Nuffield Foundation

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Key understandings in mathematics learning

Paper 1: Overview

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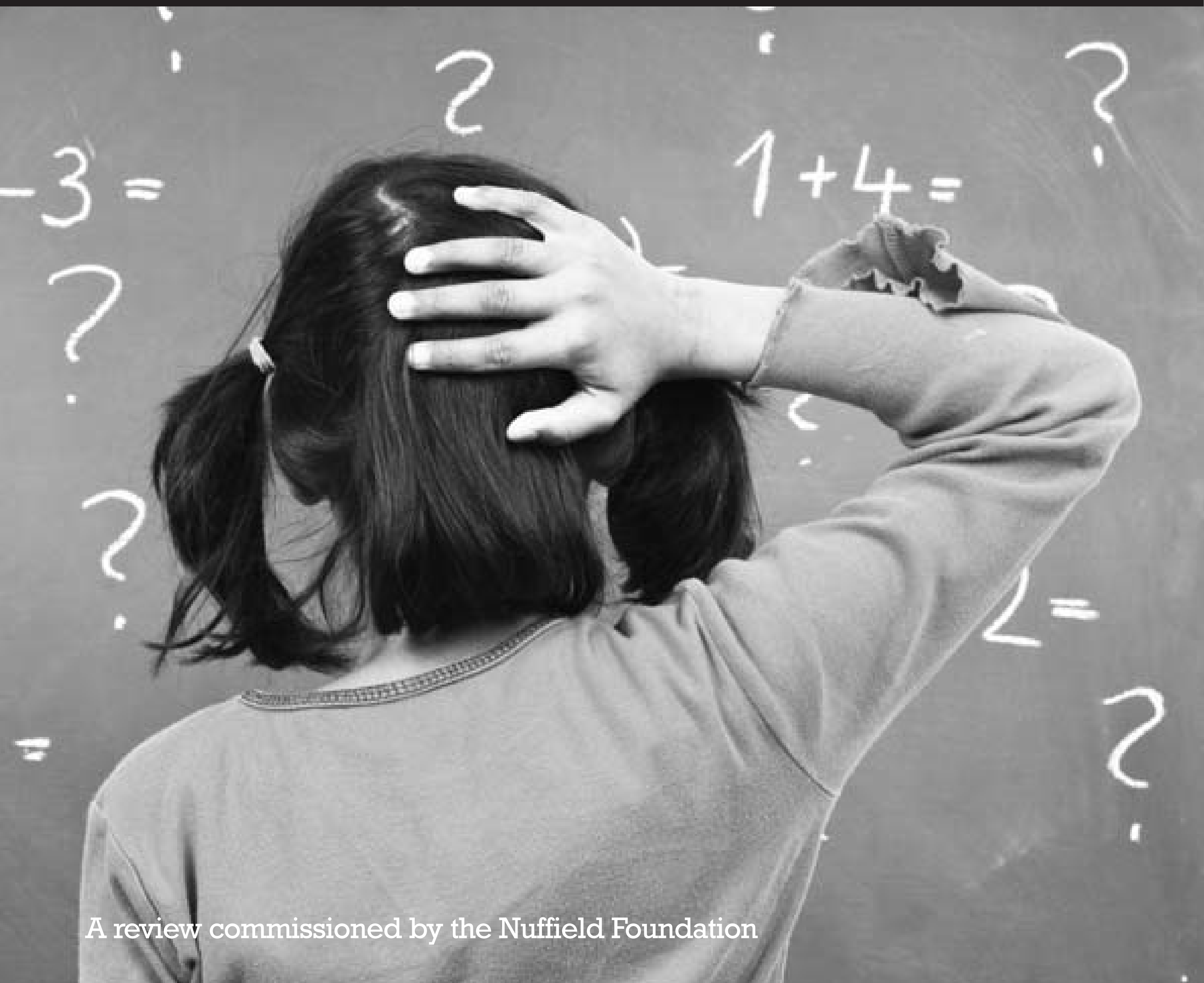

A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of findings

Aims

Our aim in the review is to present a synthesis of research on mathematics learning by children from the age of five to the age of sixteen years and to identify the issues that are fundamental to understanding children's mathematics learning. In doing so, we concentrated on three main questions regarding key understandings in mathematics.

- What insights must students have in order to understand basic mathematical concepts?
- What are the sources of these insights and how does informal mathematics knowledge relate to school learning of mathematics?
- What understandings must students have in order to build new mathematical ideas using basic concepts?

Theoretical framework

While writing the review, we concluded that there are two distinct types of theory about how children learn mathematics.

Explanatory theories set out to explain how children's mathematical thinking and knowledge change. These theories are based on empirical research on children's solutions to mathematical problems as well as on experimental and longitudinal studies. Successful theories of this sort should provide insight into the causes of children's mathematical development and worthwhile suggestions about teaching and learning mathematics.

Pragmatic theories set out to investigate what children ought to learn and understand and also identify obstacles to learning in formal educational settings.

Pragmatic theories are usually not tested for their consistency with empirical evidence, nor examined for the parsimony of their explanations vis-à-vis other existing theories; instead they are assessed in multiple contexts for their descriptive power, their credibility and their effectiveness in practice.

Our starting point in the review is that children need to learn about quantities and the relations between them and about mathematical symbols and their meanings. These meanings are based on sets of relations. Mathematics teaching should aim to ensure that students' understanding of quantities, relations and symbols go together.

Conclusions

This theoretical approach underlies the six main sections of the review. We now summarise the main conclusions of each of these sections.

Whole numbers

- Whole numbers represent both quantities and relations between quantities, such as differences and ratio. Primary school children must establish clear connections between numbers, quantities and relations.

- Children's initial understanding of quantitative relations is largely based on correspondence. One-to-one correspondence underlies their understanding of cardinality, and one-to-many correspondence gives them their first insights into multiplicative relations. Children should be encouraged to think of number in terms of these relations.

- Children start school with varying levels of ability in using different action schemes to solve arithmetic problems in the context of stories. They do not need to know arithmetic facts to solve these problems: they count in different ways depending on whether the problems they are solving involve the ideas of addition, subtraction, multiplication or division.

- Individual differences in the use of action schemes to solve problems predict children's progress in learning mathematics in school.

- Interventions that help children learn to use their action schemes to solve problems lead to better learning of mathematics in school.

- It is more difficult for children to use numbers to represent relations than to represent quantities.

Implications for the classroom

Teaching should make it possible for children to:
- connect their knowledge of counting with their knowledge of quantities
- understand additive composition and one-to-many correspondence
- understand the inverse relation between addition and subtraction
- solve problems that involve these key understandings
- develop their multiplicative understanding alongside additive reasoning.

