Nuffield Foundation

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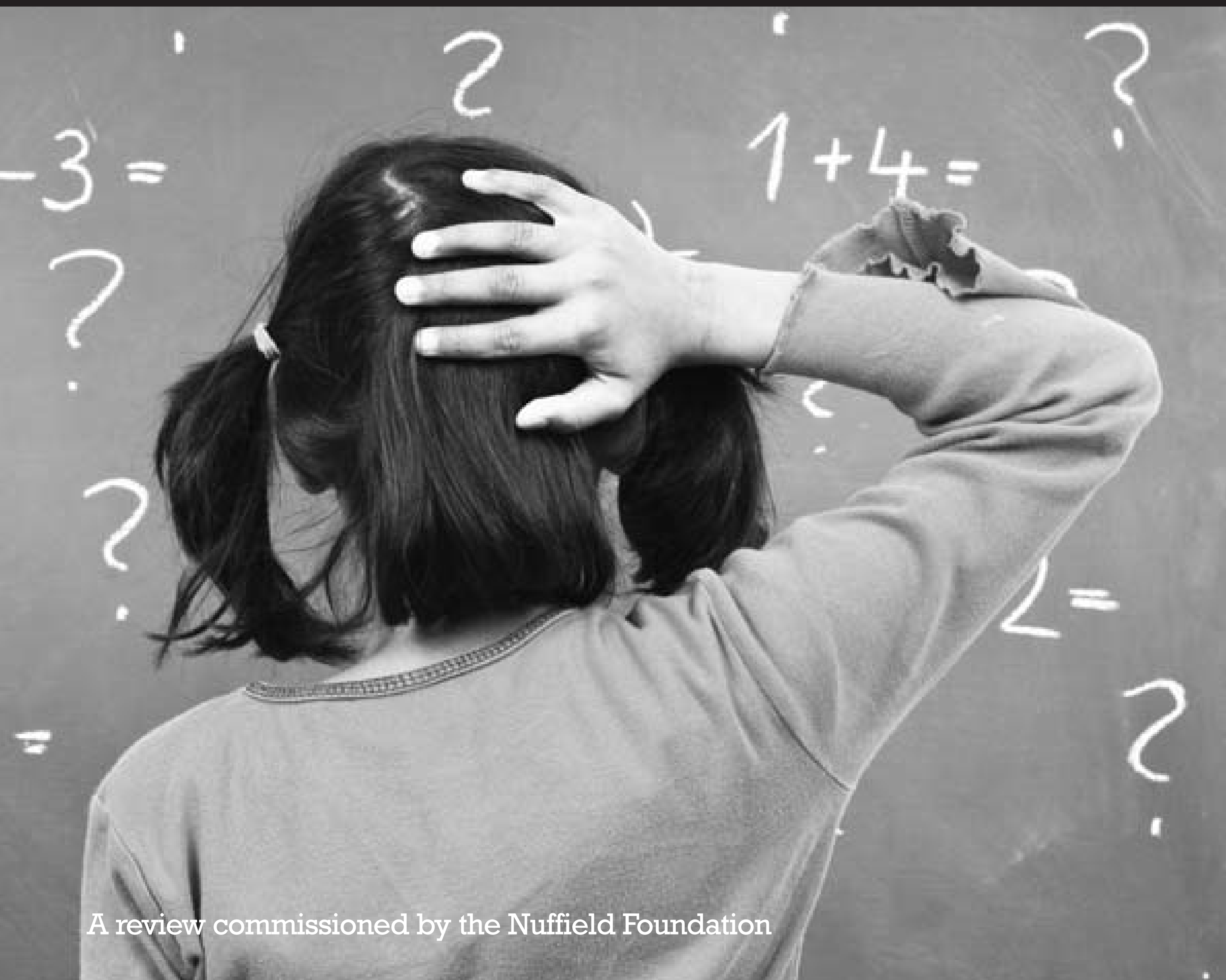
Key understandings in mathematics learning

Paper 3: Understanding rational numbers and intensive quantities

By Terezinha Nunes and Peter Bryant, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 3: Understanding rational numbers and intensive quantities

Headlines

- Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. As with whole numbers, children need to make connections between quantities and their representations in fractions in order to be able to use fractions meaningfully.

- There are two types of situation in which fractions are used in primary school. The first involves measurement: if you want to represent a quantity by means of a number and the quantity is smaller than the unit of measurement, you need a fraction – for example, a half cup or a quarter inch. The second involves division: if the dividend is smaller than the divisor, the result of the division is represented by a fraction. For example, when you share 3 cakes among 4 children, each child receives ¾ of a cake.

- Children use different schemes of action in these two different situations. In division situations, they use correspondences between the units in the numerator and the units in the denominator. In measurement situations, they use partitioning.

- Children are more successful in understanding equivalence of fractions and in ordering fractions by magnitude in situations that involve division than in measurement situations.

- It is crucial for children's understanding of fractions that they learn about fractions in both types of situation: most do not spontaneously transfer what they learned in one situation to the other.

- When a fraction is used to represent a quantity, children need to learn to think about how the numerator and the denominator relate to the value represented by the fraction. They must think about direct and inverse relations: the larger the numerator, the larger the quantity but the larger the denominator, the smaller the quantity.

- Like whole numbers, fractions can be used to represent quantities and relations between quantities, but in primary school they are rarely used to represent relations. Older students often find it difficult to use fractions to represent relations.

There is little doubt that students find fractions a challenge in mathematics. Teachers often say that it is difficult to teach fractions and some think that it would be better for everyone if children were not taught about fractions in primary school. In order to understand fractions as numbers, students must be able to know whether two fractions are equivalent or not, and if they are not, which one is the bigger number. This is similar to understanding that 8 sweets is the same number as 8 marbles and that 8 is more than 7 and less than 9, for example. These are undoubtedly key understandings about whole numbers and fractions. But even after the age of 11 many students have difficulty in knowing whether two fractions are equivalent and do not know how to order some fractions. For example, in a study carried out in London, students were asked to paint 2/3 of figures divided in 3, 6 and 9 equal parts. The majority solved the task correctly when the figure was divided into 3 parts but 40% of the 11- to 12-year-old students could not solve it when the figure was divided into 6 or 9 parts, which meant painting an equivalent fraction (4/6 and 6/9, respectively).

Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. If the teaching of fractions were to be omitted from the primary school curriculum, children would not have the support of school learning to represent these quantities. We do not believe that it would be best to just forget about teaching fractions in primary school because research shows that children have some informal knowledge that could be used as a basis for learning fractions. Thus the question is not whether to teach fractions in primary school but what do we know about their informal knowledge and how can teachers draw on this knowledge.

There are two types of situation in which fractions are used in primary school: measurement and division situations.

When we measure anything, we use a unit of measurement. Often the object we are measuring cannot be described only with whole units, and we need fractions to represent a part of the unit. In the kitchen we might need to use a ½ cup of milk and when setting the margins for a page in a document we often need to be precise and define the margin as, for example, as 3.17 cm. These two examples show that, when it comes to measurement, we use two types of notation, ordinary and decimal notation. But regardless of the notation used, we could not accurately describe the quantities in these situations without using fractions. When we speak of ¾ of a chocolate bar, we are using fractions in a measurement situation: we have less than one unit, so we need to describe the quantity using a fraction.

In division situations, we need a fraction to represent a quantity when the dividend is smaller than the divisor. For example, if 3 cakes are shared among 4 children, it is not possible for each one to have a whole cake, but it is still possible to carry out the division and to represent the amount that each child receives using a number, ¾. It would be possible to use decimal notation in division situations too, but this is rarely the case. The reason for preferring ordinary fractions in these situations is that there are two quantities in division situations: in the example, the number of cakes and the number of children. An ordinary fraction represents each of these quantities by a whole number: the dividend is represented by the numerator, the divisor by the denominator, and the operation of division by the dash between the two numbers.

Although these situations are so similar for adults, we could conclude that it is not necessary to distinguish between them, however, research shows that children think about the situations differently. Children use different schemes of action in each of these situations.

In measurement situations, they use partitioning. If a child is asked to show ¾ of a chocolate, the child will try to cut the chocolate in 4 equal parts and mark 3 parts. If a child is asked to compare ¾ and 6/8, for example, the child will partition one unit in 4 parts, the other in 8 parts, and try to compare the two. This is a difficult task because the partitioning scheme develops over a long period of time and children have to solve many problems to succeed in obtaining equal parts when partitioning. Although partitioning and comparing the parts is not the only way to solve this problem, this is the most likely solution path tried out by children, because they draw on their relevant scheme of action.

In division situations, children use a different scheme of action, correspondences. A problem analogous to the one above in a division situation is: there are 4 children sharing 3 cakes and 8 children sharing 6 identical cakes; if the two groups share the cakes fairly, will the children in one group get the same amount to eat as the children in the other group? Primary school pupils often approach this problem by establishing correspondences between cakes and children. In this way they soon realise that in both groups 3 cakes will be shared by 4 children; the difference is that in the second group there are two lots of 3 cakes and two lots of 4 children, but this difference does not affect how much each child gets.

From the beginning of primary school, many children have some informal knowledge about division that could be used to understand fractional quantities. Between the ages of five and seven years, they are very bad at partitioning wholes into equal parts but can be relatively good at thinking about the consequences of sharing. For example, in one study in London 31% of the five-year-olds, 50% of the six-year-olds and 81% of the seven-year-olds understood the inverse relation between the divisor and the shares resulting from the division: they knew that the more recipients are sharing a cake, the less each one will receive. They were even able to articulate this inverse relation when asked to justify their answers. It is unlikely that they had at this time made a connection between their understanding of quantities and fractional representation; actually, it is unlikely that they would know how to represent the quantities using fractions.

The lack of connection between students' understanding of quantities in division situations and their knowledge about the magnitude of fractions is very clearly documented in research. Students who have no doubt that recipients of a cake shared between 3 people will fare better than those of a cake shared between 5 people may, nevertheless, say that 1/5 is a bigger fraction than 1/3 because 5 is a bigger number than 3. Although they understand the inverse relation in the magnitude of quantities in a division situation, they do not seem to connect this with the magnitude of fractions. The link between their understanding of fractional quantities and fractions as numbers has to be developed through teaching in school.

There is only one well-controlled experiment which compared directly young children's understanding of quantities in measurement and division situations. In this study, carried out in Portugal, the children were six- to seven-years-old. The context of the problems in both situations was very similar: it was about children eating cakes, chocolates or pizzas. In the measurement problems, there was no sharing, only partitioning. For example, in one of the measurement problems, one girl had a chocolate bar which was too large to eat in one go. So she cut her chocolate in 3 equal pieces and ate 1. A boy had an identical bar of chocolate and decided to cut his into 6 equal parts, and eat 2. The children were asked whether the boy and the girl ate the same amount of chocolate. The analogous division problem was about 3 girls sharing one chocolate bar and 6 boys sharing 2 identical chocolate bars. The rate of correct responses in the partitioning situation was 10% for both six- and seven-year-olds and 35% and 49%, respectively, for six- and seven-year-olds in the division situation.

