem when their computation was correct, the error is explained as 'choice of the wrong operation', but the wrong choice of operation is a symptom, not an explanation for what went wrong during problem solving.

Models of situations are ways of thinking about them, and more than one way may be appropriate. It all depends on the question that we want to answer. Suppose there are 12 girls and 18 boys in a class and they are assigned to single-sex groups during French lessons. If there were not enough books for all of them and the Head Teacher decided to give 4 books to the girls and 6 books to the boys, would this be fair? If you give one book to each girl, there are 8 girls left without books; if you give one book to each boy, there are 12 boys left without books. This seems unfair. If you ask all the children to share, 3 girls will share one book and 3 boys will share one book. This seems fair. The first model is additive: the questions it answers are 'How many more girls than books?' and 'How many more boys than books?' The second model is multiplicative: it examines the ratio between girls and books and the ratio between boys and books. If the Head Teacher is planning to buy more books, she needs an additive model. If the Head Teacher is not planning to buy more books, the ratio is more informative. A model of a situation is constructed by the problem solver for a purpose; additive and multiplicative relations answer different questions about the same situation.

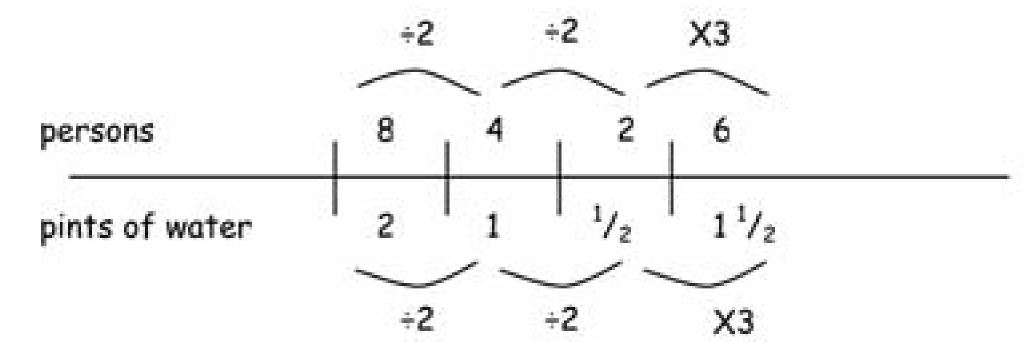
Children, but also adults, often make mistakes in the choice of operation when solving problems: they sometimes use additive reasoning when they should have used multiplicative reasoning but they can also make the converse mistake, and use multiplicative reasoning when additive reasoning would be appropriate. So, we need to examine research that explains how children can become more successful in choosing the appropriate model to answer a question.

Experts often use diagrams, tables and graphs to help them analyse situations. These resources could support children's thinking about situations. But children seem to have difficulty in using these resources and have to learn how to use them. They have to become literate in the use of these mathematical tools in order to interpret them correctly. A question that has not been addressed in the literature is whether children can learn about using these tools and about analysing situations mathematically at the same time. Research about interpreting tables and graphs has been carried out either to assess students' previous knowledge (or misconceptions) before they are taught or to test ways of making them literate in the use of these tools.

A remarkable exception is found in the work of researchers in the Freudenthal Institute. One of their explicit aims for instruction in mathematics is to help students mathematise situations: i.e. to help them build a model of a situation and later transform it into a model for other situations through their awareness of the relations in the model. They argue that we need to use diagrams, tables and graphs during the process of mathematising situations. These are built by students (with teacher guidance) as they explore the situations rather than presented to the students ready made for interpretation. Students are encouraged to use their implicit knowledge of relations; by building these representations, they can become aware of which models they are using. The process of solution is thus not to choose an operation and calculate but to analyse the relations in the problem and work towards solution. This process allows the students to become aware of the relations that are conserved throughout the different steps.

Streefland worked out in detail how this process would work if students were asked to solve Hart's famous onion soup recipe problem. In this problem, students are presented with a recipe of onion soup for 8 people and asked how much of each ingredient they would need if they were preparing the soup for 6 people. Many students use their everyday knowledge of relations in searching for a solution: they think that you need half of the original recipe (which would serve 4) plus half of this (which would serve 2 people) in order to have a recipe for 6 people. This perfectly sound reasoning is actually a mixture of additive and multiplicative thinking: half of a recipe for 8 serves 4 people (multiplicative reasoning) and half of the latter serves 2 (multiplicative reasoning); 6 people is 2 more than 4 (additive); a recipe for 6 is the same as the recipe for 4 plus the recipe for 2 (additive).

Streefland and his colleagues suggested that diagrams and tables provide the sort of representation that helps students think about the relations in the problem. It is illustrated here by the ratio table showing how much water should be used in the soup. The table can be used to help students become aware that the first two steps in their reasoning are multiplicative: they divide the number of persons in half and also the amount of water in half. Additive reasoning does not work: the transformation from 8 to 4 people would mean subtracting 4 whereas the parallel transformation in the amount of water would be to subtract 1. So the relation is not the same. If they can discover that multiplicative reasoning preserves the relation, whereas additive reasoning does not, they could be encouraged to test whether there is a multiplicative relation that they can use to find the recipe for 6; they could come up with x3, trebling the recipe for 2. Streefland's ratio table can be used as a model for testing if other situations fit this sort of multiplicative reasoning. The table can be expanded to calculate the amounts of the other ingredients.



