Nuffield Foundation

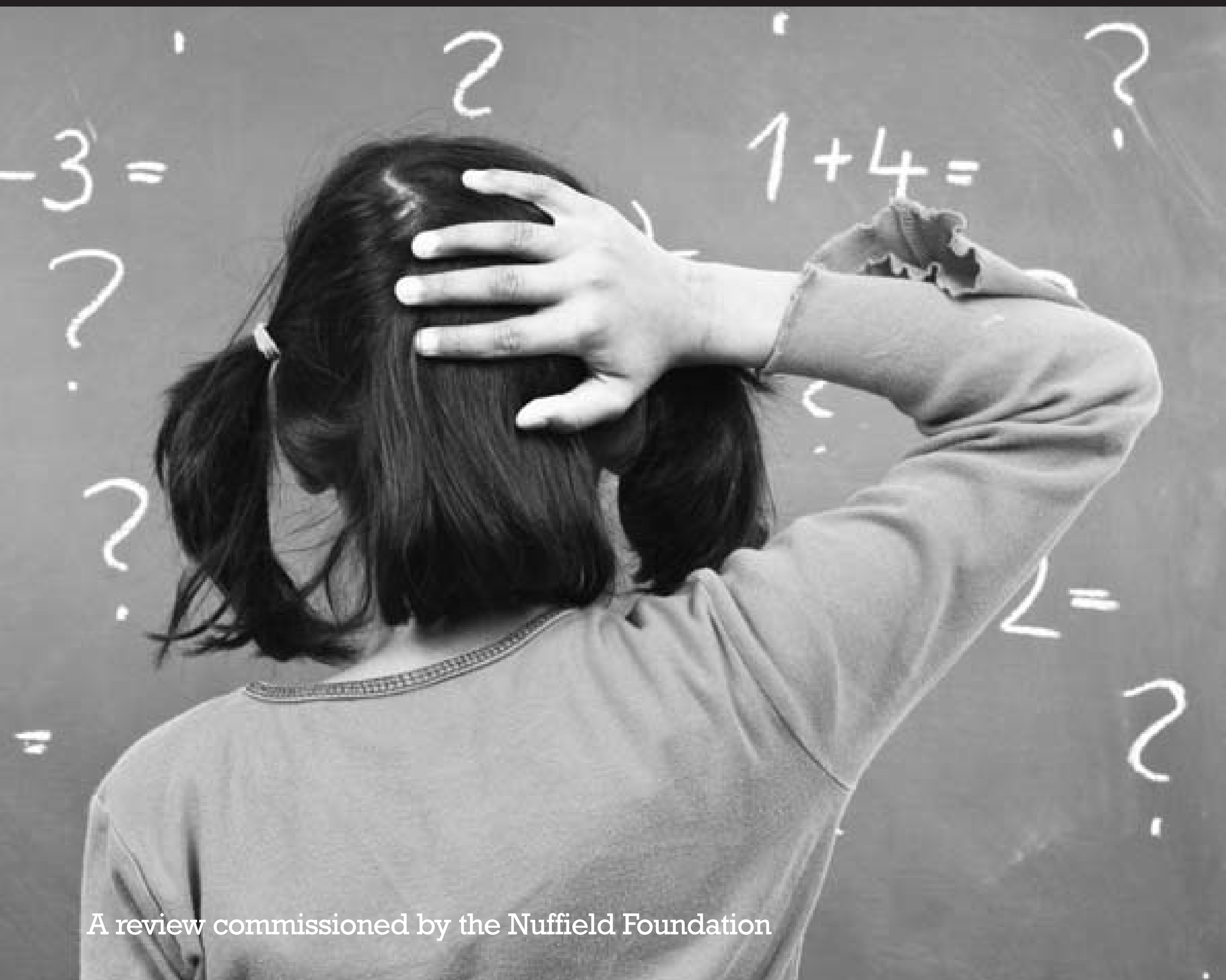
4

Key understandings in

mathematics learning

Paper 4: Understanding relations and their graphical representation

By Terezinha Nunes and Peter Bryant, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

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Contents

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 4: Understanding relations and their graphical representation

Headlines

- Children have greater difficulty in understanding relations than in understanding quantities. This is true in the context of both additive and multiplicative reasoning problems.
- Primary and secondary school students often apply additive procedures to solve multiplicative reasoning problems and also apply multiplicative procedures to solve additive reasoning problems.
- Explicit instruction to help students become aware of relations in the context of additive reasoning problems can lead to significant improvement in children's performance.
- The use of diagrams, tables and graphs to represent relations facilitates children's thinking about and discussing the nature of the relations between quantities in problems.
- Excellent curriculum development work has been carried out to design instruction to help students develop awareness of their implicit knowledge of multiplicative relations. This programme has not been systematically assessed so far.
- An alternative view is that students' implicit knowledge should not be the starting point for students to learn about proportional relations; teaching should focus on formalisations rather than informal knowledge and seek to connect mathematical formalisations with applied situations only later.
- There is no research comparing the results of these diametrically opposed ideas.

Children need to learn to co-ordinate their knowledge of numbers with their understanding of quantities. This is critical for mathematics learning in primary school so that they can use their understanding of quantities to support their knowledge of numbers and *vice versa*. But this is not all that students need to learn to be able to use mathematics sensibly. Using mathematics also involves thinking about relations between quantities. Research shows quite unambiguously that it is more difficult for children to solve problems that involve relations than to solve problems that involve only quantities.

A simple problem about quantities is: Paul had 5 marbles. He played two games with his friend. In the first game, he won 6 marbles. In the second game he lost 4 marbles. How many marbles does he have now? The same numerical information can be used differently, making the problem into one which is all about relations: Paul played three games of marbles. In the first game, he won 5 marbles. In the second game, he won 6. In the third game, he lost 4. Did he end up winning or losing marbles? How many?

The arithmetic that children need to use to solve is the same in both problems: add 5 and 6 and subtract 4. But the second problem is significantly more difficult for children because it is all about relations. They don't know how many marbles Paul actually had at any time, they only know that he had 5 more after the first game than before, and 6 more after the second game, and 4 fewer after the third game. Some children say that this problem cannot be solved because we don't know how many marbles Paul had to begin with: they recognise that it is possible to operate on quantities, but do not recognise that it is possible to operate on relations. Why should this be so?

One possible explanation is the way in which we express relations. When we speak about quantities, we say that Paul won marbles or lost marbles; these are two opposite statements. When we speak about relations, statements that use opposite words may mean the same thing: after winning 5 marbles, we can say that Paul now has 5 more marbles or that before he had 5 fewer. In order to grasp the concept of relations fully, students must be able to view these two different statements as meaning the same thing. Research shows that some students are able to treat these different statements as having the same meaning but others find this difficult. Students who realise that the two statements mean the same thing are more successful in solving problems about relations.

A second plausible explanation is that many children do not distinguish clearly between quantities and relations when they use numbers. When they are given a problem about relations, they interpret the relations as quantities. If they are given a problem like 'Tom, Fred, and Rhoda put their apples into a bag. Tom and Fred together had 17 more apples than Rhoda. Tom had 7 apples. Rhoda had 5 apples. How many apples did Fred have?', they write down that Tom and Fred had 17 apples together (instead of 17 more than Rhoda). When they make this interpretation error, the problem seems very easy: if Tom had 7, Fred had 10. The information about Rhoda seems irrelevant. But of course this is not the solution. It is possible to teach children to represent quantities and relations differently, and thus to distinguish the two: for example, they can be taught to write 'plus 17' to show that this is not a quantity but a relation. Children aged seven to nine years can adopt this notation and at the same time improve their ability to solve relational problems. However, even after this teaching, they still seem to be tempted to interpret relations as quantities. So, learning to represent relations helps children take a step towards distinguishing relations and quantities but they need plenty of opportunity to think about this distinction.

A third difficulty is that relational thinking involves building a model of a problem situation in order to treat the relations in the problem mathematically. In primary school, children have little opportunity to explore situations in their mathematics lessons before solving a problem. If they make a mistake in solving a problem when their computation was correct, the error is explained as 'choice of the wrong operation', but the wrong choice of operation is a symptom, not an explanation for what went wrong during problem solving.

Models of situations are ways of thinking about them, and more than one way may be appropriate. It all depends on the question that we want to answer. Suppose there are 12 girls and 18 boys in a class and they are assigned to single-sex groups during French lessons. If there were not enough books for all of them and the Head Teacher decided to give 4 books to the girls and 6 books to the boys, would this be fair? If you give one book to each girl, there are 8 girls left without books; if you give one book to each boy, there are 12 boys left without books. This seems unfair. If you ask all the children to share, 3 girls will share one book and 3 boys will share one book. This seems fair. The first model is additive: the questions it answers are 'How many more girls than books?' and 'How many more boys than books?' The second model is multiplicative: it examines the ratio between girls and books and the ratio between boys and books. If the Head Teacher is planning to buy more books, she needs an additive model. If the Head Teacher is not planning to buy more books, the ratio is more informative. A model of a situation is constructed by the problem solver for a purpose; additive and multiplicative relations answer different questions about the same situation.

Children, but also adults, often make mistakes in the choice of operation when solving problems: they sometimes use additive reasoning when they should have used multiplicative reasoning but they can also make the converse mistake, and use multiplicative reasoning when additive reasoning would be appropriate. So, we need to examine research that explains how children can become more successful in choosing the appropriate model to answer a question.

Experts often use diagrams, tables and graphs to help them analyse situations. These resources could support children's thinking about situations. But children seem to have difficulty in using these resources and have to learn how to use them. They have to become literate in the use of these mathematical tools in order to interpret them correctly. A question that has not been addressed in the literature is whether children can learn about using these tools and about analysing situations mathematically at the same time. Research about interpreting tables and graphs has been carried out either to assess students' previous knowledge (or misconceptions) before they are taught or to test ways of making them literate in the use of these tools.

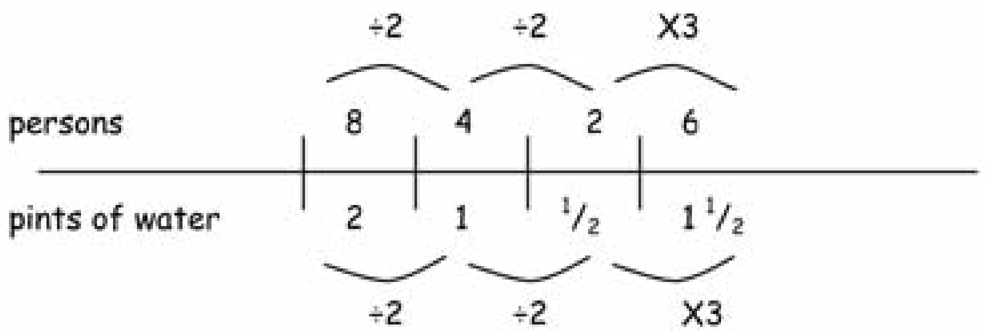
A remarkable exception is found in the work of researchers in the Freudenthal Institute. One of their explicit aims for instruction in mathematics is to help students mathematise situations: i.e. to help them build a model of a situation and later transform it into a model for other situations through their awareness of the relations in the model. They argue that we need to use diagrams, tables and graphs during the process of mathematising situations. These are built by students (with teacher guidance) as they explore the situations rather than presented to the students ready made for interpretation. Students are encouraged to use their implicit knowledge of relations; by building these representations, they can become aware of which models they are using. The process of solution is thus not to choose an operation and calculate but to analyse the relations in the problem and work towards solution. This process allows the students to become aware of the relations that are conserved throughout the different steps.

Streefland worked out in detail how this process would work if students were asked to solve Hart's famous onion soup recipe problem. In this problem, students are presented with a recipe of onion soup for 8 people and asked how much of each ingredient they would need if they were preparing the soup for 6 people. Many students use their everyday knowledge of relations in searching for a solution: they think that you need half of the original recipe (which would serve 4) plus half of this (which would serve 2 people) in order to have a recipe for 6 people. This perfectly sound reasoning is actually a mixture of additive and multiplicative thinking: half of a recipe for 8 serves 4 people (multiplicative reasoning) and half of the latter serves 2 (multiplicative reasoning); 6 people is 2 more than 4 (additive); a recipe for 6 is the same as the recipe for 4 plus the recipe for 2 (additive).

Streefland and his colleagues suggested that diagrams and tables provide the sort of representation that helps students think about the relations in the problem. It is illustrated here by the ratio table showing how much water should be used in the soup. The table can be used to help students become aware that the first two steps in their reasoning are multiplicative: they divide the number of persons in half and also the amount of water in half. Additive reasoning does not work: the transformation from 8 to 4 people would mean subtracting 4 whereas the parallel transformation in the amount of water would be to subtract 1. So the relation is not the same. If they can discover that multiplicative reasoning preserves the relation, whereas additive reasoning does not, they could be encouraged to test whether there is a multiplicative relation that they can use to find the recipe for 6; they could come up with x3, trebling the recipe for 2. Streefland's ratio table can be used as a model for testing if other situations fit this sort of multiplicative reasoning. The table can be expanded to calculate the amounts of the other ingredients.