These results are relevant to the assessment of variations in mathematics curricula. Different countries use different approaches in the initial teaching of fraction, some starting from division and others from measurement situations. There is no direct evidence from classroom studies to show whether one starting point results in higher achievement in fractions than the other. The scarce evidence from controlled studies supports the idea that division situations provide children with more insight into the equivalence and order of quantities represented by fractions and that they can learn how to connect these insights about quantities with fractional representation. The studies also indicate that there is little transfer across situations: children who succeed in comparing fractional quantities and fractions after instruction in division situations do no better in a post-test when the questions are about measurement situations than other children in a control group who received no teaching. The converse is also true: children taught in measurement situations do no better than a control group in division situations.

A major debate in mathematics teaching is the relative weight to be given to conceptual understanding and procedural knowledge in teaching. The difference between conceptual understanding and procedural knowledge in the teaching of fractions has been explored in many studies. These studies show that students can learn procedures without understanding their conceptual significance. Studies with adults show that knowledge of procedures can remain isolated from understanding for a long time: some adults who are able to implement the procedure they learned for dividing one fraction by another admit that they have no idea why the numerator and the denominator exchange places in this procedure. Learners who are able to co-ordinate their knowledge of procedures with their conceptual understanding are better at solving problems that involve fractions than other learners who seem to be good at procedures but show less understanding than expected from their knowledge of procedures. These results reinforce the idea that it is very important to try to make links between children's knowledge of fractions and their understanding of fractional quantities.

Finally, there is little, if any, use of fractions to represent relations between quantities in primary school. Secondary school students do not easily quantify relations that involve fractions. Perhaps this difficulty could be attenuated if some teaching about fractions in primary school involved quantifying relations that cannot be described by a single whole number.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's knowledge of fractional quantities starts to develop before they are taught about fractions in school.	**Teaching** Teachers should be aware of children's insights regarding quantities that are represented by fractions and make connections between their understanding of these quantities and fractions.
There are two types of situation relevant to primary school teaching in which quantities cannot be represented by a single whole number: measurement and division.	**Teaching** The primary school curriculum should include the study of both types of situation in the teaching of fractions. Teachers should be aware of the different types of reasoning used by children in each of these situations. **Research** Evidence from experimental studies with larger samples and long-term interventions in the classroom are needed to establish whether division situations are indeed a better starting point for teaching fractions.
Children do not easily transfer their understanding of fractions from division to measurement situations and *vice-versa*.	**Teaching** Teachers should consider how to establish links between children's understanding of fractions in division and measurement situations. **Research** Investigations on how links between situations can be built are needed to support curriculum development and classroom teaching.
Many students do no make links between their conceptual understanding of fractions and the procedures that they are taught to compare and operate on fractions in school.	**Teaching** Greater attention may be required in the teaching of fractions to creating links between procedures and conceptual understanding. **Research** There is a need for longitudinal studies designed to clarify whether this separation between procedural and conceptual knowledge does have important consequences for further mathematics learning.
Fractions are taught in primary school only as representations of quantities.	**Teaching** Consideration should be given to the inclusion of situations in which fractions are used to represent relations. **Research** Given the importance of understanding and representing relations numerically, studies that investigate under what circumstances primary school students can use fractions to represent relations between quantities are urgently needed.

Understanding rational numbers and intensive quantities

Introduction

Rational numbers, like natural numbers, can be used to represent quantities. There are some quantities that cannot be represented by a natural number, and to represent these quantities, we must use rational numbers. We cannot use natural numbers when the quantity that we want to represent numerically:

- is smaller than the unit used for counting, irrespective of whether this is a natural unit (e.g. we have less than one banana) or a conventional unit (e.g. a fish weighs less than a kilo)
- involves a ratio between two other quantities (e.g. the concentration of orange juice in a jar can be described by the ratio of orange concentrate to water; the probability of an event can be described by the ratio between the number of favourable cases to the total number of cases).[1]

The term 'fraction' is often identified with situations where we want to represent a quantity smaller than the unit. The expression 'rational number' usually covers both sorts of examples. In this paper, we will use the expressions 'fraction' and 'rational number' interchangeably. Fractions are considered a basic concept in mathematics learning and one of the foundations required for learning algebra (Fennell, Faulkner, Ma, Schmid, Stotsky, Wu et al. (2008); so they are important for representing quantities and also for later success in mathematics in school.

In the domain of whole numbers, it has been known for some time (e.g. Carpenter and Moser, 1982; Ginsburg, 1977; Riley, Greeno and Heller, 1983) that it is important for the development of children's mathematics knowledge that they establish connections between the numbers and the quantities that they represent. There is little comparable research about rational numbers (but see Mack, 1990), but it is reasonable to expect that the same hypothesis holds: children should learn to connect quantities that must be represented by rational numbers with their mathematical notation. However, the difficulty of learning to use rational numbers is much greater than the difficulty of learning to use natural numbers. This paper discusses why this is so and presents research that shows when and how children have significant insights into the complexities of rational numbers.

In the first section of this paper, we discuss what children must learn about rational numbers and why these might be difficult for children once they have learned about natural numbers. In the second section we describe research which shows that these are indeed difficult ideas for students even at the end of primary school. The third section compares children's reasoning across two types of situations that have been used in different countries to teach children about fractions. The fourth section presents a brief overview of research about children's understanding of intensive quantities. The fifth section considers whether children develop sound understanding of equivalence and order of magnitude of fractions when they learn procedures to compare fractions. The final section summarises our conclusions and discusses their educational implications.

What children must know in order to understand rational numbers

Piaget's (1952) studies of children's understanding of number analysed the crucial question of whether young children can understand the ideas of *equivalence* (cardinal number) and *order* (ordinal number) in the domain of natural numbers. He also pointed out that learning to count may help the children to understand both equivalence and order. All sets that are represented by the same number are equivalent; those that are represented by a different number are not equivalent. Their order of magnitude is the same as the order of the number labels we use in counting, because each number label represents one more than the previous one in the counting string.

The understanding of *equivalence* in the domain of fractions is also crucial, but it is not as simple because language does not help the children in the same way. Two fractional quantities that have different labels can be equivalent, and in fact there is an infinite number of equivalent fractions: 1/3, 2/6, 6/9, 8/12 etc. are different number labels but they represent equivalent quantities. Because rational numbers refer, although often implicitly, to a whole, it is also possible for two fractions that have the same number label to represent different quantities: 1/3 of 12 and 1/3 of 18 are not representations of equivalent quantities.

In an analogous way, it is not possible simply to transfer knowledge of *order* from natural to rational numbers. If the common fraction notation is used, there are two numbers, the numerator and the denominator, and both affect the order of magnitude of fractions, but they do so in different ways. If the denominator is constant, the larger the numerator, the larger is the magnitude of the fraction; if the numerator is constant, the larger the denominator, the smaller is the fraction. If both vary, then more knowledge is required to order the fractions, and it is not possible to tell which quantity is more by simply looking at the fraction labels.

Rational numbers differ from whole numbers also in the use of *two numerical signs* to represent a single quantity: it is the relation between the numbers, not their independent values, that represents the quantity. Stafylidou and Vosniadou (2004) analysed Greek students' understanding of this form of numerical representation and observed that most students in the age range 11 to 13 years did not seem to interpret the written representation of fractions as involving a multiplicative relation between the numerator and the denominator: 20% of the 11-year-olds, 37% of the 12-year-olds and 48% of the 13-year-olds provided this type of interpretation for fractions. Many younger students (about 38% of the 10-year-olds in grade 5) seemed to treat the numerator and denominator as independent numbers whereas others (about 20%) were able to conceive fractions as indicating a part-whole relation but many (22%) are unable to offer a clear explanation for how to interpret the numerator and the denominator.

Rational numbers are also different from natural numbers in their *density* (see, for example, Brousseau, Brousseau and Warfield, 2007; Vamvakoussi and Vosniadou, 2004): there are no natural numbers between 1 and 2, for example, but there is an infinite number of fractions between 1 and 2. This may seem unimportant but it is this difference that allows us to use rational numbers to represent quantities that are smaller than the units. This may be another source of difficulty for students.

Rational numbers have another property which is not shared by natural numbers: every non-zero rational number has a *multiplicative inverse* (e.g. the inverse of 2/3 is 3/2). This property may seem unimportant when children are taught about fractions in primary school, but it is important for the understanding of the division algorithm (i.e. we multiply the fraction which is the dividend by the inverse of the fraction that is the divisor) and will be required later in school, when students learn about algebra. Booth (1981) suggested that students often have a limited understanding of inverse relations, particularly in the domain of fractions, and this becomes an obstacle to their understanding of algebra. For example, when students think of fractions as representing the number of parts into which a whole was cut (denominator) and the number of parts taken (numerator), they find it very difficult to think that fractions indicate a division and that it has, therefore, an inverse.

Finally, rational numbers have *two common written notations*, which students should learn to connect: 1/2 and 0.5 are conceptually the same number with two different notations. There isn't a similar variation in natural number notation (Roman numerals are sometimes used in specific contexts, such as clocks and indices, but they probably play little role in the development of children's mathematical knowledge). Vergnaud (1997) hypothesized that different notations afford the understanding of different aspects of the same concept; this would imply that students should learn to use both notations for rational numbers. On the one hand, the common fractional notation 1/2 can be used to help students understand that fractions are related to the operation of division, because this notation can be interpreted as '1 divided by 2'. The connection between fractions and division is certainly less explicit when the decimal notation 0.5 is used. It is reasonable to expect that students will find it more difficult to understand what the multiplicative inverse of 0.5 is than the inverse of 1/2, but unfortunately there seems to be no evidence yet to clarify this.

On the other hand, adding 1/2 and 3/10 is a cumbersome process, whereas adding the same numbers in their decimal representation, 0.5 and 0.3, is a simpler matter. There are disagreements regarding the order in which these notations should be taught and the need for students to learn both notations in primary school (see, for example, Brousseau, Brousseau and Warfield, 2004; 2007), but, to our knowledge, no one has proposed that one notation should be the only one used and that the other one should be banned from mathematics classes. There is no evidence on whether children find it easier to understand the concepts related to rational numbers when one notation is used rather than the other.

Students' difficulties with rational numbers

Many studies have documented students' difficulties both with understanding equivalence and order of magnitude in the domain of rational numbers (e.g. Behr, Harel, Post and Lesh, 1992; Behr, Wachsmuth, Post and Lesh, 1984; Hart, 1986; Hart, Brown, Kerslake, Küchermann and Ruddock, 1985; Kamii and Clark, 1995; Kerslake, 1986). We illustrate here these difficulties with research carried out in the United Kingdom.