An alternative approach in curriculum development is to start from formalisations and not to base teaching on students' informal knowledge. The aim of this approach is to establish links between different formal representations of the same relations. A programme proposed by Adjiage and Pluvinage starts with lines divided into segments: students learn how to represent segments with the same fraction even though the lengths of the lines differ (e.g. 3/5 of lines of different lengths). Next they move to using these formal representations in other types of problems: for example, mixtures of chocolate syrup and milk where the number of cups of each ingredient differs but the ratio of chocolate to total number of cups is the same. Finally, students are asked to write abstractions that they learned in these situations and formulate rules for solving the problems that they solved during the lessons. An example of generalisation expected is 'seven divided by four is equal to seven fourths' or '7 ÷ 4 = 7/4'. An example of a rule used in problem solving would be 'Given an enlargement in which a 4 cm length becomes a 7 cm length, then any length to be enlarged has to be multiplied by 7/4.'

There is no systematic research that compares these two very different approaches. Such research would provide valuable insight into how children come to understand relations.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Numbers are used to represent quantities and relations. Primary school children often interpret statements about relations as if they were about quantities and thus make mistakes in solving problems.	**Teaching** Teachers should be aware of children's difficulties in distinguishing between quantities and relations during problem solving.
Many problem situations involve both additive and multiplicative relations; which one is used to solve a problem depends on the question being asked. Both children and adults can make mistakes in selecting additive or multiplicative reasoning to answer a question.	**Teaching** The primary school curriculum should include the study of relations in situations in a more explicit way. **Research** Evidence from experimental studies is needed on which approaches to making students aware of relations in problem situations improve problem solving.
Experts use diagrams, tables and graphs to explore the relations in a problem situation before solving a problem.	**Teaching** The use of tables and graphs in the classroom may have been hampered by the assumption that students must first be literate in interpreting these representations before they can be used as tools. Teachers should consider using these tools as part of the learning process during problem solving. **Research** Systematic research on how students use diagrams, tables and graphs to represent relations during problem solving and how this impacts their later learning is urgently needed. Experimental and longitudinal methods should be combined.
Some researchers propose that informal knowledge interferes with students' learning. They propose that teaching should start from formalisations which are only later applied to problem situations.	**Teaching** Teachers who start from formalisations should try to promote links across different types of mathematical representations through teaching. **Research** There is a need for experimental and longitudinal studies designed to investigate the progress that students make when teaching starts from formalisations rather than from students' informal knowledge and the long-term consequences of this approach to teaching students about relations.

PAPER 5: Understanding space and its representation in mathematics

By Peter Bryant, University of Oxford

Headlines

- Children come to school with a great deal of knowledge about spatial relations. One of the most important challenges in mathematical education is how best to harness this implicit knowledge in lessons about space.

- Children's pre-school implicit knowledge of space is mainly relational. Teachers should be aware of kinds of relations that young children recognise and are familiar with, such as their use of stable background to remember the position and orientation of objects and lines.

- Measuring of length and area poses particular problems for children, even though they are able to understand the underlying logic of measurement. Their difficulties concern iteration of standard units, which is a new idea for them, and also the need to apply multiplicative reasoning to the measurement of area.

- From an early age children are able to extrapolate imaginary straight lines, which allows them to learn how to use Cartesian co-ordinates to plot specific positions in space with no difficulty. However, they need instruction about how to use co-ordinates to work out the relation between different positions.

- Learning how to represent angle mathematically is a hard task for young children, even though angles are an important part of their everyday life. There is evidence that children are more aware of angle in the context of movement (turns) than in other contexts and learn about the mathematics of angle relatively easily in this context. However, children need a great deal of help from to teachers to understand how to relate angles across different contexts.

- An important aspect of learning about geometry is to recognise the relation between transformed shapes (rotation, reflection, enlargement). This also can be difficult, since children's pre-school experiences lead them to recognise the same shapes as equivalent across such transformations, rather than to be aware of the nature of the transformation. However, there is very little research on this important question.

- Another aspect of the understanding of shape is the fact that one shape can be transformed into another, by addition and subtraction of its subcomponents. For example, a parallelogram can be transformed into a rectangle of the same base and height by the addition and subtraction of equivalent triangles and adding two equivalent triangles to a rectangle creates a parallelogram. Research demonstrates that there is a danger that children might learn about these transformations only as procedures without understanding their conceptual basis.

- There is a severe dearth of psychological research on children's geometrical learning. In particular we need long-term studies of the effects of intervention and a great deal more research on children's understanding of transformations of shape.

Understanding space and its representation in mathematics

At school, children often learn formally about matters that they already know a great deal about in an informal and often quite implicit way. Sometimes their existing, informal understanding, which for the most part is based on experiences that they start to have long before going to school, fits well with what they are expected to learn in the classroom. At other times, what they know already, or what they think they know, clashes with the formal systems that they are taught at school and can even prevent them from grasping the significance of these formal systems.