Implications for further research

Long-term longitudinal and intervention studies with large samples are needed to support curriculum development and policy changes aimed at implementing these objectives. There is also a need for studies designed to promote children's competence in solving problems about relations.

Fractions

- Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. As with whole numbers, children need to make connections between quantities and their representations in fractions in order to be able to use fractions meaningfully.

- Two types of quantities that are taught in primary school must be represented by fractions. The first involves measurement: if you want to represent a quantity by means of a number and the quantity is smaller than the unit of measurement, you need a fraction; for example, a half cup or a quarter inch. The second involves division: if the dividend is smaller than the divisor, the result of the division is represented by a fraction; for example, three chocolates shared among four children.

- Children use different schemes of action in these two different situations. In division situations, they use correspondences between the units in the numerator and the units in the denominator. In measurement situations, they use partitioning.

- Children are more successful in understanding equivalence of fractions and in ordering fractions by magnitude in situations that involve division than in measurement situations.

- It is crucial for children's understanding of fractions that they learn about fractions in both types of situation: most do not spontaneously transfer what they learned in one situation to the other.

- When a fraction is used to represent a quantity, children need to learn to think about how the numerator and the denominator relate to the value represented by the fraction. They must think about direct and inverse relations: the larger the numerator, the larger the quantity, but the larger the denominator, the smaller the quantity.

- Like whole numbers, fractions can be used to represent quantities and relations between quantities, but they are rarely used to represent relations in primary school. Older students often find it difficult to use fractions to represent relations.

Implications for the classroom

Teaching should make it possible for children to:
- use their understanding of quantities in division situations to understand equivalence and order of fractions
- make links between different types of reasoning in division and measurement situations
- make links between understanding fractional quantities and procedures
- learn to use fractions to represent relations between quantities, as well as quantities.

Implications for further research
Evidence from experimental studies with larger samples and long-term interventions in the classroom are needed to establish how division situations relate to learning fractions. Investigations on how links between situations can be built are needed to support curriculum development and classroom teaching.

There is also a need for longitudinal studies designed to clarify whether separation between procedures and meaning in fractions has consequences for further mathematics learning.

Given the importance of understanding and representing relations numerically, studies that investigate under what circumstances primary school students can use fractions to represent relations between quantities, such as in proportional reasoning, are urgently needed.

Relations and their mathematical representation

- Children have greater difficulty in understanding relations than in understanding quantities. This is true in the context of both additive and multiplicative reasoning problems.

- Primary and secondary school students often apply additive procedures to solve multiplicative problems and multiplicative procedures to solve additive problems.

- Teaching designed to help students become aware of relations in the context of additive reasoning problems can lead to significant improvement.

- The use of diagrams, tables and graphs to represent relations in multiplicative reasoning problems facilitates children's thinking about the nature of the relations between quantities.

- Excellent curriculum development work has been carried out to design programmes that help students develop awareness of their implicit knowledge of multiplicative relations. This work has not been systematically assessed so far.

- An alternative view is that students' implicit knowledge should not be the starting point for students to learn about proportional relations; teaching should focus on formalisations rather than informal knowledge and only later seek to connect mathematical formalisations with applied situations. This alternative approach has also not been systematically assessed yet.

- There is no research that compares the results of these diametrically opposed ideas.

Implications for the classroom
Teaching should make it possible for children to:
- distinguish between quantities and relations
- become explicitly aware of the different types of relations in different situations
- use different mathematical representations to focus on the relevant relations in specific problems
- relate informal knowledge and formal learning.

Implications for further research
Evidence from experimental and long-term longitudinal studies is needed on which approaches to making students aware of relations in problem situations improve problem solving. A study comparing the alternative approaches – starting from informal knowledge versus starting from formalisations – would make a significant contribution to the literature.

Space and its mathematical representation

- Children come to school with a great deal of informal and often implicit knowledge about spatial relations. One challenge in mathematical education is how best to harness this knowledge in lessons about space.

- This pre-school knowledge of space is mainly relational. For example, children use a stable background to remember the position and orientation of objects and lines.

- Measuring length and area poses particular problems for children, even though they are able to understand the underlying logic of measurement. Their difficulties concern iteration of standard units and the need to apply multiplicative reasoning to the measurement of area.

- From an early age children are able to extrapolate imaginary straight lines, which allows them to learn how to use Cartesian co-ordinates to plot specific positions in space with little difficulty. However, they need help from teachers on how to use co-ordinates to work out the relation between different positions.

- Learning how to represent angle mathematically is a hard task for young children, even though angles are an important part of their everyday life. Initially children are more aware of angle in the context of movement (turns) than in other contexts. They need help from to teachers to be able to relate angles across different contexts.

- An important aspect of learning about geometry is to recognise the relation between transformed shapes (rotation, reflection, enlargement). This can be difficult, since children's preschool experiences lead them to recognise the same shapes as equivalent across such transformations, rather than to be aware of the nature of the transformation.

- Another aspect of the understanding of shape is the fact that one shape can be transformed into another by addition and subtraction of its subcomponents. For example, a parallelogram can be transformed into a rectangle of the same base and height by the addition and subtraction of equivalent triangles. Research demonstrates a danger that children learn these transformations as procedures without understanding their conceptual basis.

Implications for the classroom

Teaching should make it possible for children to:
- build on spatial relational knowledge from outside school
- relate their knowledge of relations and correspondence to the conceptual basis of measurement
- iterate with standard and non-standard units
- understand the difference between measurements which are/are not multiplicative
- relate co-ordinates to extrapolating imaginary straight lines
- distinguish between scale enlargements and area enlargements.

Implications for further research

There is a serious need for longitudinal research on the possible connections between children's pre-school spatial abilities and how well they learn about geometry at school.

Psychological research is needed on: children's ability to make and understand transformations and the additive relations in compound shapes; the exact cause of children's difficulties with iteration; how transitive inference, inversion and one-to-one correspondence relate to problems with geometry, such as measurement of length and area.

There is a need for intervention studies on methods of teaching children to work out the relation between different positions, using co-ordinates.

Algebra

- Algebra is the way we express generalisations about numbers, quantities, relations and functions. For this reason, good understanding of connections between numbers, quantities and relations is related to success in using algebra. In particular, understanding that addition and subtraction are inverses, and so are multiplication and division, helps students understand expressions and solve equations.

- To understand algebraic symbolisation, students have to (a) understand the underlying operations and (b) become fluent with the notational rules. These two kinds of learning, the meaning and the symbol, seem to be most successful when students know what is being expressed and have time to become fluent at using the notation.

- Students have to learn to recognise the different nature and roles of letters as: unknowns, variables, constants and parameters, and also the meanings of equality and equivalence. These meanings are not always distinct in algebra and do not relate unambiguously to arithmetical understandings.