An alternative approach in curriculum development is to start from formalisations and not to base teaching on students' informal knowledge. The aim of this approach is to establish links between different formal representations of the same relations. A programme proposed by Adjiage and Pluvinage starts with lines divided into segments: students learn how to represent segments with the same fraction even though the lengths of the lines differ (e.g. 3/5 of lines of different lengths). Next they move to using these formal representations in other types of problems: for example, mixtures of chocolate syrup and milk where the number of cups of each ingredient differs but the ratio of chocolate to total number of cups is the same. Finally, students are asked to write abstractions that they learned in these situations and formulate rules for solving the problems that they solved during the lessons. An example of generalisation expected is 'seven divided by four is equal to seven fourths' or '7 ÷ 4 = 7/4'. An example of a rule used in problem solving would be 'Given an enlargement in which a 4 cm length becomes a 7 cm length, then any length to be enlarged has to be multiplied by 7/4.'

There is no systematic research that compares these two very different approaches. Such research would provide valuable insight into how children come to understand relations.



Recommendations

Research about mathematical learning	Recommendations for teaching and research
Numbers are used to represent quantities and relations. Primary school children often interpret statements about relations as if they were about quantities and thus make mistakes in solving problems.	**Teaching** Teachers should be aware of children's difficulties in distinguishing between quantities and relations during problem solving.
Many problem situations involve both additive and multiplicative relations; which one is used to solve a problem depends on the question being asked. Both children and adults can make mistakes in selecting additive or multiplicative reasoning to answer a question.	**Teaching** The primary school curriculum should include the study of relations in situations in a more explicit way. **Research** Evidence from experimental studies is needed on which approaches to making students aware of relations in problem situations improve problem solving.
Experts use diagrams, tables and graphs to explore the relations in a problem situation before solving a problem.	**Teaching** The use of tables and graphs in the classroom may have been hampered by the assumption that students must first be literate in interpreting these representations before they can be used as tools. Teachers should consider using these tools as part of the learning process during problem solving. **Research** Systematic research on how students use diagrams, tables and graphs to represent relations during problem solving and how this impacts their later learning is urgently needed. Experimental and longitudinal methods should be combined.
Some researchers propose that informal knowledge interferes with students' learning. They propose that teaching should start from formalisations which are only later applied to problem situations.	**Teaching** Teachers who start from formalisations should try to promote links across different types of mathematical representations through teaching. **Research** There is a need for experimental and longitudinal studies designed to investigate the progress that students make when teaching starts from formalisations rather than from students' informal knowledge and the long-term consequences of this approach to teaching students about relations.

Understanding relations and their graphical representation

Relations and their importance in mathematics

In our analysis of how children come to understand natural and rational numbers, we examined the connections that children need to make between quantities and numbers in order to understand what numbers mean. Numbers are certainly used to represent quantities, but they are also used to represent relations. The focus of this section is on the use of numbers to represent relations. Relations do not have to be quantified: we can simply say, for example, that two quantities are equivalent or different. This is a qualitative statement about the relation between two quantities. But relations can be quantified also: if there are 20 children in the class and 17 books, we can say that there are 3 more children than books. The number 3 quantifies the additive relation between 17 and 20 and so we can say that 3 quantifies a relation.

When we use numbers to represent quantities, the numbers are the result of a measurement operation. Measures usually rely on culturally developed systems of representation. In order to measure discrete quantities, we count their units, and in order to measure continuous quantities, we use systems that have been set up to allow us to represent them by a number of conventional units. Measures are usually described by a number followed by a noun, which indicates the unit of quantity the number refers to: 5 children, 3 centimetres, 200 grams. And we can't replace the noun with another noun without changing what we are talking about. When we quantify a relation, the number does not refer to a quantity. We can say '3 more children than books' or '3 books fewer than children': it makes no difference which noun comes after the number because the number refers to the relation between the two quantities, how many more or fewer.

When we use qualitative statements about the relations between two quantities, the quantities may or may not have been expressed numerically. For example, we can look at the children and the books in the class and know that there are more children without counting them, especially if the difference is quite large. So we can say that there are more children than books without knowing how many children or how many books. But in order to quantify a relation between two quantities, the quantities need to be measurable, even if, in the case of differences, we can evaluate the relationship without actually measuring them. The ability to express the relationship quantitatively, without knowing the actual measures, is one of the roots of algebra (see Paper 5). For this reason, we will often use the term 'measures' in this section, instead of 'quantities', to refer to quantities that are represented numerically.

It is perfectly possible that when children first appear to succeed in quantifying relations, they are actually still thinking about quantities: when they say '3 children more than books', they might be thinking of the poor little things who won't have a book when the teacher shares the books out, not of the relation between the number of books and the number of children. This hypothesis is consistent with results of studies by Hudson (1983), described in Paper 2: young children are quite able to answer the question 'how many birds won't get worms' but they can't tell 'how many more birds than worms'. We, as adults, may think that they understand something about relations when they answer the first question, but they may be talking about quantities, i.e. the number of birds that won't get worms.

There is no doubt to us that children must grasp how numbers and quantities are connected in order to understand what numbers mean. But mathematics is not only about representing quantities with numbers. A major use of mathematics is to manipulate numbers that represent relations and arrive at conclusions without having to operate directly on the quantities. Attributing a number to a quantity is measuring; quantifying relations and manipulating them is quantitative reasoning. To quote Thompson (1993): 'Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships… A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities' (p. 165). Elsewhere, Thompson (1994) emphasised that 'a quantitative operation is nonnumerical; it has to do with the *comprehension* [italics in the original] of a situation. Numerical operations (which we have termed measurement operations) are used to evaluate a quantity' (p. 187–188).

In order to reach the right conclusions in quantitative reasoning, one must use an appropriate representation of the relations between the quantities, and the representation depends on what we want to know about the relation between the quantities. Suppose you want to know whether you are paying more for your favourite chocolates at one shop than another, but the boxes of chocolates in the two shops are of different sizes. Of course the bigger box costs more money, but are you paying more for each chocolate? You don't know unless you quantify the relation between price and chocolates. This relation, price per chocolate, is not quantified in the same way as the relation 'more children than books'. When you want to know how many children won't have books, you subtract the number of books from the number of children (or vice versa). When you want to know the price per chocolate, you shouldn't subtract the number of chocolates from the price (or vice versa); you should divide the price by the number of chocolates. Quantifying relations depends on the nature of the question you are asking about the quantities. If you are asking how many more, you use subtraction; if you are asking a rate question, such as price per chocolate, you use division. So quantifying relations can be done by additive or multiplicative reasoning. Additive reasoning tells us about the difference between quantities; multiplicative reasoning tells us about the ratio between quantities. The focus of this section is on multiplicative reasoning but a brief discussion of additive relations will be included at the outset to illustrate the difficulties that children face when they need to quantify and operate on relations. However, before we turn to the issue of quantification of relations, we want to say why we use the terms additive and multiplicative reasoning, instead of speaking about the four arithmetic operations.

Mathematics educators (e.g. Behr, Harel, Post and Lesh, 1994; Steffe, 1994; Vergnaud, 1983) include under the term 'additive reasoning' those problems that are solved by addition and subtraction and under the term 'multiplicative reasoning' those that are solved by multiplication and division. This way of thinking, focusing on the problem structure rather than on the arithmetic operations used to solve problems, has become dominant in mathematics education research in the last three decades or so. It is based on some assumptions about how children learn mathematics, three of which are made explicit here. First, it is assumed that in order to understand addition and subtraction properly, children must also understand the inverse relation between them; similarly, in order to understand multiplication and division, children must understand that they also are the inverse of each other. Thus a focus on specific and separate operations, which was more typical of mathematics education thinking in the past, is justified only when the focus of teaching is on computation skills. Second, it is assumed that the links between addition and subtraction, on one hand, and multiplication and division, on the other, are conceptual: they relate to the connections between quantities within each of these domains of reasoning. The connections between addition and multiplication and those between subtraction and division are procedural: you can multiply by carrying out repeated additions and divide by using repeated subtractions. Finally, it is assumed that, in spite of the procedural links between addition and multiplication, these two forms of reasoning are distinct enough to be considered as separate conceptual domains. So we will use the terms additive and multiplicative reasoning and relations rather than refer to the arithmetic operations.

Quantifying additive relations

The literature about additive reasoning consistently shows that compare problems, which involve relations between quantities, are more difficult than those that involve combining sets or transformations. This literature was reviewed in Paper 1. Our aim in taking up this theme again here is to show that there are three sources of difficulties for students in quantifying additive relations:

- to interpret relational statements as such, rather than to interpret them as statements about quantities
- to transform relational statements into equivalent statements which help them think about the problem in a different way
- to combine two relational statements into a third relational statement without falling prey to the temptation of treating the result as a statement about a quantity.

This discussion in the context of additive reasoning illustrates the role of relations in quantitative reasoning. The review is brief and selective, because the main focus of this section is on multiplicative reasoning.

Interpreting relational statements as quantitative statements

Compare problems involve two quantities and a relation between them. Their general format is: A had *x*; B has *y*; the relation between A and B is *z*. This allows for creating a number of different compare problems. For example, the simplest compare problems are of the form: Paul has 8 marbles; Alex had 5 marbles; how many more does Paul have than Alex? or How many fewer does Alex have than Paul? In these problems, the quantities are known and the relation is the unknown.

Carpenter, Hiebert and Moser (1981) observed that 53% of the first grade (estimated age about 6 years) children that they assessed in compare problems answered the question 'how many more does A have than B' by saying the number that A has. This is the most common mistake reported in the literature: the relational question is answered as a quantity mentioned in the problem. The explanation for this error cannot be children's lack of knowledge of addition and subtraction, because about 85% of the same children used correct addition and subtraction strategies when solving problems that involved joining quantities or a transformation of an initial quantity. Carpenter and Moser report that many of the children did not seem to know what to do when asked to solve a compare problem.

Transforming relational statements into equivalent relational statements

Compare problems can also state how many items A has, then the value of the relation between A's and B's quantities, and then ask how much B has. Two problems used by Vershaffel (1994) will be used to illustrate this problem type. In the problem 'Chris has 32 books. Ralph has 13 more books than Chris. How many books does Ralph have?', the relation is stated as '13 more books' and the answer is obtained by addition; this problem type is referred to by Lewis and Mayer (1987) as involving consistent language. In the problem 'Pete has 29 nuts. Pete has 14 more nuts than Rita. How many nuts does Rita have?', the relation is stated as '14 more nuts' but the answer is obtained by subtraction; this problem types is referred to as involving inconsistent language. Verschaffel found that Belgian students in sixth grade (aged about 12) gave 82% correct responses to problems with consistent language and 71% correct responses to problems with inconsistent language. The operation itself, whether it was addition or subtraction, did not affect the rate of correct responses.

Lewis and Mayer (1987) have argued that the rate of correct responses to relational statements with consistent or inconsistent language varies because there is a higher cognitive load in processing inconsistent sentences. This higher cognitive load is due to the fact that the subject of the sentence in the question 'how many nuts does Rita have?' is the object of the relational sentence 'Pete has 14 more nuts than Rita'. It takes more effort to process these two sentences than other two, in which the subject of the question is also the subject of the relational statement. They provided some evidence for this hypothesis, later confirmed by Verschaffel (1994), who also asked the students in his study to retell the problem after the students had already answered the question.

In the problems where the language was consistent, almost all the students who gave the right answer simply repeated what the researcher had said: there was no need to rephrase the problem. In the problems where the language was inconsistent, about half of the students (54%) who gave correct answers retold the problem by rephrasing it appropriately. Instead of saying that 'Pete has 14 nuts more than Rita', they said that 'Rita has 14 nuts less than Pete', and thus made Rita into the subject of both sentences. Verschaffel interviewed some of the students who had used this correct rephrasing by showing them the written problem that he had read and asking them whether they had said the same thing. Some said that they changed the phrase intentionally because it was easier to think about the question in this way; they stressed that the meaning of the two sentences was the same. Other students became confused, as if they had said something wrong, and were no longer certain of their answers. In conclusion, there is evidence that at least some students do reinterpret the sentences as hypothesised by Lewis and Mayer; some do this explicitly and others implicitly. However, almost as many students reached correct answers without seeming to rephrase the problem, and may not experience the extra cognitive load predicted.

It is likely that, under many conditions, we rephrase relational statements when solving problems. So two significant findings arise from these studies:

- rephrasing relational statements seems to be a strategy used by some people, which may place extra cognitive demands on the problem solver but nevertheless helps in the search for a solution

- rephrasing may be done intentionally and explicitly, as a strategy, but may also be carried out implicitly and apparently unintentionally, producing uncertainty in the problem solvers' minds if they are asked about the rephrasing.

Combining relational statements into a third relational statement

Compare problems typically involve two quantities and a relation between them but it is possible to have problems that require children to work with more quantities and relations than these simpler problems. In these more complex problems, it may be necessary to combine two relational statements to identify a third one.

Thompson (1993) analysed students' reasoning in complex comparison problems which involved at least three quantities and three relations. His aim was to see how children interpreted complex relational problems and how their reasoning changed as they tackled more problems of the same type. To exemplify his problems, we quote the first one: 'Tom, Fred, and Rhoda combined their apples for a fruit stand. Fred and Rhoda together had 97 more apples than Tom. Rhoda had 17 apples. Tom had 25 apples. How many apples did Fred have?' (p. 167). This problem includes three quantities (Tom's, Fred's and Rhoda's apples) and three relations (how many more Fred and Rhoda have than Tom; how many fewer Rhoda has than Tom; a combination of these two relations). He asked six children who had achieved different scores in a pre-test (three with higher and three with middle level scores) sampled from two grade levels, second (aged about seven) and fifth (aged about nine) to discuss six problems presented over four different days. The children were asked to think about the problems, represent them and discuss them.