Geometry is a good and an obvious example. Geometry lessons at school deal with the use of mathematics and logic to analyse spatial relations and the properties of shapes. The spatial relations and the shapes in question are certainly a common part of any child's environment, and psychological research has established that from a very early age children are aware of them and quite familiar with them. It has been shown that even very young babies not only discriminate regular geometric shapes but can recognise them when they see them at a tilt, thus co-ordinating information about the orientation of an object with information about the pattern of its contours.

Babies are also able to extrapolate imaginary straight lines (a key geometric skill) at any rate in social situations because they can work out what someone else is looking at and can thus construct that person's line of sight. Another major early achievement by young children is to master the logic that underlies much of the formal analysis of spatial relations that goes on in geometry. By the time they first go to school young children can make logical transitive inferences ($A > B$, $B > C$, therefore $A > C$; $A = B$, $B = C$, therefore $A = C$), which are the logical basis of all measurement. In their first few years at school they also become adept at the logic of inversion ($A + B – B$), which is a logical move that is an essential part of studying the relation between shapes.

Finally, there is strong evidence that most of the information about space that children use and remember in their everyday lives is relational in nature. One good index of this is that children's memory of the orientation of lines is largely based on the relation between these lines and the orientation of stable features in the background. For this reason children find it much easier to remember the orientation of horizontal and vertical lines than of diagonal lines, because horizontal and vertical features are quite common in the child's stable spatial environment. For the same reason, young children remember and reproduce right angles (perpendicular lines) better than acute or obtuse angles. The relational nature of children's spatial perception and memory is potentially a powerful resource for learning about geometry, since spatial relations are the basic subject matter of geometry.

With so much relevant informal knowledge about space and shape to draw on, one might think that children would have little difficulty in translating this knowledge into formal geometrical understanding. Yet, it is not always that easy. It is an unfortunate and well-documented fact that many children have persistent difficulties with many aspects of geometry.

One evidently successful link between young children's early spatial knowledge and their more formal experiences in the classroom is their learning how to use Cartesian co-ordinates to plot positions in two-dimensional space. This causes schoolchildren little difficulty, although it takes some time for them to understand how to work out the relation between two positions plotted in this way.

Other links between informal and formal knowledge are harder for young children. The apparently simple act of measuring a straight line, for example, causes them problems even though they are usually perfectly able to make the appropriate logical moves and understand the importance of one-to-one correspondence, which is an essential part of relating the units on a ruler to the line being measured. One problem here is that they find it hard to understand the idea of iteration: iteration is about repeated measurements, so that a ruler consists of a set of iterated (repeated) units like centimetres. Iteration is necessary when a particular length being measured is longer than the measuring instrument. Another problem is that the one-to-one correspondence involved in measuring a line with a ruler is asymmetrical. The units (centimetres, inches) are visible and clear in the ruler but have to be imagined on the line itself. It is less of a surprise that it also takes children a great deal of time to come to terms with the fact that measurement of area usually needs some form of multiplication, e.g. height × width with rectangles, rather than addition.

The formal concept of angle is another serious stumbling block for children even though they are familiar enough with angles in their everyday spatial environments. The main problem is that they find it hard to grasp that two angles in very different contexts are the same, e.g. themselves turning 90° and the corner of a page in a book. Abstraction is an essential part of geometry but it has very little to do with children's ordinary spatial perception and knowledge.

For much the same reason, decomposing a relatively complex shape into several simpler component shapes – again an essential activity in geometry – is something that many children find hard to do. In their ordinary lives it is usually more important for them to see shapes as unities, rather than to be able to break them up into other shapes. This difficulty makes it hard for them to work out relationships between shapes. For example, children who easily grasp that $a + b - b = a$, nevertheless often fail to understand completely the demonstration that a rectangle and a parallelogram with the same base and height are equal in area because you can transform the parallelogram into the rectangle by subtracting a triangle from one end of the parallelogram and adding an exactly equivalent triangle to the other end.

We know little about children's understanding of transformations of shape or of any difficulties that they might have when they are taught about these transformations. This is a serious gap in research on children's mathematical learning. It is well recognised, however, that children and some adults confuse scale enlargements with enlargements of area. They think that doubling the length of the contour of a geometric shape such as a square or a rectangle also doubles its area, which is a serious misconception. Teachers should be aware of this potential difficulty when they teach children about scale enlargements.

Researchers have been more successful in identifying these obstacles than in showing us how to help children to surmount them. There are some studies of ways of preparing children for geometry in the pre-school period or in the early years at school. This research, however, concentrated on short-term gains in children's geometric understanding and did not answer the question whether these early teaching programmes would actually help children when they begin to learn about geometry in the classroom.

There has also been research on teaching children about angle, mostly in the context of computer-based teaching programmes. One of the most interesting points to come out of this research is that teaching children about angle in terms of movements (turning) is successful, and there is some evidence that children taught this way are quite likely to transfer their new knowledge about angle to other contexts that do not involve movement.