- Students often get confused, misapply, or misremember rules for transforming expressions and solving equations. They often try to apply arithmetical meanings inappropriately to algebraic expressions. This is associated with over-emphasis on notational manipulation, or on 'generalised arithmetic', in which they may try to get concise answers.

Implications for the classroom

Teaching should make it possible for children to:
- read numerical and algebraic expressions relationally, rather than as instructions to calculate (as in substitution)
- describe generalisations based on properties (arithmetical rules, logical relations, structures) as well as inductive reasoning from sequences

- use symbolism to represent relations
- understand that letters and '=' have a range of meanings
- use hands-on ICT to relate representations
- use algebra purposefully in multiple experiences over time
- explore and use algebraic manipulation software.

Implications for further research

We need to know how explicit work on understanding relations between quantities enables students to move successfully between arithmetical to algebraic thinking.

Research on how expressing generality enables students to use algebra is mainly in small-scale teaching interventions, and the problems of large-scale implementation are not so well reported. We do not know the longer-term comparative effects of different teaching approaches to early algebra on students' later use of algebraic notation and thinking.

There is little research on higher algebra, except for teaching experiments involving functions. How learners synthesise their knowledge of elementary algebra to understand polynomial functions, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations is not known.

There is some research about the use of symbolic manipulators but more needs to be learned about the kinds of algebraic expertise that develops through their use.

Modelling, solving problems and learning new concepts in secondary mathematics

Students have to be fluent in understanding methods and confident about using them to know why and when to apply them, but such application does not automatically follow the learning of procedures. Students have to understand the situation as well as to be able to call on a familiar repertoire of facts, ideas and methods.

Students have to know some elementary concepts well enough to apply them and combine them to form new concepts in secondary mathematics. For example, knowing a range of functions and/or their representations seems to be necessary to understand the modelling process, and is certainly necessary to engage in modelling.

Understanding relations is necessary to solve equations meaningfully.

Students have to learn when and how to use informal, experiential reasoning and when to use formal, conventional, mathematical reasoning. Without special attention to meanings, many students tend to apply visual reasoning, or be triggered by verbal cues, rather than analyse situations to identify variables and relations.

In many mathematical situations in secondary mathematics, students have to look for relations between numbers, and variables, and relations between relations, and properties of objects, and know how to represent them.

Implications for the classroom

Teaching should make it possible for children to:

- learn new abstract understandings, which is neither achieved through learning procedures, nor through problem-solving activities, without further intervention
- use their obvious reactions to perceptions and build on them, or understand conflicts with them
- adapt to new meanings and develop from earlier methods and conceptualizations over time
- understand the meaning of new concepts 'know about', 'know how to', and 'know how to use'
- control switching between, and comparing, representations of functions in order to understand them
- use spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

Implications for further research

Existing research suggests that where contextual and exploratory mathematics, integrated through the curriculum, *do* lead to further conceptual learning it is related to conceptual learning being a rigorous focus for curriculum and textbook design, and in teacher preparation, or in specifically designed projects based around such aims. There is therefore an urgent need for research to identify the key conceptual understandings for success in secondary mathematics. There is no evidence to convince us that the new U.K. curricula will necessarily lead to better *conceptual* understanding of mathematics, either at the elementary level which is necessary to learn higher mathematics, or at higher levels which provide the confidence and foundation for further mathematical study.

We need to understand the ways in which students learn new ideas in mathematics that depend on combinations of earlier concepts, in secondary school contexts, and the characteristics of mathematics teaching at higher secondary level which contribute both to successful conceptual learning and application of mathematics.

Common themes

We reviewed different areas of mathematical activity, and noted that many of them involve common themes, which are fundamental to learning mathematics: number, logical reasoning, reflection on knowledge and tools, understanding symbol systems and mathematical modes of enquiry.

Number

Number is not a unitary idea, which children learn in a linear fashion. Number develops in complementary strands, sometimes with discontinuities and changes of meaning. Emphasis on procedures and manipulation with numbers, rather than on understanding the underlying relations and mathematical meanings, can lead to over-reliance and misapplication of methods in arithmetic, algebra, and problem-solving. For example, if children form the idea that quantities are only equal if they are represented by the same number, a principle that they could deduce from learning to count, they will have difficulty understanding the equivalence of fractions. Learning to count and to understand quantities are separate strands of development. Teaching can play a major role in helping children co-ordinate these two forms of knowledge without making counting the only procedure that can be used to think about quantities.

Successful learning of mathematics includes understanding that number describes quantity; being able to make and use distinctions between different, but related, meanings of number; being able to use relations and meanings to inform application and calculation; being able to use number relations to move away from images of quantity and use number as a structured, abstract, concept.

Logical reasoning

The evidence demonstrates beyond doubt that children must rely on logic to learn mathematics and that many of their difficulties are due to failures to make the correct logical move that would have led them to the correct solution. Four different aspects of logic have a crucial role in learning about mathematics.

The logic of correspondence (one-to-one and one-to-many correspondence) The extension of the use of one-to-one correspondence from sharing to working out the numerical equivalence or non-equivalence of two or more spatial arrays is a vastly important step in early mathematical learning. Teaching multiplication in terms of one-to-many correspondence is more effective than teaching children about multiplication as repeated addition.

The logic of inversion Longitudinal evidence shows that understanding the inverse relation between addition and subtraction is a strong predictor of children's mathematical progress. A flexible understanding of inversion is an essential element in children's geometrical reasoning as well. The concept of inversion needs a great deal more prominence than it has now in the school curriculum.

The logic of class inclusion and additive composition Class inclusion is the basis of the understanding of ordinal number and the number system. Children's ability to use this form of inclusion in learning about number and in solving mathematical problems is at first rather weak, and needs some support.

The logic of transitivity All ordered series, including number, and also forms of measurement involve transitivity ($a > c$ if $a > b$ and $b > c$: $a = c$ if $a = b$ and $b = c$). Learning how to use transitive relations in numerical measurements (for example, of area) is difficult. One reason is that children often do not grasp the importance of iteration (repeated units of measurement).

The results of longitudinal research (although there is not an exhaustive body of such work) support the idea that children's logic plays a critical part in their mathematical learning.

Reflection on knowledge and tools

Children need to re-conceptualise their intuitive models about the world in order to access the mathematical models that have been developed in the discipline. Some of the intuitive models used by children lead them to appropriate mathematical problem solving, and yet they may not know why they succeeded. Implicit models can interfere with problem solving when students rely on assumptions that lead them astray.

The fact that students use intuitive models when learning mathematics, whether the teacher recognises the models or not, is a reason for helping them to develop an awareness of their models. Students can explore their intuitive models and extend them to concepts that are less intuitive, more abstract. This pragmatic theory has been shown to have an impact in practice.