On the first day the children went directly to trying out calculations and represented the relations as quantities: the statement '97 more apples than Tom' was interpreted as '97 apples'. They did not know how to represent '97 more'. This leads to the conclusion that Fred has 80 apples because Rhoda has 17. On the second day, working with problems about marbles won or lost during the games, the researcher taught the children to use representations by writing, for example, 'plus 12' to indicate that someone had won 12 marbles and 'minus 1' to indicate that someone had lost 1 marble. The children were able to work with these representations with the researcher's support, but when they combined two statements, for example minus 8 and plus 14, they thought that the answer was 6 marbles (a quantity), instead of plus six (a relation). So at first they represented relational statements as statements about quantities, apparently because they did not know how to represent relations. However, after having learned how to represent relational statements, they continued to have difficulties in thinking only relationally, and unwittingly converted the result of operations on relations into statements about quantities. Yet, when asked whether it would always be true that someone who had won 2 marbles in a game would have 2 marbles, the children recognised that this would not necessarily be true. They did understand that relations and quantities are different but they interpreted the result of combining two relations as a quantity.

Thompson describes this tension between interpreting numbers as quantities or relations as the major difficulty that the students faced throughout his study. When they seemed to understand 'difference' as a relation between two quantities arrived at by subtraction, they found it difficult to interpret the idea of 'difference' as a relation between two relations. The children could correctly answer, when asked, that if someone has 2 marbles more than another person, this does not mean that he has two marbles; however, after combining two relations (minus 8 and plus 14), instead of saying that this person ended up with plus 6 marbles, they said that he now had 6 marbles.

Summary

1 At first, children have difficulties in using additive reasoning to quantify relations; when asked about a relation, they answer about a quantity.

2 Once they seem to conquer this, they continue to find it difficult to combine relations and stay within relational reasoning: the combination of two relations is often converted into a statement about a quantity.

3 So children's difficulties with relations are not confined to multiplicative reasoning: they are also observed in the domain of additive reasoning.

Quantifying multiplicative relations

Research on how children quantify multiplicative relations has a long tradition. Piaget and his colleagues (Inhelder and Piaget, 1958; Piaget and Inhelder, 1975) originally assumed that children first think of quantifying relations additively and can only think of relations multiplicatively at a later age. This hypothesis led to the prediction of an 'additive phase' in children's solution to multiplicative reasoning problems, before they would be able to conceive of two variables as linked by a multiplicative relation. This hypothesis led to much research on the development of proportional reasoning, which largely supported the claim that many younger students offer additive solutions to proportions problems (e.g. Hart, 1981 b; 1984; Karplus and Peterson, 1970; Karplus, Pulos and Stage, 1983; Noelting, 1980 *a* and *b*). These results are not disputed but their interpretation will be examined in the next sections of this paper because current studies suggest an alternative interpretation.

Work carried out mostly by Lieven Verschaffel and his colleagues (e.g. De Bock, Verschaffel *et al.*, 2002; 2003) shows that students also make the converse mistake, and multiply when they should be adding in order to solve some relational problems. This type of error is not confined to young students: pre-service elementary school teachers in the United States (Cramer, Post and Currier, 1993) made the same sort of mistake when asked to solve the problem: *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?* The relation between Sue's and Julie's numbers of laps should be quantified additively: because they were running at the same speed, this difference would (in principle) be constant. However, 32 of 33 pre-service teachers answered 45 (15 × 3), apparently using the ratio between the first two measures (9 and 3 laps) to calculate Sue's laps. This latter type of mistake would not be predicted by Piaget's theory.

The hypothesis that we will pursue in this chapter, following authors such as Thompson 1994) and Vergnaud (1983), is that additive and multiplicative reasoning have different origins. Additive reasoning stems from the actions of joining, separating, and placing sets in one-to-one correspondence. Multiplicative reasoning stems from the action of putting two variables in one-to-many correspondence (one-to-one is just a particular case), an action that keeps the ratio between the variables constant. Thompson (1994) made this point forcefully in his discussion of quantitative operations: 'Quantitative operations originate in actions: The quantitative operation of combining two quantities additively originates in the actions of putting together to make a whole and separating a whole to make parts; the quantitative operation of comparing two quantities additively originates in the action of matching two quantities with the goal of determining excess or deficits; the quantitative operation of comparing two quantities multiplicatively originates in matching and subdividing with the goal of sharing. As one interiorizes actions, making mental operations, these operations in the making imbue one with the ability to comprehend situations representationally and enable one to draw inferences about numerical relationships that are not present in the situation itself' (pp. 185–186).

We suggest that, if students solve additive and multiplicative reasoning problems successfully but they are guided by implicit models, they will find it difficult to distinguish between the two models. According to Fischbein (1987), implicit models and informal reasoning provide a starting point for learning, but one of the aims of mathematics teaching in primary school is to help students formalize their informal knowledge (Treffers, 1987). In this process, the models will change and become more explicitly connected to the systems of representations used in mathematics.

In this section, we analyse how students establish and quantify relations between quantities in multiplicative reasoning problems. We first discuss the nature of multiplicative reasoning and present research results that describe how children's informal knowledge of multiplicative relations develops. In the subsequent section, we discuss the representation of multiplicative relations in tables and graphs. Next we analyse how children establish other relations between measures, besides linear relations. The final section sets out some hypotheses about the nature of the difficulty in dealing with relations in mathematics and a research agenda for testing current hypotheses systematically.

The development of multiplicative reasoning

Multiplicative reasoning is important in many ways in mathematics learning. Its role in understanding numeration systems with a base and place value was already discussed in Paper 2. In this section, we focus on a different role of multiplicative reasoning in mathematics learning, its role in understanding relations between measures or quantities, which has already been recognised by different researchers (e.g. Confrey, 1994; Thompson, 1994; Vergnaud, 1983; 1994).

Additive and multiplicative reasoning problems are essentially different: additive reasoning is used in one-variable problems, when quantities of the same kind are put together, separated or compared, whereas multiplicative reasoning involves two variables in a fixed-ratio to each other. Even the simplest multiplicative reasoning problems involve two variables in a fixed ratio. For example, in the problem 'Hannah bought 6 sweets; each sweet costs 5 pence; how much did she spend?' there are two variables, number of sweets and price per sweet. The problem would be solved by a multiplication if, as in this example, the total cost is unknown. The same problem situation could be presented with a different unknown quantity, and would then be solved by means of a division: 'Hannah bought some sweets; each sweet costs 5p; she spent 30p; how many sweets did she buy?'

Even before being taught about multiplication and division in school, children can solve multiplication and division problems such as the one about Hannah. They use the schema of one-to-many correspondence.

Different researchers have investigated the use of one-to-many correspondences by children to solve multiplication and division problems before they are taught about these operations in school. Piaget's work (1952), described in Paper 3, showed that children can understand multiplicative equivalences: they can construct a set A equivalent to a set B by putting the elements in A in the same ratio that B has to a comparison set.

Frydman and Bryant (1988; 1994) also showed that young children can use one-to-many correspondences to create equivalent sets. They used sharing in their study because young children seem to have much experience with correspondence when sharing. In a sharing situation, children typically use a one-for-you one-for-me procedure, setting the shared elements (sweets) into one-to-one correspondence with the recipients (dolls). Frydman and Bryant observed that children in the age range five to seven years became progressively more competent in dealing with one-to-many correspondences and equivalences in this situation. In their task, the children were asked to construct equivalent sets but the units in the sets were of a different value. For example, one doll only liked her sweets in double units and the second doll liked his sweets in single units. The children were able to use one-to-many correspondence to share fairly in this situation: when they gave a double to the first doll, they gave two singles to the second. This flexible use of correspondence to construct equivalent sets was interpreted by Frydman and Bryant as an indication that the children's use of the procedure was not merely a copy of previously observed and rehearsed actions: it reflected an understanding of how one-to-many correspondences can result in equivalent sets. They also replicated one of Piaget's previous findings: some children who succeed with the 2:1 ratio found the 3:1 ratio difficult. So the development of the one-to-many correspondence schema does not happen in an all-or-nothing fashion.

Kouba (1989) presented young children in the United States, in first, second and third grade (aged about six to eight years), with multiplicative reasoning problems that are more typical of those used in school; for example: in a party, there were 6 cups and 5 marshmallows in each cup; how many marshmallows were there?

Kouba analysed the children's strategies in great detail, and classified them in terms of the types of actions used and the level of abstraction. The level of abstraction varied from direct representation (i.e. all the information was represented by the children with concrete materials), through partial representation (i.e. numbers replaced concrete representations for the elements in a group and the child counted in groups) up to the most abstract form of representation available to these children, i.e. multiplication facts.

For the children in first and second grade, who had not received instruction on multiplication and division, the most important factor in predicting the children's solutions was which quantity was unknown. For example, in the problem above, about the 6 cups with 5 marshmallows in each cup, when the size of the groups was known (i.e. the number of marshmallows in each cup), the children used correspondence strategies: they paired objects (or tallies to represent the objects) and counted or added, creating one-to-many correspondences between the cups and the marshmallows. For example, if they needed to find the total number of marshmallows, they pointed 5 times to a cup (or its representation) and counted to 5, paused, and then counted from 6 to 10 as they pointed to the second 'cup', until they reached the solution. Alternatively, the may have added as they pointed to the 'cup'.

In contrast, when the number of elements in each group was not known, the children used dealing strategies: they shared out one marshmallow (or its representation) to each cup, and then another, until they reached the end, and then counted the number in each cup. Here they sometimes used trial-and-error: they shared more than one at a time and then might have needed to adjust the number per cup to get to the correct distribution.

Although the actions look quite different, their aims are the same: to establish one-to-many correspondences between the marshmallows and the cups.

Kouba observed that 43% of the strategies used by the children, including first, second, and third graders, were appropriate. Among the first and second grade children, the overwhelming majority of the appropriate strategies was based on correspondences, either using direct representation or partial representation (i.e. tallies for one variable and counting or adding for the other); few used recall of multiplication facts. The recall of number facts was significantly higher after the children had received instruction, when they were in third grade.

The level of success observed by Kouba among children who had not yet received instruction is modest, compared to that observed in two subsequent studies, where the ratios were easier. Becker (1993) asked kindergarten children in the United States, aged four to five years, to solve problems in which the correspondences were 2:1 or 3:1. As reported by Piaget and by Frydman and Bryant, the children were more successful with 2:1 than 3:1 correspondences, and the level of success improved with age. The overall level of correct responses by the five-year-olds was 81%.

Carpenter, Ansell, Franke, Fennema and Weisbeck (1993) also gave multiplicative reasoning problems to U.S. kindergarten children involving correspondences of 2:1, 3:1 and 4:1. They observed 71% correct responses to these problems.

The success rates leave no doubt that many young children start school with some understanding of one-to-many correspondence, which they can use to learn to solve multiplicative reasoning problems in school. These results do not imply that children who use one-to-many correspondence to solve multiplicative reasoning problems consciously recognise that in a multiplicative situation there is a fixed ratio linking the two variables. Their actions maintain the ratio fixed but it is most likely that this invariance remains, in Vergnaud's (1997) terminology, as a 'theorem in action'.

The importance of informal knowledge

Both Fischbein (1987) and Treffers (1987) assumed that children's informal knowledge is a starting point for learning mathematics in school but it is important to consider this assumption further. If children start school with some informal knowledge that can be used for learning mathematics in school, it is necessary to consider whether this knowledge facilitates their learning or, quite the opposite, is an obstacle to learning. The action of establishing one-to-many correspondences is not the same as the concept of ratio or as multiplicative reasoning: ratio may be implicit in their actions but it is possible that the children are more aware of the methods that they used to figure out the numerical values of the quantities, i.e. they are aware of counting or adding.

Children's methods for solving multiplication problems can be seen as a starting point, if they form a basis for further learning, but also an obstacle to learning, if children stick to their counting and addition procedures instead of learning about ratio and multiplicative reasoning in school. Resnick (1983) and Kaput and West (1994) argue that an important lesson from psychological and mathematics education research is that, even after people have been taught new concepts and ideas, they still resort to their prior methods to solve problems that differ from the textbook examples on which they have applied their new knowledge. The implementation of the one-to-many correspondence schema to solve problems requires adding and counting, and students have been reported to resort to counting and adding even in secondary school, when they should be multiplying (Booth, 1981). So is this informal knowledge an obstacle to better understanding or does it provide a basis for learning?

It is possible that a precise answer to this question cannot be found: whether informal knowledge helps or hinders children's learning might depend on the pedagogy used in their classroom. However, it is possible to consider this question in principle by examining the results of longitudinal and intervention studies. If it is found in a longitudinal study that children who start school with more informal mathematical knowledge achieve better mathematics learning in school, then it can be concluded that, at least in a general manner, informal knowledge does provide a basis for learning. Similarly, if intervention studies show that increasing children's informal knowledge when they are in their first year school has a positive impact on their school learning of mathematics, there is further support for the idea that informal knowledge can offer a foundation for learning. In the case of the correspondence schema, there is clear evidence from a longitudinal study but intervention studies with the appropriate controls are still needed.