However, there has been no concerted research on how teachers could take advantage of children's considerable spatial knowledge when teaching them geometry. We badly need long-term studies of interventions that take account of children's relational approach to the spatial environment and encourage them to grasp other relations, such as the relation between shapes and the relation between shapes and their subcomponent parts, which go beyond their informal spatial knowledge.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's pre-school knowledge of space is relational. They are skilled at using stable features of the spatial framework to perceive and remember the relative orientation and position of objects in the environment. There is, however, no research on the relation between this informal knowledge and how well children learn about geometry.	**Teaching** Teachers should be aware of the research on children's considerable spatial knowledge and skills and should relate their teaching of geometrical concepts to this knowledge. **Research** There is a serious need for longitudinal research on the possible connections between children's pre-school spatial abilities and how well they learn about geometry at school.
Children already understand the logic of measurement in their early school years. They can make and understand transitive inferences, they understand the inverse relation between addition and subtraction, and they can recognise and use one-to-one correspondence. These are three essential aspects of measurement.	**Teaching** The conceptual basis of measurement and not just the procedures should be an important part of the teaching. Teachers should emphasise transitive inferences, inversion of addition and subtraction and also one-to-one correspondence and should show children their importance. **Research** Psychologists should extend their research on transitive inference, inversion and one-to-one correspondence to geometrical problems, such as measurement of length and area.
Many children have difficulties with the idea of iteration of standard units in measurement.	**Teaching** Teachers should recognise this difficulty and construct exercises which involve iteration, not just with standard units but with familiar objects like cups and hands. **Research** Psychologists should study the exact cause of children's difficulties with iteration.

Recommendations (continued)

Research about mathematical learning	Recommendations for teaching and research
Many children wrongly apply additive reasoning, instead of multiplicative reasoning, to the task of measuring area. Children understand this multiplicative reasoning better when they first think of it as the number of tiles in a row times the number of rows than when they try to use a base times height formula.	**Teaching** In lessons on area measurement, teachers can promote children's use of the reasoning 'number in a row times number of rows' by giving children a number of tiles that is insufficient to cover the area. They should also contrast measurements which do, and measurements which do not, rest on multiplication.
Even very young children can easily extrapolate straight lines and schoolchildren have no difficulty in learning how to plot positions using Cartesian co-ordinates, but it is difficult for them to work out the relation between different positions plotted in this way.	**Teaching** Teachers, using concrete material, should relate teaching about spatial co-ordinates to children's everyday experiences of extrapolating imaginary straight lines. **Research** There is a need for intervention studies on methods of teaching children to work out the relation between different positions, using co-ordinates.
Research on pre-school intervention suggests that it is possible to prepare children for learning about geometry by enhancing their understanding of space and shapes. However, this research has not included long-term testing and therefore the suggestion is still tentative.	**Research** There will have to be long-term predictive and long-term intervention studies on this crucial, but neglected, question
Children often learn about the relation between shapes (e.g. between a parallelogram and a rectangle) as a procedure without understanding the conceptual basis for these transformations.	**Teaching** Children should be taught the conceptual reasons for adding and subtracting shape components when studying the relation between shapes. **Research** Existing research on this topic was done a very long time ago and was not very systematic. We need well-designed longitudinal and intervention studies on children's ability to make and understand such transformations.
There is hardly any research on children's understanding of the transformation of shapes, but there is evidence of confusion in many children about the effects of enlargement: they consider that doubling the length of the perimeter of a square, for example, doubles its area.	**Teaching** Teachers should be aware of the risk that children might confuse scale enlargements with area enlargements. **Research** Psychologists could easily study how children understand transformations like reflection and rotation but they have not done so. We need this kind of research.

PAPER 6: Algebraic reasoning

By Anne Watson, University of Oxford

Headlines

- Algebra is the way we express generalisations about numbers, quantities, relations and functions. For this reason, good understanding of connections between numbers, quantities and relations is related to success in using algebra. In particular, students need to understand that addition and subtraction are inverses, and so are multiplication and division.

- To understand algebraic symbolisation, students have to (a) understand the underlying operations and (b) become fluent with the notational rules. These two kinds of learning, the meaning and the symbol, seem to be most successful when students know what is being expressed and have time to become fluent at using the notation.

- Students have to learn to recognise the different nature and roles of letters as: unknowns, variables, constants and parameters, and also the meanings of equality and equivalence. These meanings are not always distinct in algebra and do not relate unambiguously to arithmetical understandings, Mapping symbols to meanings is not learnt in one-off experiences.

- Students often get confused, misapply, or misremember rules for transforming expressions and solving equations. They often try to apply arithmetical meanings to algebraic expressions inappropriately. This is associated with over-emphasis on notational manipulation, or on 'generalised arithmetic', in which they may try to get concise answers.

Understanding symbolisation

The conventional symbol system is not merely an expression of generalised arithmetic; to understand it students have to understand the meanings of arithmetical operations, rather than just be able to carry them out. Students have to understand 'inverse' and know that addition and subtraction are inverses, and that division is the inverse of multiplication. Algebraic representations of relations between quantities, such as difference and ratio, encapsulate this idea of inverse. Using familiarity with symbolic expressions of these connections, rather than thinking in terms of generalising four arithmetical operations, gives students tools with which to understand commutativity and distributivity, methods of solving equations, and manipulations such as simplifying and expanding expressions.

The precise use of notation has to be learnt as well, of course, and many aspects of algebraic notation are inherently confusing (e.g. $2r$ and r^2). Over-reliance on substitution as a method of doing this can lead students to get stuck with arithmetical meanings and rules, rather than being able to recognise algebraic structures. For example, students who have been taught to see expressions such as:

$97 - 49 + 49$

as structures based on relationships between numbers, avoiding calculation, identifying variation, and having a sense of limits of variability, are able to reason with relationships more securely and at a younger age than those who have focused only on calculation. An expression such as $3x + 4$ is both the answer to a question, an object in itself, and also an algorithm or process for calculating a particular value. This has parallels in arithmetic: the answer to $3 \div 5$ is $3/5$.