Understanding symbol systems

Systems of symbols are human inventions and thus are cultural tools that have to be taught. Mathematical symbols are human-made tools that improve our ability to control and adapt to the environment. Each system makes specific cognitive demands on the learner, who has to understand the systems of representation and relations that are being represented; for example place-value notation is based on additive composition, functions depict covariance. Students can behave as if they understand how the symbols work while they do not understand them completely: they can learn routines for symbol manipulation that remain disconnected from meaning. This is true of rational numbers, for example.

Students acquire informal knowledge in their everyday lives, which can be used to give meaning to mathematical symbols learned in the classroom. Curriculum development work that takes this knowledge into account is not as widespread as one would expect given discoveries from past research.

Mathematical modes of enquiry

Some important mathematical modes of enquiry arise in the topics covered in this synthesis.

Comparison helps us make new distinctions and create new objects and relations Comparisons are related to making distinctions, sorting and classifying; students need to learn to make these distinctions based on mathematical relations and properties, rather than perceptual similarities.

Reasoning about properties and relations rather than perceptions Throughout mathematics, students have to learn to interpret representations before they think about how to respond. They need to think about the relations between different objects in the systems and schemes that are being represented.

Making and using representations Conventional number symbols, algebraic syntax, coordinate geometry, and graphing methods, all afford manipulations which might otherwise be impossible. Coordinating different representations to explore and extend meaning is a fundamental mathematical skill.

Action and reflection-on-action In mathematics, actions may be physical manipulation, or symbolic rearrangement, or our observations of a dynamic image, or use of a tool. In all these contexts, we observe what changes and what stays the same as a result of actions, and make inferences about the connections between action and effect.

Direct and inverse relations It is important in all aspects of mathematics to be able to construct and use inverse reasoning. As well as enabling more understanding of relations between quantities, this also establishes the importance of reverse chains of reasoning throughout mathematical problem-solving, algebraic and geometrical reasoning.

Informal and formal reasoning At first young children bring everyday understandings into school and mathematics can allow them to formalise these and make them more precise. Mathematics also provides formal tools, which do not describe everyday experience, but enable students to solve problems in mathematics and in the world which would be unnoticed without a mathematical perspective.

Epilogue

We have made recommendations about teaching and learning, and hope to have made the reasoning behind these recommendations clear to educationalists (in the extended review). We have also recognised that there are weaknesses in research and gaps in current knowledge, some of which can be easily solved by research enabled by significant contributions of past research. Other gaps may not be so easily solved, and we have described some pragmatic theories that are or can be used by teachers when they plan their teaching. Classroom research stemming from the exploration of these theories can provide new insights for further research in the future, alongside longitudinal studies which focus on learning mathematics from a psychological perspective.

Overview

Aims

Our aim in this review is to present a synthesis of research on key aspects of mathematics learning by children from the age of 5 to the age of 16 years: these are the ages that comprise compulsory education in the United Kingdom In preparing the review, we have considered the results of a large body of research carried out by psychologists and by mathematics educators over approximately the last six decades. Our aim has been to develop a theoretical analysis of these results in order to attain a big picture of how children learn, and sometimes fail to learn, mathematics and how they could learn it better. Our main target is not to provide an answer to any specific question, but to identify issues that are fundamental to understanding children's mathematics learning. In our view theories of mathematics learning should deal with three main questions regarding key understandings in mathematics:

- What insights must students have in order to understand basic mathematical concepts?
- What are the sources of these insights and how does informal mathematics knowledge relate to school learning of mathematics?
- What understandings must students have in order to build new mathematical ideas using basic concepts?

Theoretical analysis played a major role in this synthesis. Many theoretical ideas were already available in the literature and we sought to examine them critically for coherence and for consistency with the empirical evidence. Cooper (1998) suggests that there may be occasions when new theoretical schemes must be developed to provide an overarching understanding of the higher-order relations in the research domain; this was certainly true of some of our theoretical analysis of the evidence that we read for this review.

The answers to our questions should allow us to trace students' learning trajectories. Confrey (2008) defined a learning trajectory as 'a researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.' If students' learning trajectories towards understanding specific concepts are generally understood, teachers will be much better placed to promote their advancement.

Finally, one of our aims has been to identify a set of research questions that stem from our current knowledge about children's mathematics learning and methods that can provide relevant evidence about important, outstanding issues.

Scope of the review

As we reviewed existing research and existing theories about mathematics learning, it soon became clear to us that there are two types of theories about how children learn mathematics. The first are *explanatory* theories. These theories seek to explain how children's thinking and knowledge change. Explanatory theories are based on empirical research on the strategies that children adopt in solving mathematical problems, on the difficulties and misconceptions that affect their solutions to such problems, and on their successes and their explanations of their own solutions. They also draw on quantitative methods to describe age or school grade levels when certain forms of knowledge are attained and to make inferences about the nature of relationships observed during learning (for example, to help understand the relation between informal knowledge and school learning of mathematics).

We have called the second type of theory *pragmatic*. A pragmatic theory is rather like a road map for teachers: its aims are to set out what children must learn and understand, usually in a clear sequence, about particular topics and to identify obstacles to learning in formal educational settings and other issues which teachers should keep in mind when designing teaching. Pragmatic theories are usually not tested for their consistency with empirical evidence, nor examined for the parsimony of their explanations vis-à-vis other existing theories; instead they are assessed in multiple contexts for their descriptive power, their credibility and their effectiveness in practice.

Explanatory theories are of great importance in moving forward our understanding of phenomena and have proven helpful, for example, in the domain of literacy teaching and learning. However, with some aspects of mathematics, which tend to be those that older children have to learn about, there simply is not enough explanatory knowledge yet to guide teachers in many aspects of their mathematics teaching, but students must still be taught even when we do not know much about how they think or how their knowledge changes over time through learning. Mathematics educators have developed pragmatic theories to fill this gap and to take account of the interplay of learning theory with social and cultural aspects of educational contexts. Pragmatic theories are designed to guide teachers in domains where there are no satisfactory explanatory theories, and where explanatory theory does not provide enough information to design complex classroom teaching. We have included both types of theory in our review. We believe that both types are necessary in mathematics education but that they should not be confused with each other.

We decided to concentrate in our review on issues that are specific to mathematics learning. We recognise the significance of general pedagogical theories that stress, for example, the importance of giving learners an active role in developing their thinking and conceptual understanding, the notion of didactic transposition, the theory of situations, social theories regarding the importance of conflict and cooperation, the role of the teacher, the role and use of language, peer collaboration and argumentation in the classroom. These are important ideas but they apply to other domains of learning as well, and we decided not to provide an analysis of such theories but to mention them only in the context of specific issues about mathematical learning.