Nunes, Bryant, Evans, Bell, Gardner, Gardner and Carraher (2007) carried out the longitudinal study. In this study, British children were tested on their understanding of four aspects of logical-mathematical reasoning at the start of school; one of these was multiplicative reasoning. There were five items which were multiplicative reasoning problems that could be solved by one-to-many correspondence. The children were also given the British Abilities Scale (BAS-II; Elliott, Smith and McCulloch, 1997) as an assessment of their general cognitive ability and a Working Memory Test, Counting Recall (Pickering and Gathercole, 2001), at school entry. At the beginning of the study, the children's age ranged from five years and one month to six years and six months. About 14 months later, the children were given a state-designed and teacher-administered mathematics achievement test, which is entirely independent of the researchers and an ecologically valid measure of how much they have learned in school. The children's performance in the five items on correspondence at school entry was a significant predictor of their mathematics achievement, after controlling for: (1) age at the time of the achievement test; (2) performance on the BAS-II excluding the subtest of their knowledge of numbers at school start; (3) knowledge of number at school entry (a subtest of the BAS-II); (4) performance on the working memory measure; and (5) performance on the multiplicative reasoning, one-to-many correspondence items. Nunes *et al.* (2007) did not report the analysis of longitudinal prediction based separately on the items that assess multiplicative reasoning; so these results are reported here. The results are presented visually in Figure 4.1 and described in words subsequently.

The total variance explained in the mathematics achievement by these predictors was 66%; age explained 2% (non significant), the BAS general score (excluding the Number Skills subtest) explained a further 49% ($p < 0.001$); the sub-test on number skills explained a further 6% ($p < 0.05$); working memory explained a further 4% ($p < 0.05$), and the children's understanding of multiplicative reasoning at school entry explained a further 6% ($p = 0.005$). This result shows that children's understanding of multiplicative reasoning at school entry is a specific predictor of mathematics achievement in the first two years of school. It supports the hypothesis that, in a general way, this informal knowledge forms a basis for their school learning of mathematics: after 14 months and after controlling for general cognitive factors at school entry, performance on an assessment of multiplicative reasoning still explained a significant amount of variance in the children's mathematics achievement in school.

It is therefore quite likely that instruction will be an important factor in influencing whether students continue to use the one-to-many schema of action to solve such problems, even if replacing objects with numbers but still counting or adding instead of multiplying, or whether they go on to adopt the use of the operations of multiplication and division. Treffers (1987) and Gravemeijer (1997) argue that students do and should use their informal knowledge in the classroom when learning about multiplication and division, but that it should be one of the aims of teaching to help them formalise this knowledge, and in the process develop a better understanding of the arithmetic operations themselves. We do not review this work here but recognise the importance of their argument, particularly in view of the strength of this informal knowledge and students' likelihood of using it even after having been taught other forms of knowledge in school. However, it must be pointed out that there is no evidence that teaching students about arithmetic operations makes them more aware of the invariance of the ratio when they use one-to-many correspondences to solve problems. Kaput and West (1994) also designed a teaching programme which aimed at using students' informal knowledge of correspondences to promote their understanding of multiplicative reasoning. In contrast to the programme designed by Treffers for the operation of multiplication and by Gravemeijer for the operation of division, Kaput and West's programme used simple calculations and tried to focus the students' attention on the invariance of ratio in the correspondence situations. They used different sorts of diagrams which treated the quantities in correspondence as composite units: for example, a plate and six pieces of tableware formed a single unit, a set-place for one person. The ideas proposed in these approaches to instruction are very ingenious and merit further research with the appropriate controls and measures. The lack of control groups and appropriate pre- and post-test assessments in these intervention studies makes it difficult to reach conclusions regarding the impact of the programmes.

Park and Nunes (2001) carried out a brief intervention study where they compared children's success in multiplicative reasoning problems after the children had participated in one of two types of intervention. In the first, they were taught about multiplication as repeated addition, which is the traditional approach used in British schools and is based on the procedural connection between multiplication and addition. In the second intervention group, the children were taught about multiplication by considering one-to-many correspondence situations, where these

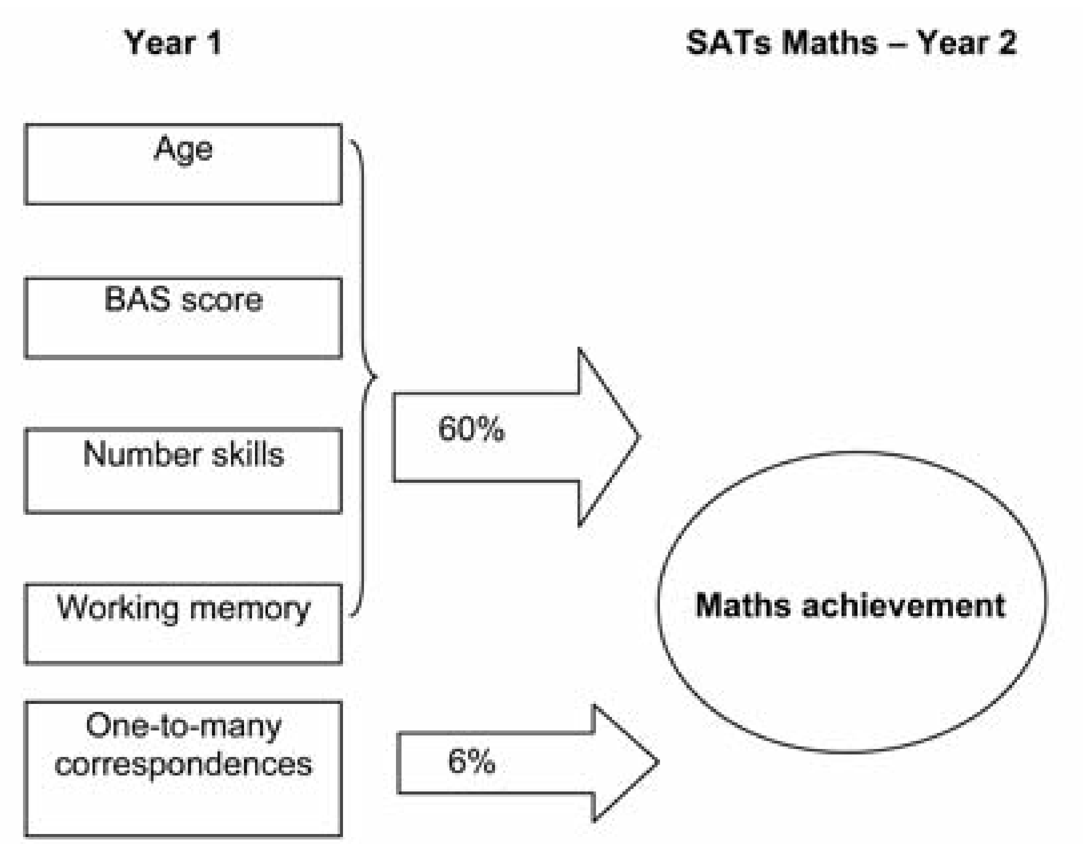


Figure 4.1: A schematic representation of the degree to which individual differences in mathematics achievement are explained by the first four factors and the additional amount of variance explained by children's informal knowledge at school entry.

correspondences were represented explicitly. A third group of children, the control group, solved addition and subtraction problems, working with the same experimenter for a similar period of time. The children in the one-to-many correspondence group made significantly more progress in solving multiplicative reasoning problems than those in the repeated addition and in the control group. This study does include the appropriate controls and provides clear evidence for more successful learning of multiplicative reasoning when instruction draws on the children's appropriate schema of action. However, this was a very brief intervention with a small sample and in one-to-one teaching sessions. It would be necessary to replicate it with larger numbers of children and to compare its level of success with other interventions, such as those used by Treffers and Gravemeijer, where the children's understanding of the arithmetic operations of multiplication and division was strengthened by working with larger numbers.

Summary

1 Additive and multiplicative reasoning have their origins in different schemas of action. There does not seem to be an order of acquisition, with young children understanding at first only additive reasoning and only later multiplicative reasoning. Children can use schemas of action appropriately both in additive and multiplicative reasoning situations from an early age.

2 The schemas of one-to-many correspondence and sharing (or dealing) allow young children to succeed in solving multiplicative reasoning problems before they are taught about multiplication and division in school.

3 There is evidence that children's knowledge of correspondences is a specific predictor of their mathematics achievement and, therefore, that their informal knowledge can provide a basis for further learning. However, this does not mean that they understand the concept of ratio: the invariance of ratio in these situations is likely to be known only as a theorem in action.

4 Two types of programmes have been proposed with the aim of bridging students' informal and formal knowledge. One type (Treffers, 1987; Gravemeijer, 1997) focuses on teaching the children more about the operations of multiplication and division, making a transition from small to large numbers easier for the students. The second type (Kaput and West, 1994; Park and Nunes, 2001) focused on making the students more aware of the schema of one-to-many correspondences and the theorems in action that it represents implicitly. There is evidence that, with younger children solving small number problems, an intervention that focuses on the schema of correspondences facilitates the development of multiplicative reasoning.

Finally, it is pointed out that all the examples presented so far dealt with problems in which the children were asked questions about quantities. None of the problems focused on the relation between quantities. In the subsequent section, we present a classification of multiplicative reasoning problems in order to aid the discussion of how quantities and relations are handled in the context of multiplicative reasoning problems.

Different types of multiplicative reasoning problems

We argued previously that many children solve problems that involve additive relations, such as compare problems, by thinking only about quantities. In this section, we examine different types of multiplicative reasoning problems and analyse students' problem solving methods with a view to understanding whether they are considering only quantities or relations in their reasoning. In order to achieve this, it is necessary to think about the different types of multiplicative reasoning problems.

Classifications of multiplicative reasoning situations vary across authors (Brown, 1981; Schwartz, 1988; Tourniaire and Pulos, 1985; Vergnaud, 1983), but there is undoubtedly agreement on what characterises multiplicative situations: in these situations there are always two (or more) variables with a fixed ratio between them. Thus, it is argued that multiplicative reasoning forms the foundation for children's understanding of proportional relations and linear functions (Kaput and West, 1994; Vergnaud, 1983).

The first classifications of problem situations considered distinct possibilities: for example, rate and ratio problem situations were distinguished initially. However, there seemed to be little agreement amongst researchers regarding which situations should be classified as rate and which as ratio. Lesh, Post and Behr (1988) wrote some time ago: 'there is disagreement about the essential characteristics that distinguish, for example rates from ratios… In fact, it is common to find a given author changing terminology from one publication to another' (p. 108). Thompson (1994) and Kaput and West (1994) consider this distinction to apply not to situations, but to the mental operations that the problem solver uses. These different mental operations could be used when thinking about the same situation: ratio refers to understanding a situation in terms of the particular values presented in the problem (e.g. travelling 150 miles over 3 hours) and rate refers to understanding the constant relation that applies to any of the pairs of values (in theory, in any of the 3 hours one would have travelled 50 miles). 'Rate is a reflectively abstracted constant ratio' (Thompson, 1994, p. 192).

In this research synthesis, we will work with the classification offered by Vergnaud (1983), who distinguished three types of problems.

- In isomorphism of measures problems, there is a simple proportional relation between two measures (i.e. quantities represented by numbers): for example, number of cakes and price paid for the cakes, or amount of corn and amount of corn flour produced.

- In product of measures problems, there is a Cartesian composition between two measures to form a third measure: for example, the number of T-shirts and number of shorts a girl has can be composed in a Cartesian product to give the number of different outfits that she can wear; the number of different coloured cloths and the number of emblems determines the number of different flags that you can produce.

- In multiple proportions problems, a measure is in simple proportion to (at least) two other measures: for example, the consumption of cereal in a Scout camp is proportional to number of persons and the number of days.

Because this classification is based on measures, it offers the opportunity to explore the difference between a quantity and its measure. Although this may seem like a digression, exploring the difference between quantities and measures is helpful in this chapter, which focuses on the quantification of relations between measures. A quantity, as defined by Thompson (1993) is constituted when we think of a quality of an object in such a way that we understand the possibility of measuring it. 'Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them' (p. 166). Two quantities, area and volume, can be used here to illustrate the difference between quantities and measures.

Hart (1981 a) pointed out that the square unit can be used to measure area by different measurement operations. We can attribute a number to the area of a rectangle, for example, by covering it with square units and counting them: this is a simple measurement operation, based on iteration of the units. If we don't have enough bricks to do this (see Nunes, Light and Mason, 1993), we can count the number of square units that make a row along the base, and establish a one-to-many correspondence between the number of rows that fit along the height and the number of square units in each row. We can calculate the area of the rectangle by conceiving of it as an isomorphism of measures problem: 1 row corresponds to x units. If we attribute a number to the area of the rectangle by multiplying its base by its height, both measured with units of length, we are conceiving this situation as a product of measures: two measures, the length of the base and that of its height, multiplied produce a third measure, the area in square units. Thus a quantity in itself is not the same as its measure, and the way it is measured can change the complexity (i.e. the number of relations to be considered) of the situation.

Nunes, Light and Mason (1993) showed that children aged 9 to 10 years were much more successful when they compared the area of two figures if they chose to use bricks to measure the areas than if they chose to use a ruler. Because the children did not have sufficient bricks to cover the areas, most used calculations. They had three quantities to consider – the number of rows that covered the height, the number of bricks in each row along the base, and the area, and the relation between number of rows and number of bricks in the row. These children worked within an isomorphism of measures situation.