The difficulty of *equivalence* questions varies across types of tasks. Kerslake (1986) noted that when students are given diagrams in which the same shapes are divided into different numbers of sections and asked to compare two fractions, this task is relatively simple because it is possible to use a perceptual comparison. However, if students are given a diagram with six or nine divisions and asked to mark 2/3 of the shape, a large proportion of them fail to mark the equivalent fractions, 4/6 and 6/9. Hart, Brown, Kerslake, Küchermann and Ruddock (1985), working with a sample of students (*N* =55) in the age range 11 to 13 years, found that about 60% of the 11- to 12-year-olds and about 65% of the 12- to 13-year-olds were able to solve this task. We (Nunes, Bryant, Pretzlik and Hurry, 2006) gave the same item more recently to a sample of 130 primary school students in Years 4 and 5 (mean ages, respectively, 8.6 and 9.6 years). The rate of correct responses across these items was 28% for the children in Year 4 and 49% for the children in Year 5. This low percentage of correct answers could not be explained by a lack of knowledge of the fraction 2/3: when the diagram was divided into three sections, 93% of the students in the study by Hart *el al.* (1985) gave a correct answer; in our study, 78% of the Year 4 and 91% of the Year 5 students' correctly shaded 2/3 of the figure.

This quantitative information is presented here to illustrate the level of difficulty of these questions. A different approach to the analysis of how the level of difficulty can vary is presented later, in the third section of this paper.

Students often have difficulty in *ordering* fractions according to their magnitude. Hart *et al.* (1985) asked students to compare two fractions with the same denominator (3/7 and 5/7) and two with the same numerator (3/5 and 3/4). When the fractions have the same denominator, students can respond correctly by considering the numerators only and ordering them as if they were natural numbers. The rate of correct responses in this case is relatively high but it does not effectively test students' understanding of rational numbers. Hart *et al.* (1985) observed approximately 90% correct responses among their students in the age range 11 to 13 years and we (Nunes *et al.*, 2006) found that 94% of the students in Year 4 and 87% of the students in Year 5 gave correct responses. In contrast, when the numerator was the same and the denominator varied (comparing 3/5 and 3/4), and the students had to consider the value of the fractions in a way that is not in agreement with the order of natural numbers, the rate of correct responses was considerably lower: in the study by Hart *et al.*, approximately 70% of the answers were correct, whereas in our study the percent of correct responses were 25% in Year 4 and 70% among in Year 5.

These difficulties are not particular to U.K. students: they have been widely reported in the literature on equivalence and order of fractions (for examples in the United States see Behr, Lesh, Post and Silver, 1983; Behr, Wachsmuth, Post and Lesh 1984; Kouba, Brown, Carpenter, Lindquist, Silver and Swafford, 1988).

Difficulties in comparing rational numbers are not confined to fractions. Resnick, Nesher, Leonard, Magone, Omanson and Peled (1989) have shown that students have difficulties in comparing decimal fractions when the number of places after the decimal point differs. The samples in their study were relatively small (varying from 17 to 38) but included students from three different countries, the United States, Israel and France, and in three grade levels (4^{th} to 6^{th}). The children were asked to compare pairs of decimals such as 0.5 and 0.36, 2.35 and 2.350, and 4.8 and 4.63. The rate of correct responses varied between 36% and 52% correct, even though all students had received instruction on decimals. A more recent study (Lachance and Confrey, 2002) of 5^{th} grade students (estimated age approximately 10 years) who had received an introduction to decimal fractions in the previous year showed that only about 43% were able to compare decimal fractions correctly.[2] Rittle-Johnson, Siegler and Alibali (2001) confirmed students' difficulties when comparing the magnitude of decimals: the rate of correct responses by the students ($N = 73$; 5^{th} grade; mean age 11 years 8 months) in their study was 19%.

In conclusion, the very basic ideas about *equivalence* and *order* of fractions by magnitude, without which we could hardly say that the students have a good sense for what fractions represent, seems to elude many students for considerable periods of time. In the section that follows, we will contrast two situations that have been used to introduce the concept of fractions in primary school in order to examine the question of whether children's learning may differ as a function of these differences between situations.

Children's schemas of action in division situations

Mathematics educators and researchers may not agree on many things, but there is a clear consensus among them on the idea that rational numbers are numbers in the domain of quotients (Brousseau, Brousseau and Warfield, 2007; Kieren, 1988; 1993; 1994; Ohlsson, 1988): that is, numbers defined by the operation of division. So, it seems reasonable to seek the origin of children's understanding of rational numbers in their understanding of division.[3]

Our hypothesis is that in division situations children can develop some insight into the equivalence and order of quantities in fractions; we will use the term fractional quantities to refer to these quantities. These insights can be developed *even in the absence of knowledge of representations for fractions*, either in written or in oral form. Two schemes of action that children use in division have been analysed in the literature: partitioning and correspondences (or dealing).

Behr, Harel, Post and Lesh (1992; 1993) pointed out that fractions represent quantitites in a different way across two types of situation. The first type is the *part-whole situation*. Here one starts with a single quantity, the whole, which is divided into a certain number of parts (y), out of which a specified number is taken (x); the symbol x/y represents this quantity in terms of part-whole relations. *Partitioning* is the scheme of action that children use in part-whole tasks. The most common type of fraction problem that teachers give to children is to ask them to partition a whole into a fixed number of parts (the denominator) and show a certain fraction with this denominator. For example, the children have to show what 3/5 of a pizza is.[4]

The second way in which fractions represent quantities is in *quotient situations*. Here one starts with two quantities, x and y, and treats x as the dividend and y as the divisor, and by the operation of *division* obtains a single quantity x/y. For example, the quantities could be 3 chocolates (x) to be shared among 5 children (y). The fractional symbol x/y represents both the division (3 divided by 5) and the quantity that each one will receive (3/5). A quotient situation calls for the use of *correspondences* as the scheme of action: the children establish correspondences between portions and recipients. The portions may be imagined by the children, not actually drawn, as they must be when the children are asked to partition a whole and show 3/5.[5]

When children use the scheme of partitioning in part-whole situations, they can gain insights about quantities that could help them understand some principles relevant to the domain of rational numbers. They can, for example, reason that, the more parts they cut the whole into, the smaller the parts will be. This could help them understand how fractions are ordered. If they can achieve a higher level of precision in reasoning about partitioning,
they could develop some understanding of the equivalence of fractions: they could come to understand that, if they have twice as many parts, each part would be halved in size. For example, you would eat the same amount of chocolate after cutting one chocolate bar into two parts and eating one part as after cutting it into four parts and eating two, because the number of parts and the size of the parts compensate for each other precisely. It is an

empirical question whether children attain these understandings in the domain of whole numbers and extend them to rational numbers.

Partitioning is the scheme that is most often used to introduce children to fractions in the United Kingdom, but it is not the only scheme of action relevant to division. Children use correspondences in quotient situations when the dividend is one quantity (or measure) and the divisor is another quantity. For example, when children share out chocolate bars to a number of recipients, the dividend is in one domain of measures – the number of chocolate bars – and the divisor is in another domain – the number of children. The difference between partitioning and correspondence division is that in partitioning there is a single whole (i.e. quantity or measure) and in correspondence there are two quantities (or measures).

Fischbein, Deri, Nello and Marino (1985) hypothesised that children develop implicit models of division situations that are related to their experiences. We use their hypothesis here to explore what sorts of implicit models of fractions children may develop from using the partitioning or the correspondence scheme in fractions situations. Fischbein and colleagues suggested, for example, that children form an implicit model of division that has a specific constraint: the dividend must be larger than the divisor. We ourselves hypothesise that this implicit model is developed only in the context of partitioning. When children use the correspondence scheme, precisely because there are two domains of measures, young children readily accept that the dividend can be smaller than the divisor: they are ready to agree that it is perfectly possible to share one chocolate bar among three children.

At first glance, the difference between these two schemes of action, partitioning and correspondence, may seem too subtle to be of interest when we are thinking of children's understanding of fractions. Certainly, research on children's understanding of fractions has not focused on this distinction so far. However, our review shows that it is a crucial distinction for children's learning, both in terms of what insights each scheme of action affords and in terms of the empirical research results.

There are at least four differences between what children might learn from using the partitioning scheme or the scheme of correspondences.

- The first is the one just pointed out: that, when children set two measures in correspondence, there is no necessary relation between the size of the dividend and of the divisor. In contrast, in partitioning children form the implicit model that the sum of the parts must not be larger than the whole. Therefore, it may be easier for children to develop an understanding of improper fractions when they form correspondences between two fields of measures than when they partition a single whole. They might have no difficulty in understanding that 3 chocolates shared between 2 children means that each child could get one chocolate plus a half. In contrast, in partitioning situations children might be puzzled if they are told that someone ate 3 parts of a chocolate divided in 2 parts.

- A second possible difference between the two schemes of action may be that, when using correspondences, children can reach the conclusion that the way in which partitioning is carried out does not matter, as long as the correspondences between the two measures are 'fair'. They can reason, for example, that if 3 chocolates are to be shared by 2 children, it is not necessary to divide all 3 chocolates in half, and then distribute the halves; giving a whole chocolate plus a half to each child would accomplish the same fairness in sharing. It was argued in the first section of this paper that this an important insight in the domain of rational numbers: different fractions can represent the same quantity.

- A third possible insight about quantities that can be obtained from correspondences more easily than from partitioning is related to ordering of quantities. When forming correspondences, children may realize that there is an inverse relation between the divisor and the quotient: the more people there are to share a cake, the less each person will get: Children might achieve the corresponding insight about this inverse relation using the scheme of partitioning: the more parts you cut the whole into, the smaller the parts. However, there is a difference between the principles that children would need to abstract from each of the schemes. In partitioning, they need to establish a within-quantity relation (the more parts, the smaller the parts) whereas in correspondence they need to establish a between-quantity relation (the more children, the less cake). It is an empirical matter to find out whether or not it is easier to achieve one of these insights than the other.

- Finally, both partitioning and correspondences could help children to understand something about the equivalence between quantities, but the reasoning required to achieve this understanding differs across the two schemes of action. When setting chocolate bars in correspondence with recipients, the children might be able to reason that, if there were twice as many chocolates and twice as many children, the shares would be equivalent, even though the dividend and the divisor are different. This may be easier than the comparable reasoning in partitioning. In partitioning, understanding equivalence is based on *inverse* proportional reasoning (twice as many pieces means that each piece is half the size) whereas in contexts where children use the correspondence scheme, the reasoning is based on a *direct* proportion (twice as many chocolates and twice as many children means that everyone still gets the same).

This exploratory and hypothetical analysis of how children can reach an understanding of equivalence and order of fractions when using partitioning or correspondences in division situations suggests that the distinction between the two schemas is worth investigating empirically. It is possible that the scheme of correspondences affords a smoother transition from natural to rational numbers, at least as far as understanding equivalence and order of fractional quantities is concerned.

We turn now to an empirical analysis of this question. The literature about these schemes of action is vast but this paper focuses on research that sheds light on whether it is possible to find continuities between children's understanding of quantities that are represented by natural numbers and fractional quantities. We review research on correspondences first and then research on partitioning.

Children's use of the correspondence scheme in judgements about quantities

Piaget (1952) pioneered the study of how and when children use the correspondence scheme to draw conclusions about quantities. In one of his studies, there were three steps in the method.