Another decision that we made about the scope of the synthesis was about how to deal with cultural differences in teaching and learning mathematics. The focus of the review is on mathematics learning by U.K. students during compulsory education. We recognise that there are many differences between learners in different parts of the world; so, we decided to include mostly research about learners who can be considered as reasonably similar to U.K. students, i.e. those living in Western cultures with a relatively high standard of living and plenty of opportunities to attend school. Thus the description of students who participated in the studies is not presented in detail and will often be indicated only in terms of the country where the research was carried out. In order to offer readers a notion of the time in students' lives when they might succeed or show difficulties with specific problems, we used age levels or school grade levels as references. These ages and years of schooling are not to be generalised to very different circumstances where, for example, children might be growing up in cultures with different number systems or largely without school participation. Occasional reference to research with other groups is used but this was purposefully limited, and it was included only when it was felt that the studies could shed light on a specific issue.

We also decided to concentrate on key understandings that offer the foundation for mathematics learning rather than on the different technologies used in mathematics. Wartofsky (1979) conceives technology as any human made tool that improves our ability to control and adapt to the environment. Mathematics uses many such tools. Some representational tools, such as counting and written numbers, are part of traditional mathematics learning in primary school. They improve our abilities in amazing ways: for example, counting allows us to represent precisely quantities which we could not discriminate perceptually and written numbers in the Hindu-Arabic system create the possibility of column arithmetic, which is not easily implemented with oral number when quantities are large or even with written Roman numerals. We have argued elsewhere (Nunes, 2002) that systems of signs enhance, structure and empower their users but learners must still construct meanings that allow them to use these systems. Our choice in this review was to consider how learners construct meanings rather than explore in depth the enabling role of mathematical representations. We discuss in much greater detail how they learn to use whole and rational numbers meaningfully than how they calculate with these numbers. Similarly, we discuss how they might learn the meaning and power of algebraic representation rather than how they might become fluent with algebraic manipulation. Psychological theories (Luria, 1973; Vygotsky, 1981) emphasise the empowering role of culturally developed systems of signs in human reasoning but stress that learners' construction of meanings for these signs undergoes a long development process in order for the signs to be truly empowering. Similarly, mathematics educators stress that technology is aimed not to replace, but to enhance mathematical reasoning (Noss and Hoyles, 1992).

Our reason for not focusing on technologies in this synthesis is that there are so many technological resources used today for doing mathematics that it is not possible to consider even those used or potentially useful in primary school in the required detail in this synthesis. We recognise this gap and strongly suggest that at least some of these issues be taken up for a synthesis at a later point, as some important comparative work already exists in the domain of column arithmetic (e.g. Anghileri, Beishuizen and Putten, 2002; Treffers, 1987) and the use of calculators (e.g. Ruthven, 2008).

We wish to emphasise, therefore, that this review is not an exhaustive one. It considers a part of today's knowledge in mathematics education. There are other, more specific aspects of the subject which, usually for reasons of space, we decided to by-pass. We shall explain the reasons for these choices as we go along.

Methods of the review

We obtained the material for the synthesis through a systematic search of peer reviewed journals, edited volumes and refereed conference proceedings.[1] We selected the papers that we read by first screening the abstracts: our main criteria for selecting articles to read were that they should be on a relevant topic and that they should report either the results of empirical research or theoretical schemes for understanding mathematics learning or both. We also consulted several books in order to read researchers' syntheses of their own empirical work and to access earlier well-established reviews of relevant research; we chose books that provide useful frameworks for research and theories in mathematics learning.

We hope that this review will become the object of discussion within the community of researchers, teachers and policy makers. We recognise that it is only one step towards making sense of the vast research on how students' thinking and knowledge of mathematics develops, and that other steps must follow, including a thorough evaluation of this contribution.

Teaching and learning mathematics: What is the nature of this task?

Learning mathematics is in some ways similar (but of course not identical) to language learning: in mathematics as well as in language it is necessary to learn symbols and their meaning, and to know how to combine them meaningfully.

Learning meanings for symbols is often more difficult than one might think. Think of learning the meaning of the word 'brother'. If Megan said to her four-year-old friend Sally 'That's my brother' and pointed to her brother, Sally might learn to say correctly and appropriately 'That's Megan's brother' but she would not necessarily know the meaning of 'brother'. 'Brother of' is a phrase that is based on a set of relationships, and in order to understand its meaning we need to understand this set of relationships, which includes 'mother of' and 'father of'. It is in this way that learning mathematics is very like learning a language: we need to learn mathematical symbols and their meanings, and the meaning of these symbols is based on sets of relations.

In the same way that Megan might point to her brother, Megan could count a set of pens and say: 'There are 15 pens here'. Sally could learn to count and say '15 pens' (or dogs, or stars). But '15' in mathematics does not just refer to the result of counting a set: it also means that this set is equivalent to all other sets with 15 objects, has fewer objects than any set with 16 or more, and has more objects than any set with 14 or fewer. Learning about numbers involves more than understanding the operations that are carried out to determine the word that represents the quantity. In the context of learning mathematics, we would like students to know, without having to count, that some operations do and others do not change a quantity. For example, we would like them to understand that there would only be more pens if we added some to the set, and fewer if we subtracted some from the set, and that there would still be 15 pens if we added and then subtracted (or vice versa) the same number of pens to and from the original set.

The basic numerical concepts that we want students to learn in primary school have these two sides to them: on the one hand, there are quantities, operations on quantities and relations between quantities, and on the other hand there are symbols, operations on symbols and relations between symbols. Mathematics teaching should aim to ensure that students' understanding of quantities, relations and symbols go together. Anything we do with the symbols has to be consistent with their underlying mathematical meaning as well as logically consistent and we are not free to play with meaning in mathematics in quite the same ways we might play with words.

This necessary connection is often neglected in theories about mathematics learning and in teaching practices. Theories that appear to be contradictory have often focused either on students' understanding of quantities or on their understanding of symbols and their manipulations. Similarly, teaching is often designed with one or the other of these two kinds of understanding in sight, and the result is that there are different ways of teaching that have different strengths and weaknesses.

Language learners eventually reach a time when they can learn the meaning of new words simply by definitions and connections with other words. Think of words like 'gene' and 'theory': we learn their meanings from descriptions provided by means of other words and from the way they are used in the language. Mathematics beyond primary school often works similarly: new mathematical meanings are learned by using previously learned mathematical meanings and ways of combining these. There are also other ways in which mathematics and language learning are similar; perhaps the most important of these other similarities is that we can use language to represent a large variety of meanings, and mathematics has a similar power. But, of course, mathematics learning differs from language learning: mathematics contains its own distinct concepts and modes of enquiry which determine the way that mathematics is used. This specificity of mathematical concepts is reflected in the themes that we chose to analyse in our synthesis.