Children who used a ruler worked within a product of measures situation and had to consider three quantities – the value of the base, the value of the height, and the area; and three relations to consider – the relation between the base and the height, the relation between base and area, and the relation between the height and the area (the area is proportional to the base if the height is constant and proportional to the height if the base is constant).

The students who developed an isomorphism of measures conception of area were able to use their conception to compare a rectangular with a triangular area, and thus expanded their understanding of how area is measured. The students who worked within a product of measures situation did not succeed in expanding their knowledge to think about the area of triangles. Nunes, Light and Mason speculated that, after this initial move, students who worked with an isomorphism of measures model might subsequently be able to re-conceptualise area once again and move on to a product of measures approach, but they did not test this hypothesis.

Hart (1981 a) and Vergnaud (1983) make a similar point with respect to the measurement of volume: it can be measured by iteration of a unit (how many litres can fit into a container) or can be conceived as a problem situation involving the relations between base, height and width, and described as product of measures. Volume as a quantity is itself neither a uni-dimensional nor a three-dimensional measure and one measure might be useful for some purposes (add 2 cups of milk to make the pancake batter) whereas a different one might be useful for other purposes (the volume of a trailer in a lorry can be easily calculated by multiplying the base, the height and the width). Different systems of representation and different measurement operations allow us to attribute different numbers to the same quantity, and to do so consistently within each system.

Vergnaud's classification of multiplicative reasoning situations is used here to simplify the discussion in this chapter. We will focus primarily on isomorphism of measures situations, because the analysis of how this type of problem is solved by students of different ages and by schooled and unschooled groups will help us understand the operations of thought used in solving them.

A diagram of isomorphism of measures situations, presented in Figure 4.2 and adapted from Vergnaud (1983), will be used to facilitate the discussion.

This simple schema shows that there are two sets of relations that can be quantified in this situation:

- the relation between *a* and *c* is the same as that between *b* and *d*; this is the scalar relation, which links two values in the same measure space
- the relation between *a* and *b* is the same as that between *c* and *d*; this is the functional relation, or the ratio, which links the two measure spaces.

The psychological difference (i.e. the difference that it makes for the students) between scalar and functional relations is very important, and it has been discussed in the literature by many authors (e.g. Kaput and West, 1994; Nunes, Schliemann and Carraher, 1993; Vergnaud, 1983). It had also been discussed previously by Noelting (1980 *a* and *b*) and Tourniaire and Pulos (1985), who used the terms within and between quantities relations. This paper will use only the terms scalar and functional relations or reasoning.

'For a mathematician, a proportion is a statement of equality of two ratios, i.e., $a/b = c/d$' (Tourniaire and Pulos, 1985, p. 181). Given this definition, there is no reason to distinguish between what has been traditionally termed multiplication and division problems and proportions problems. We think that the distinction has been based, perhaps only implicitly, on the use of a ratio with reference to the unit in multiplication and division problems. So one

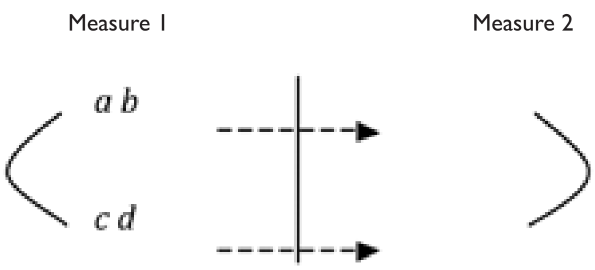


Figure 4.2: Schema of an isomorphism of measures situation. Measures 1 and 2 are connected by a proportional relation.

should not be surprised to see that one-to-many correspondences reasoning is used in the beginning of primary school by children to solve simple multiplicative reasoning problems and continues to be used by older students to solve proportions problems in which the unit ratio is not given in the problem description. Many researchers (e.g. Hart, 1981 b; Kaput and West, 1994; Lamon, 1994; Nunes and Bryant, 1996; Nunes, Schliemann and Carraher, 1993; Piaget, Grize, Szeminska and Bangh, 1977; Inhelder and Piaget, 1958; 1975; Ricco, 1982; Steffe, 1994) have described students' solutions based on correspondence procedures and many different terms have been used to refer to these, such as building up strategies, empirical strategies, halving or doubling, and replications of a composite unit. In essence, these strategies consist of using the initial values provided in the problem and changing them in one or more steps to arrive at the desired value. Hart's (1981 b) well known example of the onion soup recipe for 4 people, which has to be converted into a recipe for 6 people, illustrates this strategy well. Four people plus half of 4 makes 6 people, so the children take each of the ingredients in the recipe in turn, half the amount, and add this to the amount required for 4 people.

Students used yet another method in solving proportions problems, still related to the idea of correspondences: they first find the unit ratio and then use it to calculate the desired value. Although this method is taught in some countries (see Lave, 1988; Nunes, Schliemann and Carraher, 1993; Ricco, 1982), it is not necessarily used by all students after they have been taught; many students rely on building up strategies which change across different problems in terms of the calculations that are used, instead of using a single algorithm that aims at finding the unit ratio. Hart (1981 b) presented the following problem to a large sample of students (2257) aged 11 to 16 years in 1976: 14 metres of calico cost 63p; find the price of 24 metres. She reported that no child actually quoted the unitary method in their explanation, even though some children did essentially seek a unitary ratio. Ricco (1982), in contrast, found that some students explicitly searched for the unit ratio (e.g. 'First I need to know how much one notebook will cost and then we will see', p. 299, our translation) but others seem to search for the unit ratio without making explicit the necessity of this step in their procedure.

Building up methods and finding the unit ratio may be essentially an extension of the use of the one-to-many correspondence schema, which maintains the ratio invariant without necessarily bringing with it an awareness of the fixed relation between the variables. Unit ratio is a mathematical term but it is not clear whether the children who were explicitly searching for the price of one notebook in Ricco's study were thinking of ratio as the quantification of the relation between notebooks and money. When the child says '1 notebook costs 4 cents', the child is speaking about two quantities, not necessarily about the relation between them. A statement about the relation between the quantities would be 'the number of notebooks times 4 tells me the total cost'.

The use of these informal strategies by students in the solution of proportions problems is consistent with the hypothesis that multiplicative reasoning develops from the schema of one-to-many correspondences: students may be simply using numbers instead of objects when reasoning about the quantities in these problems. In the same way that they build up the quantities with objects, they can build up the quantities with numbers. It is unlikely that students are thinking of the scalar relation and quantifying it when they solve problems by means of building up strategies. We think that it can be concluded with some certainty that students realise that whatever transformation they make, for example, to the number of people in Hart's onion soup problem, they must also make to the quantities of ingredients. It is even less likely that they have an awareness of the ratio between the two domains of measures and have reached an understanding of a reflectively abstracted constant ratio, in Thompson's terms.

These results provoke the question of the role of teaching in developing students' understanding of functional relations. Studies of high-school students and adults with limited schooling in Brazil throw some light on this issue. They show that instruction about multiplication and division or about proportions *per se* is neither necessary for people to be able to solve proportions problems nor sufficient to promote students' thinking about functional relations. Nunes, Schliemann and Carraher (1993) have shown that fishermen and foremen in the construction industry, who have little formal school instruction, can solve proportions problems that are novel to them in three ways: (a) the problems use values that depart from the values they normally work with; (b) they are asked to calculate in a direction which they normally do not have to think about; or (c) the content of the problem is different from the problems with which they work in their everyday lives.

Foremen in the construction industry have to work with blue-prints as representations of distances in the buildings under construction. They have experience with a certain number of conventionally used scales (e.g. 1:50, 1:100 and 1:1000). When they were provided with a scale drawing that did not fit these specifications (e.g. 1:40) and did not indicate the ratio used (e.g. they were shown a distance on the blue-print and its value in the building), most foremen were able to use correspondences to figure out what the scale would be and then calculate the measure of a wall from its measure on the blueprint. They were able to do so even when fractions were involved in the calculations and the scale had an unexpected format (e.g. 3 cm:1 m uses different units whereas scales typically use the same unit) because they have extensive experience in moving across units (metres, centimetres and millimetres). Completely illiterate foremen (N = 4), who had never set foot in a school due to their life circumstances, showed 75% correct responses to these problems. In contrast, students who had been taught the formal method known as the rule of three, which involves writing an equation of the form $a/b = c/d$ and solving for the unknown value, performed significantly worse (60% correct). Thus schooling is not necessary for multiplicative reasoning to develop and proportions problems to be solved correctly, and teaching students a general formula to solve the problem is not a guarantee that they will use it when the opportunity arises.

These studies also showed that both secondary school students and adults with relatively little schooling were more successful when they could use building up strategies easily, as in problems of the type A in Figure 4.3. Problem B uses the same numbers but arranged in a way that building up strategies are not so easily implemented; the relation that is easy to quantify in problem B is the functional relation.

The difference in students' rate of success across the two types of problem was significant: they solved about 80% of type A problems correctly and only 35% of type B problems. For the adults (fishermen), there was a difference between the rate of correct responses (80% correct in type A and 75% correct in type B) but this was not statistically significant. Their success, however, was typically a result of prowess with calculations when building up a quantity, and very few answers might have resulted from a quantification of the functional relation.

These results suggest three conclusions.

- Reasoning about quantities when solving proportional problems seems to be an extension of correspondence reasoning; schooling is not necessary for this development.
- Most secondary school students seem to use the same schema of reasoning as younger students; there is little evidence of an impact of instruction on their approach to proportional problems.
- Functional reasoning is more challenging and is not guaranteed by schooling; teaching students a

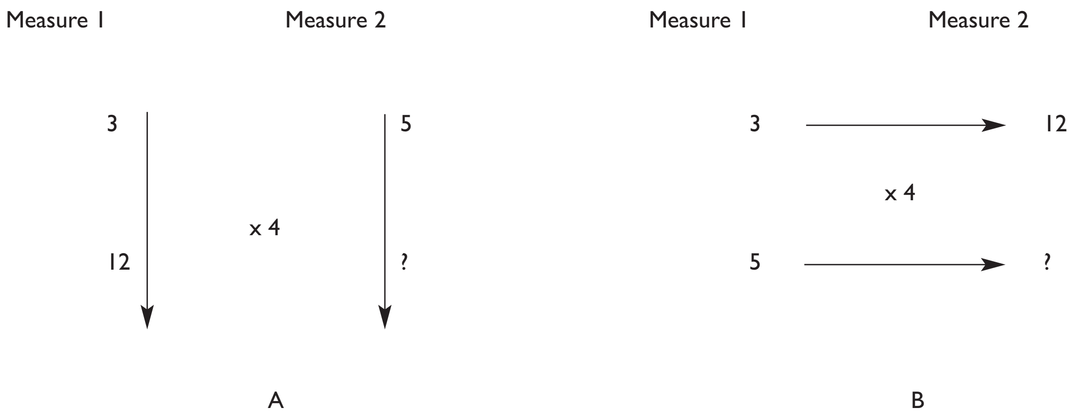


Figure 4.3: For someone who can easily think about scalar or functional relations, there should be no difference in the level of difficulty of the two problems. For those who use building up strategies and can only work with quantities, problem A is significantly easier.

formal method, which can be used as easily for both problem types, does not make functional reasoning easier (see Paper 6 for further discussion).

The results observed with Brazilian students do not differ from those observed by Vergnaud (1983) in France, and Hart (1981 b; 1984) in the United Kingdom. The novelty of these studies is the demonstration that the informal knowledge of multiplicative reasoning and the ability to solve multiplicative reasoning problems through correspondences develop into more abstract schemas that allow for calculating in the absence of concrete forms of representation, such as maniputatives and tallies. Both the students and the adults with low levels of schooling were able to calculate, for example, what should the actual distances in a building be from their size in blueprint drawings. Relatively unschooled adults who have to think about proportions in the course of their occupations and secondary school students seem to rely on these more abstract schemas to solve proportions problems. The similarity between these two groups, rather than the differences, in the forms of reasoning and rates of success is striking. These results suggest that informal knowledge of correspondences is a powerful thinking schema and that schooling does not easily transform it into a more powerful one by incorporating functional understanding into the schema.

Different hypotheses have been considered in the explanation of why this informal knowledge seems so resistant to change. Hart (1981 b) considered the possibility that this may rest on the difficulty of the calculations but the comparisons made by Nunes, Schliemann and Carraher (1993) rule out this hypothesis: the difficulty of the calculations was held constant across problems of type A and type B, and quantitative reasoning on the basis of the functional relation remained elusive.

An alternative explanation, explored by Vergnaud (1983) and Hart (personal communication), is that informal strategies are resistant to change because they are connected to reasoning about quantities, and not about relations. It makes sense to say that if I buy half as much fish, I pay half as much money: these are manipulations of quantities and their representations. But what sense does it make to divide kilos of fish by money?

There has been some discussion of the difference between reasoning about quantities and relations in the literature. However, we have not been able to find studies that establish whether the difficulty of thinking about relations might be at the root of students' difficulties in transforming their informal into formal mathematics knowledge. The educational implications of these hypotheses are considerable but there is, to our knowledge, no research that examines the issue systematically enough to provide a firm ground for pedagogical developments. The importance of the issue must not be underestimated, particularly in the United Kingdom, where students seem do to well enough in the international comparisons in additive reasoning but not in multiplicative reasoning problems (Beaton, Mullis, Martin, Gonzalez, Kelly AND Smith, 1996, p. 94–95).