Time spent relating algebraic expressions to arithmetical structures, as opposed to calculations, can make a difference to students' understanding. This is especially important when understanding that apparently different expressions can be equivalent, and that the processes of manipulation (often the main focus of algebra lessons) are actually transformations between equivalent forms.

Meanings of letters and signs

Large studies of students' interpretation and use of letters have shown a well-defined set of possible actions. Learners may, according to the task and context:

- try to evaluate them using irrelevant information
- ignore them
- used as shorthand for objects, e.g. a = apple
- treat them as objects
- use a letter as a specific unknown
- use a letter as a generalised number
- use a letter as a variable.

Teachers have to understand that students may use any one of these approaches and students need to learn when these are appropriate or inappropriate. There are conventions and uses of letters throughout mathematics that have to be understood in context, and the statement 'letters stand for numbers' is too simplistic and can lead to confusion. For example:

- it is not always true that different letters have different values
- a letter can have different values in the same problem if it stands for a variable
- the same letter does not have to have the same value in different problems.

A critical shift is from seeing a letter as representing an unknown, or 'hidden', number defined within a number sentence such as:

$3 + x = 8$

to seeing it as a variable, as in $y = 3 + x$, or $3 = y - x$. Understanding x as some kind of generalized number which can take a range of values is seen by some researchers to provide a bridge from the idea of unknown to that of variables. The use of boxes to indicate unknown numbers in simple 'missing number' statements is sometimes helpful, but can also lead to confusion when used for variables, or for more than one hidden number in a statement.

Expressions linked by the 'equals' sign might be not just numerically equal, but also equivalent, yet students need to retain the 'unknown' concept when setting up and solving equations which have finite solutions. For example, $10x - 5 = 5(2x - 1)$ is a statement about equivalence, and x is a variable, but $10x - 5 = 2x + 1$ defines a value of the variable for which this equality is true. Thus x in the second case can be seen as an unknown to be found, but in the first case is a variable. Use of graphical software can show the difference visually and powerfully because the first situation is represented by one line, and the second by two intersecting lines, i.e. one point.

Misuse of rules

Students who rely only on remembered rules often misapply them, or misremember them, or do not think about the meaning of the situations in which they might be successfully applied. Many students will use guess-and-check as a first resort when solving equations, particularly when numbers are small enough to reason about 'hidden numbers' instead of 'undoing' within the algebraic structure. Although this is sometimes a successful strategy, particularly when used in conjunction with graphs, or reasoning about spatial structures, or practical situations, over-reliance can obstruct the development of algebraic understanding and more universally applicable techniques.

Large-scale studies of U.K. school children show that, despite being taught the BIDMAS rule and its equivalents, most do not know how to decide on the order of operations represented in an algebraic expression. Some researchers believe this to be due to not fully understanding the underlying operations, others that it may be due to misinterpretation of expressions. There is evidence from Australia and the United Kingdom that students who are taught to use flow diagrams, and inverse flow diagrams, to construct and reorganise expressions are better able to decide on the order implied by expressions involving combinations of operations. However, it is not known whether students taught this way can successfully apply their knowledge of order in situations in which flow diagrams are inappropriate, such as with polynomial equations, those involving the unknown on 'both sides', and those with more than one variable. To use algebra effectively, decisions about order have to be fluent and accurate.

Misapplying arithmetical meanings to algebraic expressions

Analysis of children's algebra in clinical studies with 12- to 13-year-olds found that the main problems in moving from arithmetic to algebra arose because:

- the focus of algebra is on relations rather than calculations; the relation $a + b = c$ represents three unknown quantities in an additive relationship
- students have to understand inverses as well as operations, so that a hidden value can be found even if the answer is not obvious from knowing number bonds or multiplication facts; $7 + b = 4$ can be solved using knowledge of addition, but $c + 63 = 197$ is more easily solved if subtraction is used as the inverse of addition

- some situations have to be expressed algebraically first in order to solve them. 'My brother is two years older than me, my sister is five years younger than me; she is 12, how old will my brother be in three years' time?' requires an analysis and representation of the relationships before solution. 'Algebra' in this situation means constructing a method for keeping track of the unknown as various operations act upon it.
- letters and numbers are used together, so that numbers may have to be treated as symbols in a structure, and not evaluated. For example, the structure $2(3+b)$ is different from the structure of $6 + 2b$ although they are equivalent in computational terms. Learners have to understand that sometimes it is best to leave number as an element in an algebraic structure rather than 'work it out'.
- the equals sign has an expanded meaning; in arithmetic it is often taken to mean 'calculate' but in algebra it usually means 'is equal to' or 'is equivalent to'. It takes many experiences to recognise that an algebraic equation or equivalence is a statement about relations between quantities, or between combinations of operations on quantities. Students tend to want 'closure' by compressing algebraic expressions into one term instead of understanding what is being expressed.