The framework for this review

As we start our review, there is a general point to be made about the theoretical position that we have reached from our review of research on children's mathematics. On the whole, the teaching of the various aspects of mathematics proceeds in a clear sequence, and with a certain amount of separation in the teaching of different aspects. Children are taught first about the number sequence and then about written numbers and arithmetical operations using written numbers. The teaching of the four arithmetical operations is done separately. At school children learn about addition and subtraction separately and before they learn about multiplication and division, which also tend to be taught quite separately from each other. Lessons about arithmetic start years before lessons about proportions and the use of mathematical models.

This order of events in teaching has had a clear effect on research and theories about mathematical learning. For example, it is a commonplace that research on multiplication and division is most often (though there are exceptions; see Paper 4) carried out with children who are older than those who participate in research on addition and subtraction. Consequently, in most theories additive reasoning is hypothesised (or assumed) to precede multiplicative reasoning. Until recently there have been very few studies of children's understanding of the connection between the different arithmetical operations because they are assumed to be learned relatively independently of each other.

Our review of the relevant research has led us to us to a different position. The evidence quite clearly suggests that there is no such sequence, at any rate in the onset of children's understanding of some of these different aspects of mathematics. Much of this learning begins, as our review will show, in informal circumstances and before children go to school. Even after they begin to learn about mathematics formally, there are clear signs that they can embark on genuinely multiplicative reasoning, for example, at a time when the instruction they receive is all about addition and subtraction. Similar observations can be made about learning algebra; there are studies that show that quite young children are capable of expressing mathematical generalities in algebraic terms, but these are rare: the majority of studies focus on the ways in which learners fail to do so at the usual age at which this is taught.

Sequences do exist in children's learning, but these tend not to be about different arithmetic operations (e.g. not about addition before multiplication). Instead, they take the form of children's understanding of new quantitative relations as a result of working with and manipulating relations that have been familiar for some time. An example, which we describe in detail in Paper 2, is about the inverse relation between addition and subtraction. Young children easily understand that if you add some new items to a set of items and then subtract exactly the same items, the number of items in the set is the same as it was initially (inversion of identity), but it takes some time for them to extend their knowledge of this relation enough to understand that the number of items in the set will also remain the same if you add some new items and then subtract an equal number items from the set, which are not the same ones you had added (inversion of quantity: $a + b - b = a$). Causal sequences of this kind play an important part in the conclusions that we reach in this review.

Through our review, we identified some key understandings which we think children must achieve to be successful learners of mathematics and which became the main topics for the review. In the paragraphs that follow, we present the arguments that led us to choose the six main topics. Subsequently, each topic is summarised under a separate heading. The research on which these summaries are based is analysed in Papers 2 to 7.

The main points that are discussed here, before we turn to the summaries, guided the choice of papers in the review.

Quantity and number

The first point is that there is a distinction to be made between quantity and number and that children must make connections as well as distinctions between quantity and number in order to succeed in learning mathematics.

Thompson (1993) suggested that 'a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it. Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values' (pp. 165–166). Children experience and learn about quantities and the relations between them quite independently of learning to count. Similarly, they can learn to count quite independently from understanding quantities and relations between them. It is crucial for children to learn to make both connections and distinctions between number and quantity. There are different theories in psychology regarding how children connect quantity and number; these are discussed in Paper 2.

The review also showed that there are two different types of quantities that primary school children have to understand and that these are connected to different types of numbers. In everyday life, as well as in primary school, children learn about quantities that can be counted. Some are discrete and each item can be counted as a natural unit; other quantities are continuous and we use measurement systems, count the conventional units that are part of the system, and attribute numbers to these quantities. These quantities which are measured by the successive addition of items are termed extensive quantities. They are represented by whole numbers and give children their first insights into number.

In everyday life children also learn about quantities that cannot be counted like this. One reason why the quantity might not be countable in this way is that it may be smaller than the unit; for example, if you share three chocolate bars among four people, you cannot count how many chocolate bars each one receives. Before being taught about fractions, some primary school students are aware that you cannot say that each person would be given one chocolate bar, because they realise that each person's portion would be smaller than one: these children conclude that they do not know a number to say how much chocolate each person will receive (Nunes and Bryant, 2008). Quantities that are smaller than the unit are represented by fractions, or more generally by rational numbers.

Rational numbers are also used to represent quantities which we do not measure directly but only through a relation between two other measures. For example, if we want to say something about the concentration of orange squash in a glass, we have to say something about the ratio of concentrate used to water. This type of quantity, measurable by ratio, is termed intensive quantity and is often represented by rational numbers.

In Papers 2 and 3 we discuss how children make connections between whole and rational numbers and the different types of quantities that they represent.

Relations

Our second general point is about relations. Numbers are used to represent quantities as well as relations; this is why children must establish a connection between quantity and number but also distinguish between them. Measures are numbers that are connected to a quantity. Expressions such as 20 books, 3 centimetres, 4 kilos, and ½ a chocolate are measures. Relations, like quantities, do not have to be quantified. For example, we can simply say that two quantities are equivalent or different. This is a qualitative statement about the relation between two quantities. But we can quantify relations and we use numbers to do so: for example, when we compare two measures, we are quantifying a relation. If there are 20 children in a class and 17 books, we can say that there are 3 more children than books. The number 3 quantifies the relation. We can say 3 more children than books or 3 books fewer than children; the meaning does not change when the wording changes because the number 3 does not refer to children or to books, but to the relation between the two measures.

A major use of mathematics is to quantify relations and manipulate these representations to expand our understanding of a situation. We came to the conclusion from our review that understanding relations between quantities is at the root of understanding mathematical models. Thompson (1993) suggested that 'Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships… A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities' (p. 165). Elsewhere, Thompson (1994) emphasised that 'a quantitative operation is non-numerical; it has to do with the *comprehension* (italics in the original) of a situation.' (p. 187). So relations, like quantities, are different from numbers but we use numbers to quantify them.

Paper 4 of this synthesis discusses the quantification of relations in mathematics, with a focus on the sorts of relations that are part of learning mathematics in primary school.

The coordination of basic concepts and the development of higher order concepts

Students in secondary school have the dual task of refining what they have learned in primary school and understanding new concepts, which are based on reflections about and combinations of previous concepts. The challenge for students in secondary school is to learn to take a different perspective with respect to their mathematics knowledge and, at the same time, to learn about the power of this new perspective. Students can understand much about using mathematical representations (numbers, diagrams, graphs) for quantities and relations and how this helps them solve problems. Students who have gone this far understand the role of mathematics in representing and helping us understand phenomena, and even generalising beyond what we know. But they may not have understood a distinct and crucial aspect of the importance of mathematics: that, above and beyond helping represent and explore what you know, it can be used to discover what you do not know. In this review, we consider two related themes of this second side of mathematics: algebraic reasoning and modelling. Papers 6 and 7 summarise the research on these topics.