Summary

We draw some educational implications from these studies, which must be seen as hypotheses about what is important for successful teaching of multiplicative reasoning about relations.

1 Before children are taught about multiplication and division in school, they already have schemas of action that they use to solve multiplicative reasoning problems. These schemas of action involve setting up correspondences between two variables and do not appear to develop from the idea of repeated addition. This informal knowledge is a predictor of their success in learning mathematics and should be drawn upon explicitly in school.

2 Students' schemas of multiplicative reasoning develop sufficiently for them to apply these schemas to numbers, without the need to use objects or tallies to represent quantities. But they seem to be connected to quantities, and it appears that students do not focus on the relations between quantities in multiplicative reasoning problems. This informal knowledge seems to be resistant to change under current conditions of instruction.

3 In many previous studies, researchers drew the conclusion that students' problems in understanding proportional relations were explained by their difficulties in thinking multiplicatively. Today, it seems more likely that students' problems are based on their difficulty in thinking about relations, and not

about quantities, since even young children succeed in multiplicative reasoning problems.

4 Teaching approaches might be more successful in promoting the formalisation of students' informal knowledge if: (a) they draw on the students' informal knowledge rather than ignore it; (b) they offer the students a way of representing the relations between quantities and promote an awareness of these relations; and (c) they use a variety of situational contexts to help students extend their knowledge to new domains of multiplicative reasoning.

We examine now the conceptual underpinnings of two rather different teaching approaches to the development of multiplicative reasoning in search for more specific hypotheses regarding how greater levels of success can be achieved by U.K. students.

The challenge in attempting a synthesis of results is that there are many ways of classifying teaching approaches and there is little systematic research that can provide unambiguous evidence. The difficulty is increased by the fact that by the time students are taught about proportions, some time between their third and their sixth year in school, they have participated in a diversity of pedagogical approaches to mathematics and might already have distinct attitudes to mathematics learning. However, we consider it plausible that systematic investigation of different teaching approaches would prove invaluable in the analysis of pathways to help children understand functional relations. In the subsequent section, we explore two different pathways by considering the types of representations that are offered to students in order to help them become aware of functional relations.

Representing functional relations

The working hypothesis we will use in this section is that in order to become explicitly aware of something, we need to represent it. This hypothesis is commonplace in psychological theories: it is part of general developmental theories, such as Piaget's theory on reflective abstraction (Piaget, 1978; 2001; 2008) and Karmiloff-Smith's theory of representational re-descriptions in development (Karmiloff-Smith, 1992; Karmiloff-Smith and Inhelder, 1977). It is also used to describe development in specific domains such as language and literacy (Gombert, 1992; Karmiloff-Smith, 1992), memory (Flavell, 1971) and the understanding of others (Flavell, Green and Flavell, 1990). It is beyond the scope of this work to review the literature on whether representing something does help us become more aware of the represented meaning; we will treat this as an assumption.

The hypothesis concerning the importance of representations will be used in a different form here. Duval (2006) pointed out that 'the part played by signs in mathematics, or more exactly by semiotic systems of representation, is not only to designate mathematical objects or to communicate but also to work on mathematical objects and with them.' (p. 107). We have so far discussed the quantification of relations, and in particular of functional relations, as if the representation of functional relations could only be attained through the use of numbers. Now we wish to make explicit that this is not so. Relations, including functional relations, can be represented by numbers but there are many other ways in which relations can be represented before a number is attributed to them; to put it more forcefully, one could say that relations can be represented in different ways in order to facilitate the attribution of a number to them.

When students are taught to write an equation of the form $a/b = c/x$, for example, to represent a proportions problem in order to solve for x, this formula can be used to help them quantify the relations in the problem. Hart (1981 b) reports that this formula was taught to 100 students in one school where she carried out her investigations of proportions problems but that it was only used by 20 students, 15 of whom were amongst the high achievers in the school. This formula can be used to explore both scalar and functional relations in a proportions problem but it can also be taught as a rule to solve the problem without any exploration of the scalar or functional relations that it symbolises. In some sense, students can learn to use the formula without developing an awareness of the nature of the relations between quantities that are assumed when the formula is applied.

Researchers in mathematics education have been aware for at least two decades that one needs to explore different forms of representation in order to seek the best ways to promote students' awareness of reasoning about the relations in a proportions problem. It is likely that the large amount of research on proportional reasoning, which exposed students' difficulties as well as their reliance on their own methods even after teaching, played a crucial role in this process. It did undoubtedly raise teachers' and researchers' awareness that the representation through formulae ($a/b = c/x$) or algorithms did not work all that well. In this section, we will seek to examine the underlying assumptions of two very different approaches to teaching students about proportions.

Two approaches to the representation of functional relations

Kieren (1994) suggested that there are two approaches to research about, and to the teaching of, multiplicative reasoning in school. The first is *analytic-functional*: it is human in focus, and investigates actions, action schemes and operations used in giving meaning to multiplicative situations. The second is *algebraic*: this focuses on mathematical structures, and investigates structures used in this domain of mathematics. Although the investigation of mathematics structures is not incompatible with the analytic-functional approach, these are alternatives in the choice of starting point for instruction. They delineate radically distinct pathways for guiding students' learning trajectories.

Most of the research carried out in the past about students' difficulties did not describe what sort of teaching students had participated in; one of the exceptions is Hart's (1981 b) description of the teaching in one school, where students were taught the $a/b = c/d$, algebraic approach: the vast majority of the students did not use this formula when they were interviewed about proportions in her study, and its use was confined to the higher achievers in their tests. It is most urgent that a research programme that systematically compares these two approaches should be carried out, so that U.K. students can benefit from better understanding of the consequences of how these different pathways contribute to learning of multiplicative reasoning. In the two subsequent sections, we present one well developed programme of teaching within each approach.

The analytic-functional approach: from schematic representations of quantities in correspondence to quantifying relations

Streefland and his colleagues (Streefland, 1984; 1985 a and b; van den Brink and Streefland, 1979) highlighted the role that drawing and visualisation can play in making children aware of relations. In an initial paper, van den Brink and Streefland (1979) analysed a boy's reactions to proportions in drawings and also primary school children's reactions in the classroom when visual proportions were playfully manipulated by their teacher.

The boy's reactions were taken from a discussion between the boy and his father. They saw a poster for a film, where a man is bravely standing on a whale and trying to harpoon it. The whale's size is exaggerated for the sake of sensation. The father asked what was wrong with the picture and the boy eventually said: 'I know what you mean. That whale should be smaller. When we were in England we saw an orca and it was only as tall as three men' (van den Brink and Streefland, 1979, p. 405). In line with Bryant (1974), van den Brink and Streefland argued that visual proportions are part of the basic mechanisms of perception, which can be used in learning in a variety of situations, and suggested that this might be an excellent start for making children aware of relations between quantities.

Van den Brink and Streefland then developed classroom activities where six- to eight-year-old children explored proportional relations in drawings. Finally, the teacher showed the children a picture of a house and asked them to mark their own height on the door of the house. The children engaged in measurements of themselves and the door of the classroom in order to transpose this size relation to the drawing and mark their heights on the door. This activity generated discussions relevant to the question of proportions but it is not possible to assess the effect of this activity on their understanding of proportions, as no assessments were used. The lesson ended with the teacher showing another part of the same picture: a girl standing next to the house. The girl was much taller than the house and the children concluded that this was actually a doll house. Surprise and playfulness were considered by Streefland an important factor in children's engagement in mathematics lessons.

Van den Brink and Streefland suggested that children can use perceptual mechanisms to reflect about proportions when they judge something to be out of proportion in a picture. They argued that it is not only of psychological interest but also of mathematical-didactical interest to discover why children can reason in ratio and proportion terms in such situations, abstracting from perceptual mechanisms.

Streefland (1984) later developed further activities in a lesson series with the theme 'with a giant's regard', which started with activities that explored the children's informal sense of proportions and progressively included mathematical representations in the lesson. The children were asked, for example, to imagine how many steps would a normal man take to catch up with one of the giant's steps; later, they were asked to represent the man's and the giant's steps on a number line and subsequently by means of a table. Figure 4.4 presents one example of the type of diagram used for a visual comparison.

In a later paper, Streefland (1985 a and b) pursued this theme further and illustrated how the diagrams used to represent visual meanings could be used in a progressively more abstract way, to represent correspondences between values in other problems that did not have a visual basis. This was illustrated using, among others, Hart's (1981 b) onion soup problem, where a recipe for onion soup for 8 people is to be adapted for 4 or 6 people. The diagram proposed by Streefland, which the teacher should encourage the pupils to construct, shows both (a) the correspondences between the values, which the children can find using their own, informal building up strategies, and (b) the value of the scalar transformation. See Figure 4.5 for an example.

Streefland suggested that these schematic representations could be used later in Hart's onion soup problem in a vertical orientation, more common for tables than the above diagram, and with all ingredients listed on the same table in different rows. The top row would list the number of people, and the subsequent rows would list each ingredient. This would help students realise that the same scalar transformation is applied to all the ingredients for the taste to be preserved when the amounts are adjusted. Streefland argued that 'the ratio table is a permanent record of proportion as an equivalence relation, and in this way contributes to acquiring the correct concept. Applying the ratio table contributes to the detachment from the context… In this quality the ratio table is, as it were, a unifying model for a variety of ratio contexts, as well as for the various manifestations of ratio… The ratio table can contribute to discovering, making conscious and applying all properties that characterise ratio-preserving mappings and to their use in numerical problems' (Streefland, 1985, a, p. 91). Ratio tables are then related to graphs, where the relation between two variables can be discussed in a new way.

Streefland emphasises that 'mathematizing reality involves model building' (Streefland, 1985a, p. 86); so students must use their intuitions to develop a model and then learn how to represent it in order to assess its appropriateness. He (Streefland, 1985 *b*; in van den Heuvel-Panhuizen, 2003) argued that children's use of such schematic *models of* situations that they understand well can become a *model for* new situations that they would encounter in the future. The representation of their knowledge in such schematic form helps them understand what is implied in the model, and make explicit a relation that they had used only implicitly before.

This hypothesis is in agreement with psychological theories that propose that reflection and representation help make implicit knowledge explicit (e.g. Karmiloff-Smith, 1992; Piaget, 2001). However, the concept proposes a pedagogical strategy in Streefland's work: the model is chosen by the teacher, who guides the student to use it and adds

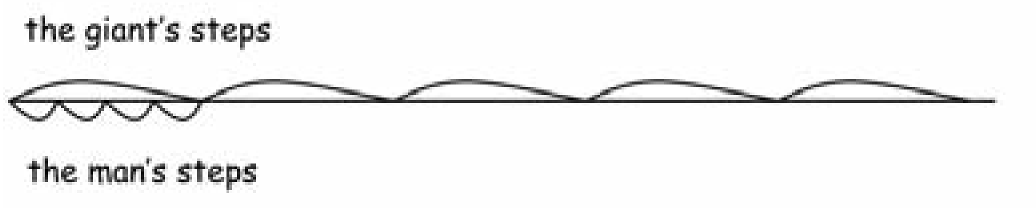


Figure 4.4: The giant's steps and the man's steps on a line; this drawing can be converted into a number line and a table which displays the numerical correspondences between the giant's and the man's steps.

elements, such as the explicit representation of the scalar factor. The model is chosen because it can be easily stripped of the specifics in the situation and because it can help the students move from thinking about the context to discussing the mathematical structures (van den Heuvel-Panhuizen, 2003). So children's informal knowledge is to be transformed into formal knowledge through changes in representation that highlight the mathematical relations that remain implicit when students focus on quantities.

Finally, Streefland also suggested that teaching children about ratio and proportions could start much earlier in primary school and should be seen as a longer project than prescribed by current practice. Starting from children's informal knowledge is a crucial aspect of his proposal, which is based on Freudenthal's (1983) and Vergnaud's (1979) argument that we need to know about children's implicit mathematical models for problem situations, not just their arithmetic skills, when we want to develop their problem solving ability. Streefland suggests that, besides the visual and spatial relations that he worked with, there are other concepts which children aged eight to ten years can grasp in primary school, such as comparisons between the density (or crowdedness) of objects in space and probabilities. Other concepts, such as percentages and fractions, were seen by him as related to proportions, and he argued that connections should be made across these concepts. However, Streefland considered that they merited their own analyses in the mathematics classroom. He argued, citing Vergnaud (1979) that 'different properties, almost equivalent to the mathematician, are not all equivalent for the child (Vergnaud, 1979, p. 264). So he also developed programmes for the teaching of percentages (Streefland and van den Heuvel-Panhuizen, 1992; van den Heuvel-Panhuizen, 2003) and fractions (Streefland, 1993; 1997). Marja van den Heuvel-Panhuizen and her colleagues (Middleton and van den Heuvel-Panhuizen, 1995; Middleton, van den Heuvel-Panhuizen and Shew, 1998) detailed the use of the ratio table in teaching students in their 3rd year in school about percentages and connecting percentages, fractions and proportions.

In all these studies, the use of the ratio table is seen as a tool for computation and also for discussion of the different relations that can be quantified in the problem situations. Their advice is that teachers should allow students to use the table at their own level of understanding but always encourage students to make their reasoning explicit. In this way, students can compare their own reasoning with their peers' approach, and seek to improve their understanding through such comparisons.