Expressing generalisations

In several studies it has been found that students understand how to use algebra if they have focused on generalizing with numerical and spatial representations in which counting is not an option. Attempts to introduce symbols to very young students as tools to be used when they have a *need* to express known general relationships, have been successful both for aiding their understanding of symbol use, and understanding the underlying quantitative relations being expressed. For example, some year 1 children first compare and discuss quantities of liquid in different vessels, and soon become able to use letters to stand for unknown amounts in relationships, such as $a > b$; $d = e$; and so on. In another example, older primary children could generalise the well-known questions of how many people can sit round a line of tables, given that there can be two on each side of each table and one at each of the extreme ends. The ways in which students count differ, so the forms of the general statement also differ and can be compared, such as: 'multiply the number of tables by 4 and add 2 or 'it is two times one more than the number of tables'.

The use of algebra to express *known* arithmetical generalities is successful with students who have developed advanced mental strategies for dealing with additive, multiplicative and proportional operations (e.g. compensation as in $82 – 17 = 87 – 17 – 5$). When students are allowed to use their own methods of calculation they often find algebraic structures for themselves. For example, expressing 13×7 as $10 \times 7 + 3 \times 7$, or as $2 \times 7^2 –7$, are enactments of distributivity and learners can represent these symbolically once they know that letters can stand for numbers, though this is not trivial and needs several experiences. Explaining a general result, or structure, in words is often a helpful precursor to algebraic representation.

Fortunately, generalising from experience is a natural human propensity, but the everyday inductive reasoning we do in other contexts is not always appropriate for mathematics. Deconstruction of diagrams and physical situations, and identification of relationships between variables, have been found to be more successful methods of developing a formula than pattern-generalisation from number sequences alone. The use of verbal descriptions has been shown to enable students to bridge between observing relations and writing them algebraically.

Further aspects of algebra arise in the companion summaries, and also in the main body of *Paper 6: Algebraic reasoning*.

Recommendations

Research about mathematical learning	Recommendations for teaching
The bases for using algebraic symbolisation successfully are (a) understanding the underlying operations and relations and (b) being able to use symbolism correctly.	Emphasis should be given to reading numerical and algebraic expressions relationally, rather than computationally. For algebraic thinking, it is more important to understand how operations combine and relate to each other than how they are performed. Teachers should avoid emphasising symbolism without understanding the relations it represents.
Children interpret 'letter stands for number' in a variety of ways, according to the task. Mathematically, letters have several meanings according to context: unknown, variable, parameter, constant.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of children's possible interpretations of letters, and also that when correctly used, letters can have a range of meanings. Teachers should avoid using materials that oversimplify this variety. Hands-on ICT can provide powerful new ways to understand these differences in several representations.
Children interpret '=' to mean 'calculate'; but mathematically '=' means either 'equal to' or 'equivalent to'.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of the difficulties about the '=' sign and use multiple contexts and explicit language. Hands-on ICT can provide powerful new ways to understand these differences in several representations.
Students often forget, misremember, misinterpret situations and misapply rules	Developers of the curriculum, advisory schemes of work and teaching methods need to take into account that algebraic understanding takes time, multiple experiences, and clarity of purpose. Teachers should emphasise situations in which generalisations can be identified and described to provide meaningful contexts for the use of algebraic expressions. Use of software which carries out algebraic manipulations should be explored.
Everyone uses 'guess-&-check' if answers are immediately obvious, once algebraic notation is understood.	Algebra is meaningful in situations for which specific arithmetic cannot be easily used, as an expression of relationships. Focusing on algebra as 'generalised arithmetic', e.g. with substitution exercises, does not give students reasons for using it.

Recommendations (continued)

Research about mathematical learning	Recommendations for teaching
Even very young students can use letters to represent unknowns and variables in situations where they have reasoned a general relationship by relating properties. Research on inductive generalisation from pattern sequences to develop algebra shows that moving from expressing simple additive patterns to relating properties has to be explicitly supported.	Algebraic expressions of relations should be a commonplace in mathematics lessons, particularly to express relations and equivalences. Students need to have multiple experiences of algebraic expressions of general relations based in properties, such as arithmetical rules, logical relations, and so on as well as the well-known inductive reasoning from sequences.

Recommendations for research

The main body of *Paper 6: Algebraic reasoning* includes a number of areas for which further research would be valuable, including the following.

- How does explicit work on understanding relations between quantities enable students to move successfully from arithmetical to algebraic thinking?
- What kinds of explicit work on expressing generality enable students to use algebra?
- What are the longer-term comparative effects of different teaching approaches to early algebra on students' later use of algebraic notation and thinking?
- How do learners' synthesise their knowledge of elementary algebra to understand polynomial functions, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations?
- What useful kinds of algebraic expertise could be developed through the use of computer algebra systems in school?

PAPER 7: Modelling, problem-solving and integrating concepts

By Anne Watson, University of Oxford

Headlines

We have assumed a general educational context which encourages thinking and problem-solving across subjects. A key difference about mathematics is that empirical approaches may solve individual problems, and offer directions for reasoning, but do not themselves lead to *new* mathematical knowledge or mathematical reasoning, or to the power that comes from applying an abstract idea to a situation.

In secondary mathematics, the major issue is not how children learn elementary concepts, but what experiences they have had and how these enable or limit what else can be learnt. That is why we have combined several aspects of secondary mathematics which could be exemplified by particular topics.