- First, Piaget asked the children to place one pink flower into each one of a set of vases;
- next, he removed the pink flowers and asked the children to place a blue flower into each one of the same vases;
- then, he set all the flowers aside, leaving on the table only the vases, and asked the children to take from a box the exact number of straws required if they wanted to put one flower into each straw.

Without counting, and only using correspondences, five- and six-year old children were able to make inferences about the equivalence between straws and flowers: by setting two straws in correspondence with each vase, they constructed a set of straws equivalent to the set of vases. Piaget concluded that the children's judgements were based on 'multiplicative equivalences' (p. 219) established by the use of the correspondence scheme: the children reasoned that, if there is a 2-to-1 correspondence between flowers and vases and a 2-to-1 correspondence between straws and vases, the number of flowers and straws must be the same.

In Piaget's study, the scheme of correspondence was used in a situation that involved ratio but not division. Frydman and Bryant (1988) carried out a series of studies where children established correspondences between sets in a division situation which we have described in more detail in Paper 2, Understanding whole numbers. The studies showed that children aged four often shared pretend sweets fairly, using a one-for-you one-for-me type of procedure. After the children had distributed the sweets, Frydman and Bryant asked them to count the number of sweets that one doll had and then deduce the number of sweets that the other doll had. About 40% of the four-year-olds were able to make the necessary inference and say the exact number of sweets that the second doll had; this proportion increased with age. This result extends Piaget's observations that children can make equivalence judgements not only in multiplication but also in division problems by using correspondence.

Frydman and Bryant's results were replicated in a number of studies by Davis and his colleagues (Davis and Hunting, 1990; Davis and Pepper, 1992; Pitkethly and Hunting, 1996), who refer to this scheme of action as 'dealing'. They used a variety of situations, including redistribution when a new recipient comes, to study children's ability to use correspondences in division situations and to make inferences about equality and order of magnitude of quantities. They also argue that this scheme is basic to children's understanding of fractions (Davis and Pepper, 1992).

Correa, Nunes and Bryant (1998) extended these studies by showing that children can make inferences about quantities resulting from a division not only when the divisors are the same but also when they are different. In order to circumvent the possibility that children feel the need to count the sets after division because they think that they could have made a mistake in sharing, Bryant and his colleagues did not ask the children to do the sharing: the sweets were shared by the experimenter, outside the children's view, after the children had seen that the number of sweets to be shared was the same.

There were two conditions in this study: same dividend and same divisor versus same dividend and different divisors. In the same dividend and same divisor condition, the children should be able to conclude for the equivalence between the sets that result from the division; in the same dividend and different divisor condition, the children should conclude that the more recipients there are, the fewer sweets they receive; i.e. in order to answer correctly, they would need to use the inverse relation between the divisor and the result as a principle, even if implicitly.

About two-thirds of the five-year-olds, the vast majority of the six-year-olds, and all the seven-year-olds concluded that the recipients had equivalent shares when the dividend and the divisor were the same. Equivalence was easier than the inverse relation between divisor and quotient: 34%, 53% and 81% of the children in these three age levels, respectively, were able to conclude that the more recipients there are, the smaller each one's share will be. Correa (1994) also found that children's success in making these inferences improved if they solved these problems after practising sharing sweets between dolls; this indicates that thinking about how to establish correspondences improves their ability to make inferences about the relations between the quantities resulting from sharing.

In all the previous studies, the dividend was composed of discrete quantities and was larger than the divisor. The next question to consider is whether children can make similar judgements about equivalence when the situations involve continuous quantities and the dividend is smaller than the divisor: that is, when children have to think about fractional quantities.

Kornilaki and Nunes (2005) investigated this possibility by comparing children's inferences in division situations in which the quantities were discrete and the dividends were larger than the divisors to their inferences in situations in which the quantities were continuous and dividends smaller than the divisors. In the discrete quantities tasks, the children were shown one set of small toy fishes to be distributed fairly among a group of white cats and another set of fishes to be distributed to a group of brown cats; the number of fish was always greater than the number of cats. In the continuous quantities tasks, the dividend was made up of fish-cakes, to be distributed fairly among the cats: the number of cakes was always smaller than the number of cats, and varied between 1 and 3 cakes, whereas the number of cats to receive a portion in each group varied between 2 and 9. Following the paradigm devised by Correa, Nunes and Bryant (1998), the children were neither asked to distribute the fish nor to partition the fish cakes. They were asked whether, after a fair distribution in each group, each cat in one group would receive the same amount to eat as each cat in the other group.[6]

In some trials, the number of fish (dividend) and cats (divisor) was the same; in other trials, the dividend was the same but the divisor was different. So in the first type of trials the children were asked about equivalence after sharing and in the second type the children were asked to order the quantities obtained after sharing.

The majority of the children succeeded in all the items where the dividend and the divisor were the same: 62% of the five-year-olds, 84% of the six-year-olds and all the seven-year-olds answered all the questions correctly. When the dividend was the same and the divisors differed, the rate of success was 31%, 50% and 81%, respectively, for the three age levels. There was no difference in the level of success attained by the children with discrete versus continuous quantities.

In almost all the items, the children explained their answers by referring to the type of relation between the dividends and the divisors: same divisor, same share or, with different divisors, the more cats receiving a share, the smaller their share. The use of numbers as an explanation for the relative size of the recipients' shares was observed in 6% of answers by the seven-year-olds when the quantities were discrete and less often than this by the younger children. Attempts to use numbers to speak about the shares in the continuous quantities trials were practically non-existent (3% of the seven-year-olds' explanations). Thus, the analysis of justifications supports the idea that the children were reasoning about relations between quantities rather than using counting when they made their judgments of equivalence or ordered the quantities that would be obtained after division.

This study replicated the previous findings, which we have mentioned already, that young children can use correspondences to make inferences about equivalences and also added new evidence relevant to children's understanding of fractional quantities: many young children who have never been taught about fractions used correspondences to order fractional quantities. They did so successfully when the division would have resulted in unitary fractions and also when the dividend was greater than 1 and the result would not be a unitary fraction (e.g. 2 fish cakes to be shared by 3, 4 or 5 cats).

A study by knowledge Mamede, Nunes and Bryant (2005) confirmed that children can make inferences about the order of magnitude of fractions in sharing situations where the dividend is smaller than the divisor (e.g. 1 cake shared by 3 children compared to 1 cake shared by 5 children). She worked with Portuguese children in their first year in school, who had received no school instruction about fractions. Their performance was only slightly weaker than that of British children: 55% of the six-year-olds and 71% of the seven-year-olds were able to make the inference that the larger the divisor, the smaller the share that each recipient would receive.

These studies strongly suggest that children can learn principles about the relationship between dividend and divisor from experiences with sharing when they establish correspondences between the two domains of measures, the shared quantities and the recipients. They also suggest that children can make a relatively smooth transition from natural numbers to rational numbers when they use correspondences to understand the relations between quantities. This argument is central to Streefland's (1987; 1993; 1997) hypothesis about what is the best starting point for teaching fractions to children and has been advanced by others also (Davis and Pepper, 1992; Kieren, 1993; Vergnaud, 1983).

This research tell an encouraging story about children's understanding of the logic of division even when the dividend is smaller than the divisor, but there is one further point that should be considered in the transition between natural and rational numbers. In the domain of rational numbers there is an infinite set of equivalences (e.g. 1/2 = 2/4 = 3/6 etc) and in the studies that we have described so far the children were only asked to make equivalence judgements when the dividend and the divisor in the equivalent fractions were the same. Can they still make the inference of equivalence in sharing situations when the dividend and the divisor are different across situations, but the dividend-divisor ratio is the same?

Nunes, Bryant, Pretzlik, Bell, Evans and Wade (2007) asked British children aged between 7.5 and 10 years, who were in Years 4 and 5 in school, to make comparisons between the shares that would be received by children in sharing situations where the dividend and divisor were different but their ratio was the same. Previous research (see, for example, Behr, Harel, Post and Lesh, 1992; Kerslake, 1986) shows that children in these age levels have difficulty with the equivalence of fractions. The children in this study had received some instruction on fractions: they had been taught about halves and quarters in problems about partitioning. They had only been taught about one pair of equivalent fractions: they were taught that one half is the same as two quarters. In the correspondence item in this study, the children were presented with two pictures: in the first, a group of 4 girls was going to share fairly 1 pie; in the second, a group of 8 boys was going to share fairly 2 pies that were exactly the same as the pie that the girls had. The question was whether each girl would receive the same share as each boy. The overall rate of correct responses was 73% (78% in Year 4 and 70% in Year 5; this difference was not significant). This is an encouraging result: the children had only been taught about halves and quarters; nevertheless, they were able to attain a high rate of correct responses for fractional quantities that could be represented as 1/4 and 2/8.

In the studies reviewed so far the children were asked about quantities that resulted from division and always included two domains of measures; thus the children's correspondence reasoning was engaged in these studies. However, they did not involve asking the children to represent these quantities through fractions. The final study reviewed here is a brief teaching study (Nunes, Bryant, Pretzlik, Evans, Wade and Bell, 2008), where the children were taught to represent fractions in the context of two domains of measures, shared quantities and recipients, and were asked about the equivalence between fractions. The types of arguments that the children produced to justify the equivalence of the fractions were then analyzed and compared to the insights that we hypothesized would emerge in the context of sharing from the use of the correspondence scheme.

Brief teaching studies are of great value in research because they allow the researchers to know what understandings children can construct if they are given a specific type of guidance in the interaction with an adult (Cooney, Grouws and Jones, 1988; Steffe and Tzur, 1994; Tzur, 1999; Yackel, Cobb, Wood, Wheatley and Merkel, 1990). They also have compelling ecological validity: children spend much of their time in school trying to use what they have been taught to solve new mathematics problems. Because this study has only been published in a summary form (Nunes, Bryant, Pretzlik and Hurry, 2006), some detail is presented here.

The children ($N = 62$) were in the age range from 7.5 to 10 years, in Years 4 and 5 in school. Children in Year 4 had only been taught about half and quarters and the equivalence between half and two quarters; children in Year 5 had been taught also about thirds. They worked with a researcher outside the classroom in small groups (12 groups of between 4 and 6 children, depending on the class size) and were asked to solve each problem first individually, and then to discuss their answers in the group. The sessions were audio- and video-recorded. The children's arguments were transcribed *verbatim;* the information from the video-tapes was later coordinated with the transcripts in order to help the researchers understand the children's arguments.