Streefland's proposal is consistent with many of the educational implications that we drew from previous research. It starts from the representation of the correspondences between quantities and moves to the representation of relations. It uses schematic drawings and tables that bring to the fore of each student's activity the explicit representation of the two (or more) measures that are involved in the problem. It is grounded on students' informal knowledge because students use their building up solutions in order to construct tables and schematic drawings. It systematizes the students' solutions in tables and re-represents them by means of graphs. After exploring students' work on quantities, students' attention is focused on scalar relations, which they are asked to represent explicitly using the same visual records. It draws on a variety of contexts

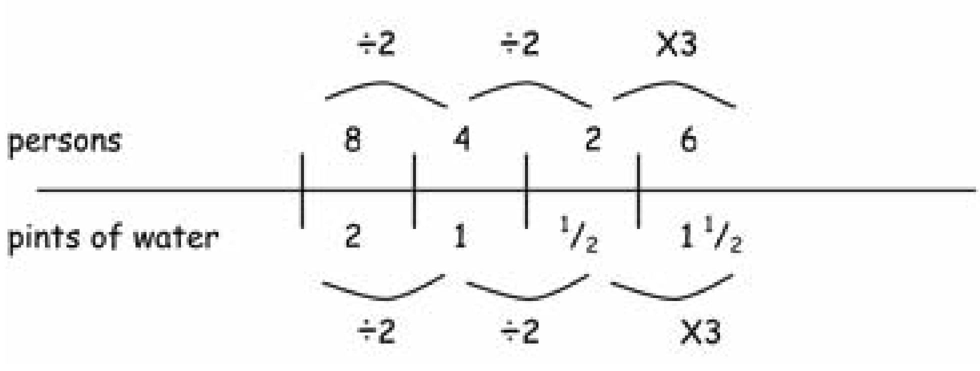


Figure 4.5: A table showing the answers that the children can build up and the representation of the scalar transformations.

that have been previously investigated and which students have been able to handle successfully. Finally, it uses graphs to explore the linear relations that are implied in proportional reasoning.

To our knowledge, there is no systematic investigation of how this proposal actually works when implemented either experimentally or in the classroom. The work by Treffers (1987) and Gravemeijer (1997) on the formalisation of students' understanding of multiplication and division focused on the transition from computation with small to large numbers. The work by van den Heuvel-Panhuizen and colleagues focused on the use of ratio tables in the teaching of percentages and equivalence of fractions. In these papers, the authors offer a clear description of how teachers can guide students' transition from their own intuitions to a more formal mathematical representation of the situations. However, there is no assessment of how the programmes work and limited systematic description of how students' reasoning changes as the programmes develop.

The approach by researchers at the Freudenthal Institute is described as developmental research and aims at constructing a curriculum that is designed and improved on the basis of students' responses (Gravemeijer, 1994). This work is crucial to the development of mathematics education. However, it does not allow for the assessment of the effects of specific teaching approaches, as more experimental intervention research does. It leaves us with the sense that the key to formalising students' multiplicative reasoning may be already to hand but we do not know this yet. Systematic research at this stage would offer an invaluable contribution to the understanding of how students learn and to education.

Streefland was not the only researcher to propose that teaching students about multiplicative relations should start from their informal understanding of the relations between quantities and measures. Kaput and West (1994) developed an experimental programme that took into account students' building up methods and sought to formalise them through connecting them with tables. Their aim was to help students create composite units of quantity, where the correspondences between the measures were represented iconically on a computer screen. For example, if in a problem the quantities are 3 umbrellas for 2 animals, the computer screen would display cells with images for 3 umbrellas and 2 animals in each cell, so that the group of umbrellas and animals became a higher-level unit. The cells in the computer screen were linked up with tables, which showed the values corresponding to the cells that had been filled with these composite units: for example, if 9 cells had been filled in with the iconic representations, the table displayed the values for 1 through 9 of the composite units in columns headed by the icons for umbrellas and animals. Subsequently, students worked with non-integer values for the ratios between the quantities: for example, they could be asked to enlarge a shape and the corresponding sides of the two figures had a non-integer ratio between them (e.g. one figure had a side 21 cm long and the other had the corresponding side 35 cm long).

Kaput and West's programme was delivered over 11 lessons in two experimental classes, which included 31 students. Two comparison classes, with a total of 29 students, followed the instruction previously used by their teachers and adopted from textbooks. One comparison class had 13 lessons: the first five lessons were based on a textbook and covered exercises involving ratio and proportion; the last eight consisted of computer-based activities using function machines with problems about rate and profit. The second comparison class had only three lessons; the content of these is not described by the authors. The classes were not assigned randomly to these treatments and it is not clear how the teachers were recruited to participate in the study.

At pre-test, the students in the experimental and comparison classes did not differ in the percentage of correct solutions in a multiplicative reasoning test. At post-test, the students in the experimental group significantly out-performed those in both comparison classes. They also showed a larger increase in the use of multiplicative strategies than students in the comparison classes. It is not possible from Kaput and West's report to know whether these were building up, scalar or functional solutions, as they are considered together as multiplicative solutions.

In spite of the limitations pointed out, the study does provide evidence that students benefit from teaching that develops their building up strategies into more formalised approaches to solution, by linking the quantities represented by icons of objects to tables that represent the same quantities. This result goes against the view that informal methods are an obstacle to students' learning in and of themselves; it is more likely that they are an obstacle *if* the teaching they are exposed to does not build on the students' informal strategies and does not help students connect what they know with the new forms of mathematical representation that the teacher wants them to learn.

The algebraic approach: representing ratios and equivalences

In contrast to the functional approach to the teaching of proportions that was described in the previous section, some researchers have proposed that teaching should not start from students' understanding of multiplicative reasoning, but from a formal mathematical definition of proportions as the equality of two ratios. We found the most explicit justification for this approach in a recent paper by Adjiage and Pluvinage (2007). Adjiage and Pluvinage, citing several authors (Hart, 1981 b); Karplus, Pulos and Stage, 1983; Lesh, Post, and Behr, 1988), argue that building up strategies are a weak indicator of proportionality reasoning and that the link between 'interwoven physical and mathematical considerations, present in the build-up strategy' (2007, p. 151) should be the representation of problems through rational numbers. For example, a mixture that contains 3 parts concentrate and 2 parts water should be represented as 3/5, using numbers or marks on a number line. The level 1, which corresponds to an iconic representation of the parts used in the mixture, should be transformed into a level 2, numerical representation, and students should spend time working on such transformations. Similarly, a scale drawing of a figure where one side is reduced from a length of 5 cm to 3 cm should be represented as 3/5, also allowing for the move from an iconic to a numerical representation. Finally, the representation by means of an equivalence of ratios, as in 3/5 = 6/10, should be introduced, to transform the level 2 into a level 3 representation. The same results could be obtained by using decimals rather than ordinary fractions representations.

In brief, level 1 allows for an articulation between physical quantities: the students may realise that a mixture with 3 parts concentrate and 2 parts water tastes the same as another with 6 parts concentrate and 4 parts water. Level 2 allows for articulations between the physical quantities and a mathematical representation: students may realise that two different situations are represented by the same number. Level 3 allows for articulations within the mathematical domain as well as conversions from one system of representation to another: 3/5 = 0.6 or 6/10.

Adjiage and Pluvinage (2007) argue that it is important to separate the physical from the mathematical initially in order to articulate them later, and propose that three rational registers should be used to facilitate students' attainment of level 3: linear scale (a number line with resources such as subdividing, sliding along the line, zooming), fractional writing, and decimal writing should be used in the teaching of ratio and proportions.

It seems quite clear to us that this proposal does not start from students' intuitions or strategies for solving multiplicative reasoning problems, but rather aims to formalise the representation of physical situations from the start and to teach students how to work with these formalisations. The authors indicate that their programme is inspired by Duval's (1995) theory of the role of representations in mathematical thinking but we believe that there is no necessary link between the theory and this particular approach to teaching students about ratio and proportions.

In order to convey a sense for the programme, Adjiage and Pluvinage (2007) describe five moments experienced by students. The researchers worked with two conditions of implementation, which they termed the full experiment and the partial experiment. Students in the partial experiment did not participate in the first moment using a computer; they worked with pencil and paper tasks in moments 1 and 2.

- *Moment 1* The students are presented with three lines, divided into equal spaces. They are told that the lines are drawn in different scales. The lines have different numbers of subdivisions – 5, 3 and 4, respectively. Points equivalent to 3/5, 2/3 and 1/4 are marked on the line. The students are asked to compare the segments from the origin to the point on the line. This is seen as a purely mathematical question, executed in the computer by students in the full experiment condition. The computer has resources such as dividing the lines into equal segments, which the students can use to execute the task.

- *Moment 2* A similar task is presented with paper and pencil.

- *Moment 3* The students are shown two pictures that represent two mixtures: one is made with 3 cups of chocolate and 2 of milk (the cups are shown in the pictures in different shades) and the other with 2 cups of chocolate and 1 of milk. The students are asked which mixture tastes more chocolaty. This problem aims to link the physical and the mathematical elements.

- *Moment 4* The students are asked in what way are the problems in moments 2 and 3 similar. Students are expected to show on the segmented line which portion corresponds to the cups of chocolate and which to the cups of milk.

- *Moment 5* This is described as institutionalization in Douady's (1984) sense: the students are asked to make abstractions and express rules. For instance, expressions such as these are expected: '7 divided by 4 is equal to seven fourths (7 ÷ 4 = 7/4)'; 'Given an enlargement in which a 4 cm length becomes a 7 cm length, then any length to be enlarged has to be multiplied by 7/4.' (Adjiage and Pluvinage, 2007, pp. 160–161).

The teaching programme was implemented over two school years, starting when the students were in their 6^{th} year (estimated age about 11 years) in school. A pre-test was given to them before they started the programme; the post-test was carried out at the end of the students' 7^{th} year (estimated age about 12 years) in school.

Adjiage and Pluvinage (2007) worked with an experienced French mathematics teacher, who taught two classes using their experimental programme. In both classes, the students solved the same problems but in one class, referred to as a partial experimental, the students did not use the computer-based set of activities whereas in the other one, referred to as full experimental, they had access to the computer activities. The teacher modified only his approach to teaching ratio and proportions; other topics in the year were taught as previously, before his engagement in the experiment.

The performance of students in these two experimental classes was compared to results obtained by French students in the same region (the baseline group) in a national assessment and also to the performance of non-specialist, prospective school-teachers on a ratio and proportions task. The tasks given to the three groups were not the same but the researchers considered them comparable.

Adjiage and Pluvinage reported positive results from their teaching programme. When the pupils in the experimental classes were in grade 6 they had a low rate of success in ratio and proportions problems: about 13%. At the end of grade 7, they attained 39% correct answers whereas the students in the sample from the same region (baseline group) attained 15% correct responses in the national assessment. The students in the full experimental classes obtained significantly better results than those in the partial experimental classes but the researchers did not provide separate percentages for the two groups. Prospective teachers attained 83% on similar problems. The researchers were not satisfied with these results because, as they point out, the students performed significantly worse than the prospective teachers, who were taken to represent educated adults.

Although there are limitations to this study, it documents some progress among the students in the experimental classes. However, it is difficult to know from their report how much time was devoted to the teaching programme over the two years and how this compares to the instruction received by the baseline group.

In brief, this approach assumes that students' main difficulties in solving proportions problems result from their inability to co-ordinate different forms of mathematical representations and to manipulate them. There is no discussion of the question of quantities and relations and there is no attempt to make students aware of the relations between quantities in the problems. The aims of teaching are to:

- develop students' understanding of how to use number line and numerical representations together in order to compare rational numbers
- promote students' reflection on how the numerical and linear representations relate to problem situations that involve physical elements (3 cups of chocolate and 2 of milk)
- promote students' understanding of the relations between the different mathematical representations and their use in solving problems.

A comparison between this example of the algebraic approach and the functional approach as exemplified by Streefland's work suggests that this algebraic approach does not offer students the opportunity to distinguish between quantities and relations. The three forms of representation offered in the Adjiage and Pluvinage programme focus on quantities; the relations between quantities are left implicit. Students are expected to recognise that mixtures of concentrate that are numerically represented as 3/5 and 6/10 are equivalent. In the number line, they are expected to manipulate the representations of quantities in order to compare them. We found no evidence in the description of their teaching programme that students were asked to think about their implicit models of the situations and explicitly discuss the transformations that would maintain the equivalences.

Summary

1 It is possible to identify in the literature two rather different views of how students can best be taught about multiplicative reasoning. Kieren identified these as the functional and the algebraic approach.

2 The functional approach proposes that teaching should start from students' understanding of quantities and seek to make their implicit models of relations between quantities explicit.

3 The algebraic approach seeks to represent quantities with mathematical symbols and lead students to work with symbols as soon as possible, disentangling physical and mathematical knowledge.

4 There is no systematic comparison between these two approaches. Because their explicit description is relatively recent, this paper is the first detailed comparison of their characteristics and provides a basis for future research.

Graphs and functional relations

The previous sections focused on the visual and numerical representations of relations. This section will briefly consider the question of the representation of relations in the Cartesian plane. We believe that this is a form of representation that merits further discussion because of the additional power that it can add to students' reflections, if properly explored.