- Students have to be fluent in understanding methods and confident about using them to know why and when to apply them, but such application does not automatically follow learning procedures. Students have to understand the situation as well as being able to call on a familiar repertoire of ideas and methods.

- Students have to know some elementary concepts well enough to apply them and combine them to form new concepts in secondary mathematics, but little is known from research about what concepts are essential in this way. Knowledge of a range of functions is necessary for modelling situations.

- Students have to learn when and how to use informal, experiential reasoning and when to use formal, conventional, mathematical reasoning. Without special attention to meanings many students tend to apply visual reasoning, or be triggered by verbal cues, rather than to analyse situations mathematically.

- In many mathematical situations in secondary mathematics, students have to look for relations between numbers and variables and relations between relations and properties of objects, and know how to represent them.

How secondary learners tackle new situations

In new situations students first respond to familiarity in appearance, or language, or context. They bring earlier understandings to bear on new situations, sometimes erroneously. They naturally generalise from what they are offered, and they often over-generalise and apply inappropriate ideas to new situations. They can learn new mathematical concepts either as extensions or integrations of earlier concepts, and/or as inductive generalisations from examples, and/or as abstractions from solutions to problems.

Routine or context?

One question is whether mathematics is learnt better from routines, or from complex contextual situations. Analysis of research which compares how children learn mathematics through being taught routines efficiently (such as with computerised and other learning packages designed to minimise cognitive load) to learning through problem-solving in complex situations (such as through Realistic Mathematics Education) shows that the significant difference is not about the speed and retention of learning but *what* is being learnt. In each approach the main question for progression is whether the student learns new concepts well enough to use and adapt them in future learning and outside mathematics. Both approaches have inherent weaknesses in this respect. These weaknesses will become clear in what follows. However, there are several studies which show that those who develop mathematical methods of enquiry over time can then learn procedures easily and do as well, or better, in general tests.

Problem-solving and modelling

To learn mathematics one has to learn to solve mathematical problems or model situations mathematically. Studies of students' problem solving mainly focus in the successful solution of contextually-worded problems using mathematical methods, rather than using problem solving as a context for learning new concepts and developing mathematical thinking. To solve unfamiliar problems in mathematics, a meta-analysis of 487 studies concluded that for students to be maximally successful:

- problems need to be fully stated with supportive diagrams
- students need to have previous extensive experience in using the representations used
- they have to have relevant basic mathematical skills to use
- teachers have to understand problem-solving methods.

This implies that fluency with representations and skills is important, but also depends on how clearly the problem is stated. In some studies the difficulty is also to do with the underlying concept, for example, in APU tests area problems were difficult with or without diagrams.

To be able to solve problems whose wording does not indicate what to do, students have to be able to read the problem in two ways: firstly, their technical reading skills and understanding of notation have to be good enough; secondly, they have to be able to interpret it to understand the contextual and mathematical meanings. They have to decide whether and how to bring informal knowledge to bear on the situation, or, if they approach it formally, what are the variables and how do they relate. If they are approaching it formally, they then have to represent the relationships in some way and decide how to operate on them.

International research into the use of ICT to provide new ways to represent situations and to see relationships, such as by comparing spreadsheets, graph plotters and dynamic images appear to speed up the process of relating representations through isomorphic reasoning about covariation, and hence the development of understanding about mathematical structures and relations.

Application of earlier learning

Knowing methods

Students who have only routine knowledge may not recognize that it is relevant to the situation. Or they can react to verbal or visual cues without reference to context, such as 'how much?' triggering multiplication rather than division, and 'how many?' always triggering addition. A further problem is that they may not understand the underlying relationships they are using and how these relate to each other. For example, a routine approach to $2 \times 1/3 \times 3/2$ may neither exploit the meaning of fractions nor the multiplicative relation.

Students who have only experience of applying generic problem-solving skills in a range of situations sometimes do not recognize underlying mathematical structures to which they can apply methods used in the past. Indeed, given the well-documented tendency for people to use *ad hoc* arithmetical trial-and-adjustment methods wherever these will lead to reasonable results, it is possible that problem-solving experience may not result in learning new mathematical concepts or working with mathematical structures, or in becoming fluent with efficient methods.

Knowing concepts

Students who have been helped to learn concepts, and can define, recognise and exemplify elementary ideas are better able to use and combine these ideas in new situations and while learning new concepts. However, many difficulties appear to be due to having too limited a range of understanding. Their understanding may be based on examples which have irrelevant features in common, such as the parallel sides of parallelograms always being parallel to the edges of a page. Understanding is also limited by examples being similar to a prototype, rather than extreme cases. Another problem is that students may recognize examples of a concept by focusing too much on visual or verbal aspects, rather than their properties, such as believing that it is possible to construct an equilateral triangle on a nine-pin geoboard because it 'looks like one'.

Robust connections between and within earlier ideas can make it easier to engage with new ideas, but can also hinder if the earlier ideas are limited and inflexible. For example, learning trigonometry involves understanding: the definition of triangle; right-angles; recognizing them in different orientations; what angle means and how it is measured; typical units for measuring lines; what ratio means; similarity of triangles; how ratio is written as a fraction; how to manipulate a multiplicative relationship; what 'sin' (etc.) means as a symbolic representation of a function and so on. Thus knowing about ratios can support learning trigonometry, but if the understanding of 'ratio' is limited to mixing cake recipes it won't help much. To be successful students have to have had enough experience to be fluent, and enough knowledge to use methods wisely.