In this study the researchers used problems developed by Streefland (1990). The children solved two of his sharing tasks on the first day and an equivalence task on the second day of the teaching study. The tasks were presented in booklets with pictures, where the children also wrote their answers. The tasks used in the first day were:

- Six girls are going to share a packet of biscuits. The packet is closed; we don't know how many biscuits are in the packet. (a) If each girl received one biscuit and there were no biscuits left, how many biscuits were in the packet? (b) If each girl received a half biscuit and there were no biscuits left, how many biscuits were in the packet? (c) If some more girls join the group, what will happen when the biscuits are shared? Do the girls now receive more or less each than the six girls did?
- Four children will be sharing three chocolates. (a) Will each child be able to get one bar of chocolate? (b) Will each child be able to get at least a half bar of chocolate? (c) How would you share the chocolate? (The booklets contained a picture with three chocolate bars and four children and the children were asked to show how they would share the chocolates) Write what fraction each one gets.

After the children had completed these tasks, the researcher told them that they were going to practice writing fractions which they had not yet learned in school. The children were asked to write 'half' with numerical symbols, which they knew already. The researcher taught the children to write fractions that they had not yet learned in school in order to help the children re-interpret the meaning of fractions. The numerator was to be used to represent the number of items to be divided, the denominator should represent the number of recipients, and the dash between them two numbers should represent the sign for division (for a discussion of children's interpretation of fraction symbols in this situation, see Charles and Nason, 2000, and Empson, Junk, Dominguez and Turner, 2005).

The equivalence task, presented on the second day, was:

- Six children went to a pizzeria and ordered two pizzas to share between them. The waiter brought one first and said they could start on it because it would take time for the next one to come. (a) How much will each child get from the first pizza that the waiter brought? Write the fraction that shows this. (b) How much will each child get from the second pizza? Write your answer. (c) If you add the two pieces together, what fraction of a pizza will each child get? You can write a plus sign between the first fraction and the second fraction, and write the answer for the share each child gets in the end. (d) If the two pizzas came at the same time, how could they share it differently? (e) Are these fractions (the ones that the children wrote for answers c and d equivalent?

According to the hypotheses presented in the previous section, we would expect children to develop some insights into rational numbers by thinking about different ways of sharing the same amount. It was expected that they would realize that:

1 it is possible to divide a smaller number by a larger number

2 different fractions might represent the same amount

3 twice as many things to be divided and twice as many recipients would result in equivalent amounts

4 the larger the divisor, the smaller the quotient.

The children's explanations for why they thought that the fractions were equivalent provided evidence for all these insights, and more, as described below.

It is possible to divide a smaller number by a larger number

There was no difficulty among the students in attempting to divide 1 pizza among 6 children. In response to part a of the equivalence problem, all children wrote at least one fraction correctly (some children wrote more than one fraction for the same answer, always correctly).

In response to part c, when the children were asked how they could share the 2 pizzas if both pizzas came at the same time and what fraction would each one receive, some children answered 1/3 and others answered 2/12 from each pizza, giving a total share of 4/12. The latter children, instead of sharing 1 pizza among 3 girls, decided to cut each pizza in 12 parts: i.e. they cut the sixths in half.

Different fractions can represent the same amount

The insight that different fractions can represent the same amount was expressed in all groups. For example, one child said that, 'They're the same amount of people, the same amount of pizzas, and that means the same amount of fractions. It doesn't matter how you cut it.' Another child said, 'Because it wouldn't really matter when they shared it, they'd get that [3 girls would get 1 pizza], and then they'd get that [3 girls would get the other pizza], and then it would be the same.' Another child said, 'It's the same amount of pizza. They might be different fractions but the same amount [this child had offered 4/12 as an alternative to 2/6].' Another child said: 'Erm, well basically just the time doesn't make much difference, the main thing is the number of things.'

When the dividend is twice as large and the divisor is also twice as large, the result is an equivalent amount

The principle that when the dividend is twice as large and the divisor is also twice as large, the result is an equivalent amount was expressed in 11 of the 12 groups. For example, one child said, 'It's half the girls and half the pizzas; three is a half of six and one is a half of two.' Another child said, 'If they have two pizzas, then they could give the first pizza to three girls and then the next one to another three girls. (…) If they all get one piece of that each, and they get the same amount, they all get the same amount'.

So all three ideas we thought that could appear in this context were expressed by the children. But two other principles, which we did not expect to observe in this correspondence problem, were also made explicit by the children.

The number of parts and size of parts are inversely proportional

The principle that the number of parts and size of parts are inversely proportional was enunciated in 8 of the 12 groups. For example, one child who cut the pizzas the second time around in 12 parts each said, 'Because it's double the one of that [total number of pieces] and it's double the one of that [number of pieces for each], they cut it twice and each is half the size; they will be the same'. Another child said, 'Because one sixth and one sixth is actually a different way in fractions [from 1 third] and it doubled [the number of pieces] to make it [the size of the piece] littler, and halving [the number of pieces] makes it [the size of the piece] bigger, so I halved it and it became one third'.

The fractions show the same part-whole relation

The reasoning that the fractions show the same part-whole relation, which we had not expected to emerge from the use of the correspondence scheme, was enunciated in only one group (out of 12), initially by one child, and was then reiterated by a second child in her own terms. The first child said, 'You need three two sixths to make six [6/6 – he shows the 6 pieces marked on one pizza], and you need three one thirds to make three (3/3 – shows the 3 pieces marked on one pizza). [He wrote the computation and continued] There's two sixths, add two sixths three times to make six sixths. With one third, you need to add one third three times to make three thirds.'

To summarize: this brief teaching experiment was carried out to elicit discussions between the children in situations where they could use the correspondence scheme in division. The first set of problems, in which they are asked about sharing discrete quantities, created a background for the children to use this scheme of action. The researchers then helped them to construct an interpretation for written fractions where the numerator is the dividend, the denominator is the divisor, and the line indicates the operation of division. This interpretation did not replace their original interpretation of number of parts taken from the whole; the two meanings co-existed and appeared in the children's arguments as they explained their answers. In the subsequent problems, where the quantity to be shared was continuous and the dividend was smaller than the divisor, the children had the opportunity to explore the different ways in which continuous quantities can be shared. They were not asked to actually partition the pizzas, and some made marks on the pizzas whereas others did not. The most salient feature of the children's drawings was that they were not concerned with partitioning per se, even when the parts were marked, but with the correspondences between pizzas and recipients. Sometimes the correspondences were carried out mentally and expressed verbally and sometimes the children used drawings and gestures which indicated the correspondences.

Other researchers have identified children's use of correspondences to solve problems that involve fractions, although they did not necessarily use this label in describing the children's answers. Empson (1999), for example, presented the following problem to children aged about six to seven years (first graders in the USA): 4 children got 3 pancakes to share; how many pancakes are needed for 12 children in order for the children to have the same amount of pancake as the first group? She reported that 3 children solved this problem by partitioning and 3 solved it by placing 3 pancakes in correspondence to each group of 4 children. Similar strategies were reported when children solved another problem that involved 2 candy bars shared among 3 children.

Kieren (1993) also documented children's use of correspondences to compare fractions. In his problem, the fractions were not equivalent: there were 7 recipients and 4 items in Group A and 4 recipients and 2 items in Group B. The children were asked how much each recipient would get in each group and whether the recipients in both groups would get the same amount. Kieren presents a drawing by an eight-year-old, where the items are partitioned in half and the correspondences between the halves and the recipients are shown; in Group A, a line without a recipient shows that there is an extra half in that group and the child argues that there should be one more person in Group A for the amounts to be the same. Kieren termed this solution 'corresponding or 'ratiolike' thinking' (p. 54).

Conclusion

The scheme of correspondences develops relatively early: about one-third of the five-year-olds, half of six-year-olds and most seven-year-olds can use correspondences to make inferences about equivalence and order in tasks that involve fractional quantities. Children can use the scheme of correspondences to:

- establish equivalences between sets that have the same ratio to a reference set (Piaget, 1952)
- re-distribute things after having carried out one distribution (Davis and Hunting, 1990; Davis and Pepper, 1992; Davis and Pitkethly, 1990; Pitkethly and Hunting, 1996);
- reason about equivalences resulting from division both when the dividend is larger or smaller than the divisor (Bryant and colleagues: Correa, Nunes and Bryant, 1994; Frydman and Bryant, 1988; 1994; Empson, 1999; Nunes, Bryant, Pretzlik and Hurry, 2006; Nunes, Bryant, Pretzlik, Bell, Evans and Wade, 2007; Mamede, Nunes and Bryant, 2005);
- order fractional quantities (Kieren, 1993; Kornilaki and Nunes, 2005; Mamede, 2007).

These studies were carried out with children up to the age of ten years and all of them produced positive results. This stands in clear contrast with the literature on children's difficulties with fractions and prompts the question whether the difficulties might stem from the use of partitioning as the starting point for the teaching of fractions (see also Lamon, 1996; Streefland, 1987). The next section examines the development of children's partitioning action and its connection with children's concepts of fractions.

Children's use of the scheme of partitioning in making judgements about quantities

The scheme of partitioning has been also named subdivision and dissection (Pothier and Sawada, 1983), and is consistently defined as the process of dividing a whole into parts. This process is understood not as the activity of cutting something into parts in any way, but as a process that must be guided from the outset by the aim of obtaining a pre-determined number of equal parts.

Piaget, Inhelder and Szeminska (1960) pioneered the study of the connection between partitioning and fractions. They spelled out a number of ideas that they thought were necessary for children to develop an understanding of fractions, and analysed them in partitioning tasks. The motivation for partitioning was sharing a cake between a number of recipients, but the task was one of partitioning. They suggested that 'the notion of fraction depends on two fundamental relations: the relation of part to whole (…) and the relation of part to part' (p. 309). Piaget and colleagues identified a number of insights that children need to achieve in order to understand fractions:

1 the whole must be conceived as divisible, an idea that children under the age of about two do not seem to attain

2 the number of parts to be achieved is determined from the outset

3 the parts must exhaust the whole (i.e. there should be no second round of partitioning and no remainders)

4 the number of cuts and the number of parts are related (e.g. if you want to divide something in 2 parts, you should use only 1 cut)

5 all the parts should be equal

6 each part can be seen as a whole in itself, nested into the whole but also susceptible of further division

7 the whole remains invariant and is equal to the sum of the parts.

Piaget and colleagues observed that children rarely achieved correct partitioning (sharing a cake) before the age of about six years. There is variation in the level of success depending on the shape of the whole (circular areas are more difficult to partition than rectangles) and on the number of parts. A major strategy in carrying out successful partitioning was the use of successive divisions in two: so children are able to succeed in dividing a whole into fourths before they can succeed with thirds. Successive halving helped the children with some fractions: dividing something into eighths is easier this way. However, it interfered with success with other fractions: some children, attempting to divide a whole into fifths, ended up with sixths by dividing the whole first in halves and then subdividing each half in three parts.

Piaget and colleagues also investigated children's understanding of their seventh criterion for a true concept of fraction, i.e. the conservation of the whole. This conservation, they argued, would require the children to understand that each piece could not be counted simply as one piece, but had to be understood in its relation to the whole. Some children aged six and even seven years failed to understand this, and argued that if someone ate a cake cut into 1/2 + 2/4 and a second person ate a cake cut into 4/4, the second one would eat more because he had four parts and the first one only had three. Although these children would recognise that if the pieces were put together in each case they would form one whole cake, they still maintained that 4/4 was more than 1/2 + 2/4. Finally, Piaget and colleagues also observed that children did not have to achieve the highest level of development in the scheme of partitioning in order to understand the conservation of the whole.