Much research on how students interpret graphs has shown that graph reading has to be learned, just as one must learn how to read words or numbers. Similarly to other aspects of mathematics learning, students have some ideas about reading graphs before they are taught, and researchers agree that these ideas should be considered when one designs instruction about graph reading. Several papers can be of interest in this context but this research is not reviewed here, as it does not contribute to the discussion of how graphs can be used to help students understand functional relations (for complementary reviews, see Friel, Curcio and Bright, 2001; Mevarech and Kramarsky, 1997). We focus here on the possibilities of using graphs to help students understand functional relations.

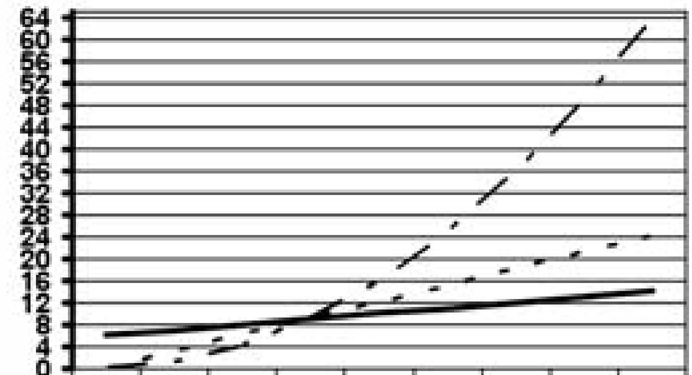
As reported earlier in this chapter, Lieven Verschaffel and his colleagues have shown that students make multiplicative reasoning errors in additive situations as well as additive errors in multiplicative situations, and so there is a need for students to be offered opportunities to reflect on the nature of the relation between quantities in problems. Van Dooren, Bock, Hessels, Janssens and Verschaffel (2004) go as far as suggesting that students fall prey to what they call an illusion of linearity, but we think that they have overstated their case in this respect. In fact, some of the examples that they use to illustrate the so-called illusion of linearity are indeed examples of linear functions, but perhaps not as simple as the typical linear functions used in school. In two examples of their 'illusion of linearity' discussed here, there is a linear function connecting the two variables but the problem situation is more complex than many of the problems used in schools when students are taught about ratio and proportions. In our view, these problems demonstrate the importance of working with students to help them reflect about the relations between the quantities in the situations.

In one example, taken from Cramer, Post and Currier (1993) and discussed earlier on in this paper, two girls, Sue and Julie, are supposed to be running on a track at the same speed. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run? Although prospective teachers wrongly quantified the relation between the number of laps in a multiplicative way, we do not think that they fell for the 'illusion of linearity', as argued by De Bock, Verschaffel *et al.*, (2002; 2003). The function actually is linear, as illustrated in Figure 4.6. However, the intercept between Sue's and Julie's numbers of laps is not at zero, because Sue must have run 6 laps before Julie starts. So the prospective teachers' error is not an illusion of linearity but an inability to deal with intercepts different from zero.

Figure 4.6 shows that three different curves would be obtained: (1) if the girls were running at the same speed but one started before the other, as in the Cramer, Post and Currier problem; (2) if one were running faster than the other and this difference in speed were constant; and (3) if they started out running at the same speed but one girl became progressively more tired whereas the other was able to speed up as she warmed up. Students might hypothesise that this latter example is better described by a quadratic than a linear function, if the girl who was getting tired went from jogging to walking, but they could find that the quadratic function would exaggerate the difference between the girls: how could the strong girl run 25 laps while the weak one ran 5?

The aim of this illustration is to show that relations between quantities in the same context can vary and that students can best investigate the nature of the relation between quantities is if they have a tool to do so. Streefland suggested that tables and graphs can be seen as tools that allow students to explore relations between quantities; even though they could be used to help students' reasoning in this problem, we do not know of research where it has been used.

Van Dooren, *et al.* (2004) used graphs and tables in an intervention programme designed to help student overcome the 'illusion of linearity' in a second problem, which we argue also involve mislabelling of the phenomenon under study. In several studies, De Bock, Van Dooren and their colleagues (De Bock, Verschaffel and Janssens, 1998; De Bock, Van Dooren, Janssens and Verschaffel, 2002; De Bock, Verschaffel and Janssens, 2002; De Bock, Verschaffel, Janssens, Van Dooren and Claes, 2003) claim to have identified this illusion in questions exemplified in this problem: 'Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?' De Bock, Van Dooren and colleagues worked with relatively large numbers of Belgian students across their many studies, in the age range 12 to 16 years. They summarise their findings by indicating that 'the vast majority of students (even 16-year-olds) failed on this type of problem because of their alarmingly strong tendency to apply linear methods' (Van Dooren, Bock, Hessels, Janssens and Verschaffel, 2004, p. 487) and that even with considerable support many students were not able to overcome this difficulty. Some students who did become more cautious about over-using a linear model, subsequently failed to use it when it was appropriate.



	0	1	2	3	4	5	6	7	8
Same speed, different starts	6	7	8	9	10	11	12	13	14
Multiplicative relation	0	3	6	9	12	15	18	21	24
Quadratic function	0	1	4	9	16	25	36	49	64

Same speed, different starts
Multiplicative relation
Quadratic function

Figure 4.6: Three graphs showing different relations between the number of laps run by two people over time

We emphasise here that in this problem, as in the previous one, students were not falling prey to an illusion of linearity. The area of a rectangular figure is indeed proportional to its side *when the other side is held constant*; this is a case of multiple proportions and thus the linear relation between the side and the area can only be appreciated if the other side does not change. Because the rectangle in their problem is the particular case of a square, if one side changes, so does the other; with both measures changing at the same time, the area is not a simple linear function of one of the measures.

Van Dooren *et al.* (2004) describe an intervention programme, in which students used graphs and tables to explore the relation between the measure of the side of a square, its area and its perimeter. The intervention contains interesting examples in which students have the opportunity to examine diagrams that display squares progressively larger by 1 cm, in which the square units (1 cm^2) are clearly marked. Students thus can see that when the side of a square increases, for example, from 1 cm to 2 cm, its area increases from 1 cm^2 to 4 cm^2, and when the increase is from 2 cm to 3 cm, the area increases to 9 cm^2. The graph associated with this table displays a quadratic function whereas the graph associate with the perimeter displays a linear function.

Their programme was not successful in promoting students' progress: the experimental group significantly decreased the rate of responses using simple proportional reasoning to the area problems but also decreased the rate of correct responses to perimeter problems, although the perimeter of a square is connected to its sides by a simple proportion.

We believe that the lack of success of their programme may be due not to a lack of effectiveness of the use of graphs and tables in promoting students' reasoning but from their use of an inadequate mathematical analysis of the problems. Because the graphs and tables used only two variables, measure of the side and measure of the area, the students had no opportunity to appreciate that in the area problem there is a proportional relation between area and each the two sides. The two sides vary at the same time in the particular case of the square but in other rectangular figures there isn't a quadratic relation between side and area. The relation between sides and perimeter is additive, not multiplicative: it happens to be multiplicative in the case of the square because all sides are equal; so to each increase by 1 cm in one side corresponds a 4 cm increase in the perimeter.

We think that it would be surprising if the students had made significant progress in understanding the relations between the quantities through the instruction that they received in these problems: they were not guided to an appropriate model of the situation, and worked with one measure, side, instead of two measures, base and height. One of the students remarked at the end of the intervention programme, after ten experimental lessons over a two week period: 'I really do understand now why the area of a square increases 9 times if the sides are tripled in length, since the enlargement of the area goes in two dimensions. But suddenly I start to wonder why this does not hold for the perimeter. The perimeter also increases in two directions, doesn't it?' (Van Dooren *et al.*, 2004, p. 496). This student seems to have understood that the increase in one dimension of the square implies a similar increase along the other dimension and that these are multiplicatively related to the area but apparently missed the opportunity to understand that sides are additively related to the perimeter.

In spite of the shortcomings of this study, the intervention illustrates that it is possible to relate problem situations to tables and graphs systematically to stimulate students' reflection about the implicit models. It is a current hypothesis by many researchers (e.g. Carlson, Jacobs, Coe, Larsen and Hsu, 2002; Hamilton, Lesh, Lester and Yoon, 2007; Lesh, Middleton, Caylor and Gupta, 2008) that modelling data, testing the adequacy of models through graphs, and comparing different model fits can make an important contribution to students' understanding of the relations between quantities. It is consistently acknowledged that this process must be carefully designed: powerful situations must be chosen, clear means of hypothesis testing must be available, and appropriate teacher guidance should be provided. Shortcomings in any of these aspects of teaching experiments could easily result in negative results.

The hypothesis that modelling data, testing the adequacy of models through graphs, and comparing different model fits can promote student's understanding of different types of relations between quantities seems entirely plausible but, to our knowledge, there is no research to provide clear support to it. We think that there are now many ideas in the literature that can be implemented to assess systematically how effective the use of graphs and tables is as tools to support students' understanding of the different types of relation that can exist between measures. This research has the potential to make a huge contribution to the improvement of mathematics education in the United Kingdom.

Conclusions and implications

This review has identified results in the domain of how children learn mathematics that have significant implications for education. The main points are highlighted here.

1 Children form concepts about quantities from their everyday experiences and can use their schemas of action with diverse representations of the quantities (iconic, numerical) to solve problems. They often develop sufficient awareness of quantities to discuss their equivalence and order as well as how they can be combined.

2 It is significantly more difficult for them to become aware of the relations between quantities and operate on relations. Even after being taught how to represent relations, they often interpret the results of operations on relations as if they were quantities. Children find both additive and multiplicative relations significantly more difficult than understanding quantities.

3 There is little evidence that the design of instruction has so far taken into account the importance of helping students become aware of the difference between quantities and relations. Some researchers have carried out experimental teaching studies that suggest that it is possible to promote students' awareness of relations. Further research must be carried out to analyse how this knowledge affects mathematics learning. If positive results are found, there will be strong policy implications.

4 Previous research had led to the conclusion that students' problems with proportional reasoning stemmed from their difficulties with multiplicative reasoning. However, there is presently much evidence to show that, from a relatively early age (about five to six years in the United Kingdom), children already have informal knowledge that allows them to solve multiplicative reasoning problems. We suggest that students' problems with proportional reasoning stems from their difficulties in becoming explicitly aware of relations between quantities. This awareness would help them distinguish between situations that involve different types of relations: additive, proportional or quadratic, for example.

5 Multiplicative reasoning problems are defined by the fact that they involve two (or more) measures linked by a fixed ratio. Students' informal knowledge of multiplicative reasoning stems from the schema of one-to-many correspondence, which they use both in multiplication and division problems. When the product is unknown, children set the elements in the two measures in correspondence (e.g. 1 sweet costs 4p) and figure out the product (how much 5 sweets will cost). When the correspondence is unknown (e.g. if you pay 20p for 5 sweets, how much does each sweet cost), the children share out the elements (20p shared in 5 groups) to find what the correspondence is.

6 This informal knowledge is currently ignored in U.K. schools, probably due to the theory that multiplication is essentially repeated addition and division is repeated subtraction. However, the connections between addition and multiplication, on one hand, and subtraction and division, on the other hand, are procedural and not conceptual. So students' informal knowledge of multiplicative reasoning could be developed in school from an earlier age.

7 A considerable amount of research carried out independently in different countries has shown that students sometimes use additive reasoning about relations when the appropriate model is a multiplicative one. Some recent research has shown that students also use multiplicative reasoning in situations where the appropriate model is additive. These results suggest that children use additive and multiplicative models implicitly and do not make conscious decisions regarding which model is appropriate in a specific situation. The educational implication from these findings is that schools should take up the task of helping students become more aware of the models that they use implicitly and of ways of testing their appropriateness to particular situations.

8 Proportional reasoning stems from children's use of the schema of one-to-many correspondences, which is expressed in calculations as building-up strategies. Evidence suggests that many students who use these strategies are not aware of functional relations that characterises a linear function. This result reinforces the importance of the role that schools could play in helping students become aware of functional relations in proportions problems.

9 Two radically different approaches to teaching proportions and linear functions in schools can be identified in the literature. One, identified as functional and human in focus, is based on the notion that students' schemas of action should be the starting point for this teaching. Through instruction, they should become progressively more aware of the scalar and functional relations that can be identified in such problems. Diagrams, tables and graphs are seen as tools that could help students understand the models that they are using of situations and make them into models for other situations later. The second approach, identified as algebraic, proposes that there should be a sharp separation between students' intuitive knowledge, in which physical and mathematical knowledge are intertwined, and mathematical knowledge. Students should be led to formalisations early on in instruction and re-establish the connections between mathematical structures and physical knowledge at a later point. Representations using fractions, ordinary and decimal, and the number line are seen as the tools that can allow students to abstract early on from the physical situations. There is no unambiguous evidence to show how either of these approaches to teaching succeeds in promoting students' progress, nor that either of them is more successful than the less clearly articulated ideas that are implicit in current teaching in the classroom. Research that can clarify this issue is urgently needed and could have a major impact by promoting better learning in U.K. students.

10 Students need to learn to read graphs in order to be able to use them as tools for thinking about functions. Research has shown that students have ideas about how to read graphs before instruction and that these ideas should be taken into account when graphs are used in the classroom. It is possible to teach students to read graphs and to use them in order to think about relations but much more research is needed to show how students' thinking changes if they do learn to use graphs in order to analyse the type of relation that is most relevant in specific situations.

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