In the rest of this opening paper we shall summarise our main conclusions from our review. In other words, Papers 2 to 7 contain our detailed reviews of research on mathematics learning; each of the six subsequent sections about a central topic in mathematics learning is a summary of Papers 2 to 7.

Key understandings in mathematics: A summary of the topics reviewed

Understanding extensive quantities and whole numbers

Natural numbers are a way of representing quantities that can be counted. When children learn numbers, they must find out not just about the counting sequence and how to count, but also about how the numbers in the counting system represent quantities and relations between them. We found a great deal of evidence that children are aware of quantities such as the size of objects or the amount of items in groups of objects long before they learn to count or understand anything about the number system. This is quite clear in their ability to discriminate objects by size and sets by number when these discriminations can be made perceptually.

Our review also showed that children learn to count with surprisingly little difficulty. Counting is an activity organised by principles such as the order invariance of number labels, one-to-one correspondence between items and counting labels, and the use of the last label to say how many items are in the set. There is no evidence of children being taught these principles systematically before they go to school and yet most children starting school at the age of five years are already able to respect these principles when counting and identify other people's errors when they violate counting principles.

However, research on children's numerical understanding has consistently shown that at first they make very little connection between the number words that they learn and their existing knowledge about quantities such as size and the amounts of items. Our review showed that Thompson's (1993) theoretical distinction between quantities and number is hugely relevant to understanding children's mathematics. For example, many four-year-old children understand how to share objects equally between two or more people, on a one-for-A, one-for-B basis, but have some difficulty in understanding that the number of items in two equally shared sets must be the same, i.e. that if there are six sweets in one set, there must be six in the other set as well. To make the connection between number words and quantities, children have to grasp two aspects of number, which are *cardinal number* and *ordinal number*. By *cardinal number*, we mean that two sets with the same number of items in them are equal in amount. The term *ordinal number* refers to the fact that numbers are arranged in an ordered series of increasing magnitude: successive numbers in the counting sequence are greater than the preceding number by 1. Thus, 2 is a greater quantity than 1 and 3 than 2 and it follows that 3 must also greater than 1.

There are three different theories about how children come to co-ordinate their knowledge of quantities with their knowledge of counting.

The first is Piaget's theory, which maintains that this development is based on children's schemas of action and the coordination of the schemas with each other. Three schemas of action are relevant to natural number: adding, taking away, and setting objects in correspondence. Children must also understand how these schemas relate to each other. They must, for example, understand that a quantity increases by addition, decreases by subtraction, and that if you add and take away the same amount to an original quantity, that quantity stays the same. They must also understand the additive composition of number, which involves the coordination of one-to-one correspondence with addition and subtraction: if the elements of two sets are placed in correspondence but one has more elements than the other, the larger set is the sum of the smaller set plus the number of elements for which there is no corresponding item in the smaller set. Research has shown that this insight is not attained by young children, who think that adding elements to the smaller set will make it larger than the larger set without considering the number added.

A second view, in the form of a nativist theory, has been suggested by Gelman and Butterworth (Gelman & Butterworth, 2005). They propose that from birth children have access to an innate, inexact but powerful 'analog' system, whose magnitude increases directly with the number of objects in an array, and they attach the number words to the properties occasioning these magnitudes. According to this view both the system for knowing about quantities and the principles of counting are innate and are naturally coordinated.

A third theoretical alternative, proposed by Carey (2004), starts from a standpoint in agreement with Gelman's theory with respect to the innate analog system and counting principles. However, Carey does not think that these systems are coordinated naturally: they become so through a 'parallel individuation' system, which allows very young children to make precise discriminations between sets of one and two objects, and a little later, between two and three objects. During the same period, these children also learn number words and, through their recognition of 1, 2 and 3 as distinct quantities, they manage to associate the right count words ('one', 'two' and 'three') with the right quantities. This association between parallel individuation and the count list eventually leads to what Carey (2004) calls 'bootstrapping': the children lift themselves up by their own intellectual bootstraps by inducing a rule that the next count word in the counting system is exactly one more than the previous one. They do so, some time between the age of three – and five years and, therefore, before they go to school.

One important point to note about these three theories is that they use different definitions of cardinal number, and therefore different criteria for assessing whether children understand cardinality or not. Piaget's criterion is the one that we have mentioned already and which we ourselves think to be right: it is the understanding that two or more sets are equal in quantity when the number of items in them is the same (and vice-versa). Gelman's and Carey's less demanding criterion for understanding cardinality is the knowledge that the last count word for the set represents the set's quantity: if I count 'one, two, three' items and realise that means that that there are three in the set, I understand cardinal number. In our view, this second view of cardinality is inadequate for two reasons: first, it is actually based on the position of the count word and is thus more related to ordinal than cardinal number; second, it does not include any consideration of the fact that cardinal number involves inferences regarding the equivalence of sets. Piaget's definition of cardinal and ordinal number is much more stringent and it has not been disputed by mathematics educators. He was sceptical of the idea that children would understand cardinal and ordinal number concepts simply from learning how to count and the evidence we reviewed definitely shows that learning about quantities and numbers develop independently of each other in young children.

This conclusion has important educational implications. Schools must not be satisfied with teaching children how to count: they must ensure that children learn not only to count but also to establish connections between counting and their understanding of quantities.

Piaget's studies concentrated on children's ability to reason logically about quantitative relations. He argued that children must understand the inverse relation between addition and subtraction and also additive composition (which he termed class-inclusion and was later investigated under the label of part-whole relations) in order to truly understand number. The best way to test this sort of causal hypothesis is through a combination of longitudinal and intervention studies. Longitudinal studies with the appropriate controls can suggest that A is causally related to B if it is a specific predictor of B at a later time. Intervention studies can test these causal ideas: if children are successfully taught A and, as a consequence, their learning of B improves, it is safe to conclude that the natural, longitudinal connection between A and B is also a causal one.

It had been difficult in the past to use this combination of methods in the analysis of children's mathematics learning for a variety of reasons. First, researchers were not clear on what sorts of logical reasoning were vital to learning mathematics. There are now clearer hypothesis about this: the inverse relation between addition and subtraction and additive composition of number appear as key concepts in the work of different researchers. Second, outcome measures of mathematics learning were difficult to find. The current availability of standardised assessments, either developed for research or by policy makers for monitoring the performance of educational systems, makes both longitudinal and intervention studies possible, as these can be seen as valid outcome measures. Our own research has shown that researcher designed and government designed standardised assessments are highly correlated and, when used as outcome measures in longitudinal and intervention studies, lead to convergent conclusions. Finally, in order to carry out intervention studies, it is necessary to develop ways of teaching children the key concepts on which mathematics learning is grounded. Fortunately, there are currently successful interventions that can be used for further research to test the effect that learning about these key concepts has on children's mathematics learning.