Children's difficulties with partitioning continuous wholes into equal parts have been confirmed many times in studies with pre-schoolers and children in their first years in school (e.g. Hiebert and Tonnessen, 1978; Hunting and Sharpley, 1988 *b*,) observed that children often do not anticipate the number of cuts and fail to cut the whole extensively, leaving a part of the whole un-cut. These studies also extended our knowledge of how children's expertise in partitioning develops. For example, Pothier and Sawada (1983) and Lamon (1996) proposed more detailed schemes for the analysis of the development of partitioning schemes and other researchers (Hiebert and Tonnessen, 1978; Hunting and Sharpley, 1988 *a* and *b*; Miller, 1984; Novillis, 1976) found that the difficulty of partitioning discrete and continuous quantities is not the same, as hypothesized by Piaget. Children can use a procedure for partitioning discrete quantities that is not applicable to continuous quantities: they can 'deal out' the discrete quantities but not the continuous ones. Thus they perform significantly better with the former than the latter. This means that the transition from discrete to continuous quantities in the use of partitioning is difficult, in contrast to the smooth transition noted in the case of the correspondence scheme.

These studies showed that the scheme of partitioning continuous quantities develops slowly, over a longer period of time. The next question to consider is whether partitioning can promote the understanding of equivalence and ordering of fractions once the scheme has developed.

Many studies investigated children's understanding of equivalence of fractions in partitioning contexts (e.g. Behr, Lesh, Post and Silver, 1983; Behr, Wachsmuth, Post and Lesh, 1984; Larson, 1980; Kerslake, 1986), but differences in the methods used in these studies render the comparisons between partitioning and correspondence studies ambiguous. For example, if the studies start with a representation of the fractions, rather than with a problem about quantities, they cannot be compared to the studies reviewed in the previous section, in which children were asked to think about quantities without necessarily using fractional representation. We shall not review all studies but only those that use comparable methods.

Kamii and Clark (1995) presented children with identical rectangles and cut them into fractions using different cuts. For example, one rectangle was cut horizontally in half and the second was cut across a diagonal. The children had the opportunity to verify that the rectangles were the same size and that the two parts from each rectangle were the same in size. They asked the children: if these were chocolate cakes, and the researcher ate a part cut from the first rectangle and the child ate a part cut from the second, would they eat the same amount? This method is highly comparable to the studies by Kornilaki and Nunes (2005) and by Mamede (2007), where the children do not have to carry out the actions, so their difficulty with partitioning does not influence their judgements. They also use similarly motivated contexts, ending in the question of whether recipients would eat the same amount. However, the question posed by Kamii and Clark draws on the child's understanding of partitioning and the relations between the parts of the two wholes because each whole corresponds to a single recipient.

The children in Kamii's study were considerably older than those in the correspondence studies: they were in the fifth or sixth year in school (approximately 11 and 12 years). Both groups of children had been taught about equivalent fractions. In spite of having received instruction, the children's rate of success was rather low: only 44% of the fifth graders and 51% of the sixth graders reasoned that they would eat the same amount of chocolate cake because these were halves of identical wholes.

Kamii and Clark then showed the children two identical wholes, cut one in fourths using a horizontal and a vertical cut, and the other in eighths, using only horizontal cuts. They discarded one fourth from the first 'chocolate cake', leaving three fourths be eaten, and asked the children to take the same amount from the other cake, which had been cut into eighths, for themselves. The percentage of correct answers was this time even lower: 13% of the fifth graders and 32% of the sixth graders correctly identified the number of eighths required to take the same amount as three fourths.

Recently, we (Nunes and Bryant, 2004) included a similar question about halves in a survey of English children's knowledge of fractions. The children in our study were in their fourth and fifth year (mean ages eight and a half and nine and a half, respectively) in school. The children were shown pictures of a boy and a girl and two identical rectangular areas, the 'chocolate cakes'. The boy cut his cake along the diagonal and the girl cut hers horizontally. The children were asked to indicate whether they ate the same amount of cake and, if not, to mark the child who ate more. Our results were more positive than Kamii and Clark's: 55% of the children in year four (eight and a half year olds) in our study answered correctly. However, these results are weak by comparison to children's rate of correct responses when the problem draws on their understanding of correspondences. In the Kornilaki and Nunes study, 100% of the seven-year-olds (third graders) realized that two divisions that have the same dividend and the same divisor result in equivalent shares. Our results with fourth graders, when both the dividend and the divisor were different, still shows a higher rate of correct responses when correspondences are used: 78% of the fourth graders gave correct answers when comparing one fourth and two eighths.

In the preceding studies, the students had to think about the quantities ignoring their perceptual appearance. Hart *et al.* (1985) and Nunes *et al.*

(2004) presented students with verbal questions, which did not contain drawings that could lead to incorrect conclusions based on perception. In both studies, the children were told that two boys had identical chocolate bars; one cut his into 8 parts and ate 4 and the other cut his into 4 parts and ate 2. Combining the results of these two studies, it is possible to see how the rate of correct responses changed across age: 40% at ages 8 to 9 years, 74% at 10 to11 years, 60% at 11 to 12 years, and 64% at 12 to 13 years. The students aged 8 to 10 were assessed by Nunes *et al.* and the older ones by Hart *et al*. These results show modest progress on the understanding of equivalence questions presented in the context of partitioning even though the quantities eaten were all equivalent to half.

Mamede (2007) carried out a direct comparison between children's use of the correspondence and the partitioning scheme in solving equivalence and order problems with fractional quantities. In this well-controlled study, she used story problems involving chocolates and children, similar pictures and mathematically identical questions; the division scheme relevant to the situation was the only variable distinguishing the problems. In correspondence problems, for example, she asked the children: in one party, three girls are going to share fairly one chocolate cake; in another party, six boys are going to share fairly two chocolate cakes. The children were asked to decide whether each boy would eat more than each girl, each girl would eat more than each boy, or whether they would have the same amount to eat. In the partitioning problems, she set the following scenario: This girl and this boy have identical chocolate cakes; the cakes are too big to eat at once so the girl cuts her cake in 3 identical parts and eats one and the boy cuts his cake in 6 identical parts and eats 2. The children were asked whether the girl and the boy ate the same amount or whether one ate more than the other. The children (age range six to seven) were Portuguese and in their first year in school; they had received no instruction about fractions.

In the correspondence questions, the responses of 35% of the six-year-olds and 49% of the seven-year-olds were correct; in the partitioning questions, 10% of the answers of children in both age levels were correct. These highly significant differences suggest that the use of correspondence reasoning supports children's understanding of equivalence between fractions whereas partitioning did not seem to afford the same insights.

Finally, it is important to compare students' arguments for the equivalence and order of quantities represented by fractions in teaching studies where partitioning is used as the basis for teaching. Many teaching studies that aim at promoting students' understanding of fractions through partitioning have been reported in the literature (e.g., Behr, Wachsmuth, Post and Lesh, 1984; Brousseau, Brousseau and Warfield, 2004; 2007; Empson, 1999; Kerslake, 1986; Olive and Steffe, 2002; Olive and Vomvoridi, 2006; Saenz-Ludlow, 1994; Steffe, 2002). In most of these studies, students' difficulties with partitioning are circumvented either by using pre-divided materials (e.g. Behr, Wachsmuth, Post and Lesh, 1984) or by using computer tools where the computer carries out the division as instructed by the student (e.g. Olive and Steffe, 2002; Olive and Vomvoridi, 2006).

Many studies combine partitioning with correspondence during instruction, either because the researchers do not use this distinction (e.g. Saenz-Ludlow, 1994) or because they wish to construct instruction that combines both schemes in order to achieve a better instructional program (e.g. Brousseau, Brousseau and Warfield, 2004; 2007). These studies will not be discussed here. Two studies that analysed student's arguments focus the instruction on partitioning and are presented here.

The first study was carried out by Berhr, Wachsmuth, Post and Lesh (1984). The researchers used objects of different types that could be manipulated during instruction (e.g. counters, rectangles of the same size and in different colours, pre-divided into fractions such as halves, quarters, thirds, eighths) but also taught the students how to use algorithms (division of the denominator by the numerator to find a ratio) to check on the equivalence of fractions. The students were in fourth grade (age about 9) and received instruction over 18 weeks. Behr *et al.* provided a detailed analysis of children's arguments regarding the ordering of fractions. In summary, they report the following insights after instruction.

- When ordering fractions with the same numerator and different denominators, students seem to be able to argue that there is an inverse relation between the number of parts into which the whole was cut and the size of the parts. This argument appears either with explicit reference to the numerator ('there are two pieces in each, but the pieces in two fifths are smaller.' p. 328) or without it ('the bigger the number is, the smaller the pieces get.' p. 328).

- A third fraction can be used as a reference point when two fractions are compared: three ninths is less than three sixths because 'three ninths is … less than half and three sixths is one half' (p. 328). It is not clear how the students had learned that 3/6 and 1/2 are equivalent but they can use this knowledge to solve another comparison.

- Students used the ratio algorithm to verify whether the fractions were equivalent: 3/5 is not equivalent to 6/8 because 'if they were equal, three goes into six, but five doesn't go into eight.' (p. 331).

- Students learned to use the manipulative materials in order to carry out perceptual comparisons: 6/8 equals 3/4 because 'I started with four parts. Then I didn't have to change the size of the paper at all. I just folded it, and then I got eight.' (p. 331).

Behr *et al.* report that, after 18 weeks of instruction, a large proportion of the students (27%) continued to use the manipulatives in order to carry our perceptual comparisons; the same proportion (27%) used a third fraction as a reference point and a similar proportion (23%) used the ratio algorithm that they had been taught to compare fractions.

Finally, there is no evidence that the students were able to understand that the number of parts and size of parts could compensate for each other precisely in a proportional manner. For example, in the comparison between 6/8 and 3/4 the students could have argued that there were twice as many parts when the whole was cut into 8 parts in comparison with cutting in 4 parts, so you need to take twice as many (6) in order to have the same amount.

In conclusion, students seemed to develop some insight into the inverse relation between the divisor and the quantity but this only helped them when the dividend was kept constant: they could not extend this understanding to other situations where the numerator and the denominator differed.

The second set of studies that focused on partitioning was carried out by Steffe and his colleagues (Olive and Steffe, 2002; Olive and Vomvoridi, 2006; Steffe, 2002). Because the aim of much of the instruction was to help the children learn to label fractions or compose fractions that would be appropriate for the label, it is not possible to extract from their reports the children's arguments for equivalence of fractions.