They become better at problem-solving and modelling when they can:

- draw on knowledge of the contextual situation to identify variables and relationships and/or, through imagery, construct mathematical representations which can be manipulated further
- draw on a repertoire of representations, functions, and methods of operation on these
- have a purpose for the modelling process, so that the relationship between manipulations in the model and changes in the situation can be meaningfully understood and checked for reasonableness.

Knowing how to approach mathematical tasks

To be able to decide when and how to use informal or formal approaches, and how to use prior knowledge, students need to be able to think mathematically about all situations in mathematics lessons. This develops best as an all-encompassing perspective in mathematics lessons, rather than through isolated experiences.

Students have to:

- learn to avoid instant reactions based in superficial visual or verbal similarity
- practice using typical methods of mathematical enquiry explicitly over time
- have experience of mulling problems over time in order to gain insight.

With suitable environments, tools, images and encouragement, learners can and do use their general perceptual, comparative and reasoning powers in mathematics lessons to:

- generalise from what is offered and experienced
- look for analogies
- identify variables
- choose the most efficient variables, those with most connections
- see simultaneous variations
- understand change
- reason verbally before symbolising
- develop mental models and other imagery
- use past experience of successful and unsuccessful attempts
- accumulate knowledge of operations and situations to do all the above successfully

Of course, all the tendencies just described can also go in unhelpful directions and in particular people tend to:

- persist in using past methods and applying procedures without meaning, if that has been their previous mathematical experience
- get locked into the specific situation and do not, by themselves, know what new mathematical ideas can be abstracted from these experiences
- be unable to interpret symbols, text, and other representations in ways the teacher expects
- use additive methods; assume that if one variable increases so will another; assume that all change is linear; confuse quantities.

Recommendations

Research about mathematical learning	Recommendations for teaching
Learning routine methods and learning through complex exploration lead to different kinds of knowledge and cannot be directly compared; neither method necessarily enables learning new concepts or application of powerful mathematics ideas. However, those who have the habit of complex exploration are often able to learn procedures quickly.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of the importance of understanding new concepts, and avoid teaching solely to pass test questions, or using solely problem-solving mathematical activities which do not lead to new abstract understandings. Students should be helped to balance the need for fluency with the need to work with meaning.
Students naturally respond to familiar aspects of mathematics; try to apply prior knowledge and methods, and generalize from their experience.	Teaching should take into account students' natural ways of dealing with new perceptual and verbal information, and the likely misapplications. Schemes of work and assessment should allow enough time for students to adapt to new meanings and move on from earlier methods and conceptualizations.
Students are more successful if they have a fluent repertoire of conceptual knowledge and methods, including representations, on which to draw.	Developers of the curriculum, advisory schemes of work and teaching methods should give time for new experiences and mathematical ways of working to become familiar in several representations and contexts before moving on. Students need time and multiple experiences to develop a repertoire of appropriate functions, operations, representations and mathematical methods in order to solve problems and model situations. Teaching should ensure conceptual understanding as well as 'knowing about', 'knowing how to', and 'knowing how to use'.
Multiple experiences over time enable students to develop new ways to work on mathematical tasks, and to develop the ability to choose what and how to apply earlier learning.	Schemes of work should allow for students to have multiple experiences, with multiple representations over time to develop mathematically appropriate 'habits of mind'.

Recommendations (continued)

Research about mathematical learning	Recommendations for teaching
Students who work in computer-supported multiple representational contexts over time can understand and use graphs, variables, functions and the modelling process. Students who can choose to use available technology are better at problem solving, and have complex understanding of relations, and have more positive views of mathematics.	There are resource implications about the use of ICT. Students need to be in control of switching between representations and comparisons of symbolic expression in order to understand the syntax and the concept of functions. The United Kingdom may be lagging behind the developed world in exploring the use of spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

Recommendations for research

Application of research findings about problem-solving, modelling and conceptual learning to current curriculum developments in the United Kingdom suggests that there may be different outcomes in terms of students' ability to solve quantitative and spatial problems in realistic contexts. However, there is no evidence to convince us that the new National Curriculum in England will lead to better *conceptual* understanding of mathematics, either at the elementary levels, which are necessary to learn higher mathematics, or at higher levels which provide the confidence and foundation for further mathematical study. Where contextual and exploratory mathematics, integrated through the curriculum, *do* lead to further conceptual learning it is related to conceptual learning being a rigorous focus for curriculum and textbook design, and in teacher preparation, such as in China, Japan, Singapore, and the Netherlands, or in specifically designed projects based around such aims.

In the main body of *Paper 7: Modelling, solving problems and learning new concepts in secondary mathematics* there are several questions for future research, including the following.

- What are the key conceptual understandings for success in secondary mathematics, from the point of view of learning?
- How do students learn new ideas in mathematics at secondary level that depend on combinations of earlier concepts?
- What evidence is there of the characteristics of mathematics teaching at higher secondary level which contribute both to successful conceptual learning and application of mathematics?

Notes

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