However, one of the protocols (Olive and Steffe, 2002) provides evidence for the student's difficulty with improper fractions, which, we hypothesise, could be a consequence of using partitioning as the basis for the concept of fractions. The researcher asked Joe to make a stick 6/5 long. Joe said that he could not because 'there are only five of them'. After prompting, Joe physically adds one more fifth to the five already used, but it is not clear whether this physical action convinces him that 6/5 is mathematically appropriate. In a subsequent example, Joe labels a stick made with 9 sticks, which had been defined as 'one seventh' of an original stick, 9/7, but according to the researchers 'an important perturbation' remains. Joe later counts 8 of a stick that had been designated as 'one seventh' but doesn't use the label 'eight sevenths'. When the researcher proposes this label, he questions it: 'How can it be EIGHT sevenths?' (Olive and Steffe, 2002, p. 426). Joe later refused to make a stick that is 10/7, even though the procedure is physically possible. Subsequently, on another day, Joe's reaction to another improper fraction is: 'I still don't understand how you could do it. *How can a fraction be bigger than itself?* (Olive and Steffe, 2002, p. 428; emphasis in the original).

According to the researchers, Joe only sees that improper fractions are acceptable when they presented a problem where pizzas were to be shared by people. When 12 friends ordered 2 slices each of pizzas cut into 8 slices, Joe realized immediately that more than one pizza would be required; the traditional partitioning situation, where one whole is divided into equal parts, was transformed into a less usual one, where two wholes are required but the size of the part remains fixed.

This example illustrates the difficulty that students have with improper fractions in the context of partitioning but which they can overcome by thinking of more than one whole.

Conclusion

Partitioning, defined as the action of cutting a whole into a pre-determined number of equal parts, shows a slower developmental process than correspondence. In order for children to succeed, they need to anticipate the solution so that the right number of cuts produces the right number of equal parts and exhausts the whole. Its accomplishment, however, does not seem to produce immediate insights into equivalence and order of fractional quantities. Apparently, many children do not see it as necessary that halves from two identical wholes are equivalent, even if they have been taught about the equivalence of fractions in school.

In order to use this scheme of action as the basis for learning about fractions, teaching schemes and researchers rely on pre-cut wholes or computer tools to avoid the difficulties of accurate partitioning. Students can develop insight into the inverse relation between the number of parts and the size of the parts through the partitioning scheme but there is no evidence that they realize that if you cut a whole in twice as many parts each one will be half in size. Finally, improper fractions seem to cause uneasiness to students who have developed their conception of fractions in the context of partitioning; it is important to be aware of this uneasiness if this is the scheme chosen in order to teach fractions.

Rational numbers and children's understanding of intensive quantities

In the introduction, we suggested that rational numbers are necessary to represent quantities that are measured by a relation between two other quantities. These are called intensive quantities and there are many examples of such quantities both in everyday life and in science. In everyday life, we often mix liquids to obtain a certain taste. If you mix fruit concentrate with water to make juice, the concentration of this mixture is described by a rational number: for example, 1/3 concentrate and 2/3 water. Probability is an intensive quantity that is important both in mathematics and science and is measured as the number of favourable cases divided by the number of total cases.[7]

The conceptual difficulties involved in understanding intensive quantities are largely similar to those involved in understanding the representation of quantities that are smaller than the whole. In order to understand intensive quantities, students must form a concept that takes two variables simultaneously into account and realise that there is an inverse relation between the denominator and the quantity represented.

Piaget and Inhelder described children's thinking about intensive quantities as one of the many examples of the development of the scheme of proportionality, which they saw as one of the hallmarks of adolescent thinking and formal operations. They devoted a book to the analysis of children's understanding of probabilities (Piaget and Inhelder, 1975) and described in great detail the steps that children take in order to understand the quantification of probabilities. In the most comprehensive of their studies, the children were shown pairs of decks of cards with different numbers of cards, some marked with a cross and others unmarked. The children were asked to judge which deck they would choose to draw from if they wanted to have a better chance of drawing a card marked with a cross.

Piaget and Inhelder observed that many of the young children treated the number of marked and unmarked cards as if they were independent: sometimes they chose one deck because it had more marked cards than the other and sometimes they chose a deck because it had fewer blank cards than the other. This approach can lead to correct responses when either the number of marked cards or the number of unmarked cards is the same in both decks, and children aged about seven years were able to make correct choices in such problems. This is rather similar to the observations of children's successes and difficulties in comparing fractions reported earlier on: they can reach the correct answer when the denominator is constant or when the numerator is constant, as this allow them to focus on the other value. When they must think of different denominators and numerators, the questions become more difficult.

Around the age of nine, children started making correspondences between marked and unmarked cards within each deck and were able to identify equivalences using this type of procedure. For example, if asked to compare a deck with one marked and two unmarked cards (1/3 probability) with another deck with two marked and four unmarked cards (2/6 probability), the children would re-organise the second deck in two lots, setting one marked card in correspondence with two unmarked, and conclude that it did not make any difference which deck they picked a card from. Piaget and Inhelder saw these as empirical proportional solutions, which were a step towards the abstraction that characterises proportional reasoning.

Noelting (1980 *a* and *b*) replicated these results with another intensive quantity, the taste of orange juice made from a mixture of concentrate and water. In broad terms, he described children's thinking and its development in the same way as Piaget and Inhelder had done. This is an important replication of Piaget's results results considering that the content of the problems differed marked across the studies, probability and concentration of juice.

Nunes, Desli and Bell (2003) compared students' ability to solve problems about extensive and intensive quantities that involved the same type of reasoning. Extensive quantities can be represented by a single whole number (e.g. 5 kilos, 7 cows, 4 days) whereas intensive quantities are represented by a ratio between two numbers. In spite of these differences, it is possible to create problems which are comparable in other aspects but differ with respect to whether the quantities are extensive or intensive. Intensive quantities problems always involve three variables. For example, three variables might be amount of orange concentrate, amount of water, and the taste of the orange juice, which is the intensive quantity. The amount of orange concentrate is directly related to how orangey the juice tastes whereas the amount of water is inversely related to how orangey the juice tastes. A comparable extensive quantities problem would involve three extensive quantities, with the one under scrutiny being inversely proportional to one of the variables and directly proportional to the other. For example, the number of days that the food bought by a farmer lasts is directly proportional to the amount of food purchased and inversely proportional to the number of animals she has to feed. In our study, we analysed students' performance in comparison problems where they had to consider either intensive quantities (e.g. how orangey a juice would taste) or extensive quantities (e.g. the number of days the farmer's food supply would last). Students performed significantly better in the extensive quantities problems even though both types of problem involved proportional reasoning and the same number of variables. So, although the difficulties shown by children across the two types of problem are similar, their level of success was higher with extensive than intensive quantities. This indicates that students find it difficult to form a concept where two variables must be coordinated into a single construct,[8] and therefore it may be important for schools and teachers to consider how they might promote this development in the classroom.

We shall not review the large literature on intensive quantities here (see, for example, Erickson, 1979; Kaput, 1985; Schwartz, 1988; Stavy, Strauss, Orpaz and Carmi, 1982; Stavy and Tirosh, 2000), but there is little doubt that students' difficulties in understanding intensive quantities are very similar to those that they have when thinking about fractions which represent quantities smaller than the unit. They treat the values independently, they find it difficult to think about inverse relations, and they might think of the relations between the numbers as additive instead of multiplicative.

There is presently little information to indicate whether students can transfer what they have learned about fractions in the context of representing quantities smaller than the unit to the representation and understanding of intensive quantities. Brousseau, Brousseau, and Warfield (2004) suggest both that teachers believe that students will easily go from one use of fractions to another, and that nonetheless the differences between these two types of situation could actually result in interference rather than in easy transfer of insights across situations. In contrast, Lachance and Confrey (2002) developed a curriculum for teaching third grade students (estimated age about 8 years) about ratios in a variety of problems, including intensive quantities problems, and then taught the same students in fourth grade (estimated age about 9 years) about decimals. Their hypothesis is that students would show positive transfer from learning about ratios to learning about decimals. They claimed that their students learned significantly more about decimals than students who had not participated in a similar curriculum and whose performance in the same questions had been described in other studies.

We believe that it is not possible at the moment to form clear conclusions on whether knowledge of fractions developed in one type of situation transfers easily to the other, shows no transfer, or actually interferes with learning about the other type of situation. In order to settle this issue, we must carry out the appropriate teaching studies and comparisons.

However, there is good reason to conclude that the use of rational numbers to represent intensive quantities should be explicitly included in the curriculum. This is an important concept in everyday life and science, and causes difficulties for students.

Learning to use mathematical procedures to determine the equivalence and order of rational numbers

Piaget's (1952) research on children's understanding of natural numbers shows that young children, aged about four, might be able to count two sets of objects, establish that they have the same number, and still not conclude that they are equivalent if the sets are displayed in very different perceptual arrangements. Conversely, they might establish the equivalence between two sets by placing their elements in correspondence and, after counting the elements in one set, be unable to infer what the number in the other set is (Piaget, 1952; Frydman and Bryant, 1988). As we noted in Paper 2, Understanding whole numbers, counting is a procedure for creating equivalent sets and placing sets in order but many young children who know how to count do not use counting when asked to compare or create equivalent sets (see, for example, Michie, 1984, Cowan and Daniels, 1989; Cowan, 1987; Cowan, Foster and Al-Zubaidi, 1993; Saxe, Guberman and Gearhart, 1987).

Procedures to establish the equivalence and order of fractional quantities are much more complex than counting, particularly when both the denominator and the numerator differ. Students are taught different procedures in different countries. The procedure that seems most commonly taught in England is to check the equivalence by analysing the multiplicative relation between or within the fractions. For example, when comparing 1/3 with 4/12, students are taught to find the factor that connects the numerators (1 and 4) and then apply the same factor to the denominators. If the numerator and the denominator of the second fraction are the product of the numerator and the denominator of the first fraction by the same number, 4 in this case, they are equivalent. An alternative approach is to find whether the multiplicative relation between the numerator and the denominator of each fraction is the same (3 in this case): if it is, the fractions are equivalent.

If students learned this procedure and applied it consistently, it should not matter whether the factor is, for example, 2, 3 or 5, because these are well-known multiplication associations. It should also not matter whether the fraction with larger numerator and denominator is the first or the second. However, research shows that these variations affect students' performance. Hart *et al.* (1985) presented students with the task of identifying the missing values in equivalent fractions. The children were presented with the item below and asked which numbers should replace the square and the triangle:

2/7 = □/14 = 10/△

The rate of correct responses by 11 to 12 and 12- to 13-year-olds for the second question was about half that for the first one: about 56% for the first question and 24% for the second. The within-fraction method cannot be easily applied in these cases but the factors are 2 and 5, and these multiplication tables should be quite easy for students at this age level.

We recently replicated these different levels of difficulty in a study with 8- to 10-year-olds. The easiest questions were those where the common factor was 2; the rates of correct responses for 1/3 = 2/□ and 6/8 = 3/□ were 52% and 45%, respectively. The most difficult question was 4/12 = 1/□ this was only answered correctly by 16% of the students. It is unlikely that the difficulty of computation could explain the differences in performance: even weak students in this age range should be able to identify 3 as the factor connecting 4 and 12, if they had been taught the within-fraction method, or 4 as the factor connecting 1 and 4, if they were taught the between-fractions method.

A noteworthy aspect of our results was the low correlations between the different items: although most were significant (due to the large sample size; N = 188), only two of the nine correlations were above .4. This suggests that the students were not able to use the procedure that they learned consistently to solve five items that had the same format and could be solved by the same procedure.

Our assessment, like the one by Hart *et al.* (1985), also included an equivalence question set in the context of a story: two boys have identical chocolate bars, one cuts his into 8 equal parts and eats 4 and the other cuts his into 4 equal parts and eats 2; the children are asked to indicate whether the boys eat the same amount of chocolate and, if not, who eats more. This item is usually seen as assessing children's understanding of quantities as it is not expressed in fraction terms. In our sample, no student wrote the fractions 4/8 and 2/4 and compared them by means of a procedure. We analysed the correlations between this item and the five items described in the previous paragraph. If the students used the same reasoning or the same procedure to solve the items, there should be a high correlation between them. This was not so: the highest of the correlations between this item and each of the five previous ones was 0.32, which is low. This result exemplifies the separation between understanding fractional quantities and knowledge of procedures in the domain of rational numbers. This is much the same as observed in the domain of natural numbers.

It is possible that understanding the relations between quantities gives students an advantage in learning the procedures to establish the equivalence of fractions, but it may not guarantee that they will actually learn it if teachers do connect their understanding with the procedure. When we separated the students into two groups, one that answered the question about the boys and the chocolates correctly and the other that did not, there was a highly significant difference between the two groups in the rate of correct responses in the procedural items: the group who succeeded in the chocolate question showed 38% correct responses to the procedural items whereas the group who failed only answered 18% of the procedural questions correctly.

A combination of longitudinal and intervention studies is required to clarify whether students who understand fractional quantities benefit more when taught how to represent and compare fractions. There are presently no studies to clarify this matter.

Research that analyses students' knowledge of procedures used to find equivalent fractions and its connection with conceptual knowledge of fractions has shown that there can be discrepancies between these two forms of knowledge. Rittle-Johnson, Siegler and Alibali (2001) argued that procedural and conceptual knowledge develop *in tandem* but Kerslake (1986) and Byrnes and colleagues (Byrnes, 1992; Byrnes and Wasik, 1991), among others, identified clear discrepancies between students' conceptual and procedural knowledge of fractions. Recently we (Hallett, Nunes and Bryant, 2007) analysed a large data set (N = 318 children in Years 4 and 5) and observed different profiles of relative performance in items that assess knowledge of procedures to compare fractions and understanding of fractional quantities. Some children show greater success in procedural questions than would be expected from their performance in conceptual items, others show better performance in conceptual items than expected from their performance in the procedural items, and still others do not show any discrepancy between the two. Thus, some students seem to learn procedures for finding equivalent fractions without an understanding of why the procedures work, others base their approach to fractions on their understanding of quantities without mastering the relevant procedures, and yet others seem able to co-ordinate the two forms of knowledge. Our results show that the third group is more successful not only in a test about fractions but also in a test about intensive quantities, which did not require the use of fractions in the representation of the quantities.

Finally, we ask whether students are better at using procedures to compare decimals than to compare ordinary fractions. The students in some of the grade levels studied by Resnick and colleagues (1989) would have been taught how to add and subtract decimals: they were in grades 5 and 6 (the estimated age for U.S. students is about 10 and 11 years) and one of the early uses of decimals in the curriculum in the three participating countries is addition and subtraction of decimals. When students are taught to align the decimal numbers by placing the decimal points one under the other before adding – for example, when adding 0.8, 0.26 and 0.361 you need to align the decimal points before carrying out the addition – they may not realise that they are using a procedure that automatically converts the values to the same denominator: in this case, x/1000. It is possible that students may use this procedure of aligning the decimal point without fully understanding that this is a conversion to the same denominator and thus that it should help them to compare the value of the fractions: after learning how to add and subtract with decimals, they may still think that 0.8 is less than 0.36 but probably would not have said that 0.80 is less than 0.36.

To conclude, we find in the domain of rational numbers a similar separation between understanding quantities and learning to operate with representations when judging the equivalence and order of magnitude of quantities. Students are taught procedures to test whether fractions are equivalent but their knowledge of these procedures is limited, and they do not apply it across items consistently. Similarly, students who solve equivalence problems in context are not necessarily experts in solving problems when the fractions are presented without context.

The significance of children's difficulties in understanding equivalence of fractions cannot be overstressed: in the domain of rational numbers, students cannot learn to add and subtract with understanding if they do not realise that fractions must be equivalent in order to be added. Adding 1/3 and 2/5 without transforming one of these into an equivalent fraction with the same denominator as the other is like adding bananas and tins of soup: it makes no sense. Above and beyond the fact that one cannot be said to understand numbers without understanding their equivalence and order, in the domain of rational numbers equivalence is a core concept for computing addition and subtraction. Kerslake (1986) has shown that students learn to implement the procedures for adding and subtracting fractions without having a glimpse at why they convert the fractions into common denominators first. This separation between the meaning of fractions and the procedures cannot bode well for the future of these learners.

Conclusions and educational implications

- Rational numbers are essential for the representation of quantities that cannot be represented by a single natural number. For this reason, they are needed in everyday life as well as science, and should be part of the curriculum in the age range 5 to 16.

- Children learn mathematical concepts by applying schemes of action to problem solving and reflecting about them. Two types of action schemes are available in division situations: partitioning, which involves dividing a whole into equal parts, and correspondence situations, where two quantities (or measures) are involved, a quantity to be shared and a number of recipients of the shares.

- Children as young as five or six years in age are quite good at establishing correspondences to produce equal shares, whereas they experience much difficulty in partitioning continuous quantities. Reflecting about these schemes and drawing insights from them places children in different paths for understanding rational number. When they use the correspondence scheme, they can achieve some insight into the equivalence of fractions by thinking that, if there are twice as many things to be shared and twice as many recipients, then each one's share is the same. This involves thinking about a direct relation between the quantities. The partitioning scheme leads to understanding equivalence in a different way: if a whole is cut into twice as many parts, the size of each part will be halved. This involves thinking about an inverse relation between the quantities in the problem. Research consistently shows that children understand direct relations better than inverse relations.

- There are no systematic and controlled comparisons to allow for unambiguous conclusions about the outcomes of instruction based on correspondences or partitioning. The available evidence suggests that testing this hypothesis appropriately could result in more successful teaching and learning of rational numbers.

- Children's understanding of quantities is often ahead of their knowledge of fractional representations when they solve problems using the correspondence scheme. Schools could make use of children's informal knowledge of fractional quantities and work with problems about situations, without requiring them to use formal representations, to help them consolidate this reasoning and prepare them for formalization.

- Research has identified the arguments that children use when comparing fractions and trying to see whether they are equivalent or to order them by magnitude. It would be important to investigate next whether increasing teachers' awareness of children's own arguments would help teachers guide children's learning more effectively.

- In some countries, greater attention is given to decimal representation than to ordinary fractions in primary school whereas in others ordinary fractions continue to play an important role. The argument that decimals are easier to understand than ordinary fractions does not find support in surveys of students' performance: students find it difficult to make judgements of equivalence and order both with decimals and with ordinary fractions.

- Some researchers (e.g. Nunes, 1997; Tall, 1992; Vergnaud, 1997) argue that different representations shed light onto the same concepts from different perspectives. This would suggest that a way to strengthen students' learning of rational numbers is to help them connect both representations. Moss and Case (1999) analysed this possibility in the context of a curriculum based on measurements, where ordinary fractions and percentages were used to represent the same information. Their results are encouraging, but the study does not include the appropriate controls that would allow for establishing firmer conclusions.

- Students can learn procedures for comparing, adding and subtracting fractions without connecting these procedures with their understanding of equivalence and order of fractional quantities, independently of whether they are taught with ordinary or decimal representation. This is not a desired outcome of instruction, but seems to be a quite common one. Research that focuses on the use of children's informal knowledge suggests that it is possible to help students make connections (e.g. Mack, 1990), but the evidence is limited. There is now considerably more information regarding children's informal strategies to allow for new teaching programmes to be designed and assessed.

- Finally, this review opens the way for a fresh research agenda in the teaching and learning of fractions. The source for the new research questions is the finding that children achieve insights into relations between fractional quantities before knowing how to represent them. It is possible to envisage a research agenda that would not focus on children's misconceptions about fractions, but on children's possibilities of success when teaching starts from thinking about quantities rather than from learning fractional representations.

Endnotes

1 Rational numbers can also be used to represent relations that cannot be described by a single whole number but the representation of relations will not be discussed here.

2 The authors report a successful programme of instruction where they taught the students to establish connections between their understanding of ratios and decimals. The students had received two years of instruction on ratios. A full discussion of this very interesting work is not possible here as the information provided in the paper is insufficient.

3 There are different hypotheses regarding what types of subconstructs or meanings for rational numbers should be distinguished (see, for example, Behr, Harel, Post and Lesh, 1992; Kieren, 1988) and how many distinctions are justifiable. Mathematicians and psychologists may well use different criteria and consequently reach different conclusions. Mathematicians might be looking for conceptual issues in mathematics and psychologists for distinctions that have an impact on children's learning (i.e. show different levels of difficulty or no transfer of learning across situations). We have decided not to pursue this in detail but will consider this question in the final section of the paper.

4 Steffe and his colleagues have used a different type of problem, where the size of the part is fixed and the children have to identify how many times it fits into the whole.

5 This classification should not be confused with the classification of division problems in the mathematics education literature. Fischbein, Deri, Nello and Marino (1985) define partitive division (which they also term sharing division) as a model for situations in which 'an object or collection of objects is divided into a number of equal fragments or sub-collections. The dividend must be larger than the divisor; the divisor (operator) must be a whole number; the quotient must be smaller than the dividend (operand)… In quotative division or measurement division, one seeks to determine how many times a given quantity is contained in a larger quantity. In this case, the only constraint is that the dividend must be larger than the divisor. If the quotient is a whole number, the model can be seen as repeated subtraction.' (Fischbein, Deri, Nello and Marino, 1985, p.7). In both types of problems discussed by Fishbein et al., the scheme used in division is the same, partitioning, and the situations are of the same type, part-whole.

6 Empson, Junk, Dominguez and Turner (2005) have stressed that 'the depiction of equal shares of, for example, sevenths in a part–whole representation is not a necessary step to understanding the fraction 1/7 (for contrasting views, see Charles and Nason, 2000; Lamon, 1996; Pothier and Sawada, 1983). What is necessary, however, is understanding that 1/7 is the amount one gets when 1 is divided into 7 same-sized parts.'

7 Not all intensive quantities are represented by fractions; speed, for example, is represented by a ratio, such as in 70 miles per hour.

8 Vergnaud (1983) proposed this hypothesis in his comparison between isomorphism of measures and product of measures problems. This issue is discussed in greater detail in another paper 4 of this review.

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