Our review identified two longitudinal studies that show that children's understanding of logical aspects of number is vital for their mathematics learning. One was carried out in the United Kingdom and showed that children's understanding of the inverse relation between addition and subtraction and of additive composition at the beginning of school are specific predictors of their results in National Curriculum maths tests (a government designed and administered measure of children's mathematics learning) about 14 months later, after controlling for their general cognitive ability, their knowledge of number at school entry, and their working memory.

The second study, carried out in Germany, showed that a measure of children's understanding of the inverse relation between addition and subtraction when they were eight years old was a predictor of their performance in an algebra test when they were in university; controlling for the children's performance in an intelligence test given at age eight had no effect on the strength of the connection between their understanding of the inverse relation and their performance in the algebra measure.

Our review also showed that it is possible to improve children's understanding of these logical aspects of number knowledge. Children who were weak in this understanding at the beginning of school and improved this understanding through a short intervention performed significantly better than a control group that did not receive this teaching. Together, these studies allow us to conclude that it is crucial for children to coordinate their understanding of these logical aspects of quantities with their learning of numbers in order to make good progress in mathematics learning.

Our final step in this summary of research on whole numbers considered how children use additive reasoning to solve word problems. Additive reasoning is the logical analysis of problems that involve addition and subtraction, and of course the key concepts of additive composition and the inverse relation between addition and subtraction play an essential role in this reasoning. The chief tool used to investigate additive reasoning is the word problem. In word problems a scene is set, usually in one or two sentences, and then a question is posed. We will give three examples.

A Bob has three marbles and Bill has four: how many marbles do they have altogether? *Combine problem.*

B Wendy had four pictures on her wall and her parents gave her three more: how many does she have now? *Change problem.*

C Tom has seven books: Jane has five: how many more books does Tom have than Jane? *Compare problem*.

The main interest of these problems is that, although they all involve very simple and similar additions and subtractions, there are vast differences in the level of their difficulty. When the three kinds of problem are given in the form that we have just illustrated, the Compare problems are very much harder than the Combine and Change problems. This is not because it is too difficult for the children to subtract 5 from 7, which is how to solve this particular Compare problem, but because they find it hard to work out what to do so solve the problem. Compare problems require reasoning about relations between quantities, which children find a lot more difficult than reasoning about quantities.

Thus the difficulty of these problems rests on how well children manage to work out the arithmetical relations that they involve. This conclusion is supported by the fact that the relatively easy problems become a great deal more difficult if the mathematical relations are less transparent. For example, the usually easy Change problem is a lot harder if the result is given and the children have to work out the starting point. For example, Wendy had some pictures on her wall but then took 3 of them down: now she has 4 pictures left on the wall: how many were there in the first place? The reason that children find this problem a relatively hard one is that the story is about subtraction, but the solution is an addition. Pupils therefore have to call on their understanding of the inverse relation between adding and subtracting to solve this problem.

One way of analysing children's reactions to word problems is with the framework devised by Vergnaud, who argued that these problems involve quantities, transformations and relations. A Change problem, for example, involves the initial quantity and a transformation (the addition or subtraction) which leads to a new quantity, while Compare problems involve two quantities and the relation between them. On the whole, problems that involve relations are harder than those involving transformations, but other factors, such as the story being about addition and the solution being a subtraction or vice versa also have an effect.

The main impact of research on word problems has been to reinforce the idea with which we began this section. This idea is that in teaching children arithmetic we must make a clear distinction between numerical analysis and the children's understanding of quantitative relations. We must remember that there is a great deal more to arithmetical learning than knowing how to carry out numerical procedures. The children have to understand the quantitative relations in the problems that they are asked to solve and how to analyse these relations with numbers.

Understanding rational numbers and intensive quantities

Rational numbers, like whole numbers, can be used to represent quantities. There are some quantities that cannot be represented by a whole number, and to represent these quantities, we must use rational numbers. Quantities that are represented by whole numbers are formed by addition and subtraction: as argued in the previous section, as we add elements to a set and count them (or conventional units, in the case of continuous quantities), we find out what number will be used to represent these quantities. Quantities that cannot be represented by whole numbers are measured not by addition but by division: if we cut one chocolate, for example, in equal parts, and want to have a number to represent the parts, we cannot use a whole number.

We cannot use whole numbers when the quantity that we want to represent numerically:

- is smaller than the unit used for counting, irrespective of whether this is a natural unit (e.g. we have less than one banana) or a conventional unit (e.g. a fish weighs less than a kilo)
- involves a ratio between two other quantities (e.g. the concentration of orange juice in a jar can be described by the ratio of orange concentrate to water; the probability of an event can be described by the ratio between the number of favourable cases to the total number of cases). These quantities are called intensive quantities.

We have concluded from our review that there are serious problems in teaching children about fractions and that intensive quantities are not explicitly considered in the curriculum.

Children learn about quantities that are smaller than the unit through division. Two types of action schemes are used by children in division situations: partitioning, which involves dividing a whole into equal parts, and correspondence situations, where two quantities are involved, a quantity to be shared and a number of recipients of the shares.

Partitioning is the scheme of action most often used in primary schools in the United Kingdom to introduce the concept of fractions. Research shows that children have quite a few problems to solve when they partition continuous quantities: for example, they need to anticipate the connection between number of cuts and number of parts, and some children find themselves with an even number of parts (e.g. 6) when they wanted to have an odd number (e.g. 5) because they start out by partitioning the whole in half. Children also find it very difficult to understand the equivalence between fractions when the parts they are asked to compare do not look the same. For example, if they are shown two identical rectangles, each cut in half but in different ways (e.g. horizontally and diagonally), many 9- and 10-year-olds might say that the fractions are not equivalent; in some studies, almost half of the children in these age levels did not recognize the equivalence of two halves that looked rather different due to being the result of different cuts. Also, if students are asked to paint 2/3 of a figure divided into 9 parts, many 11- to 12-year-olds may be unable to do so, even though they can paint 2/3 of a figure divided into 3 parts; in a study in the United Kingdom, about 40% of the students did not successfully paint 2/3 of figures that had been divided into 6 or 9 sections.

Different studies that we reviewed showed that students who learn about fractions through the engagement of the partitioning schema in division tend to reply on perception rather than on the logic of division when solving problems: they are much more successful with items that can be solved perceptually than with those that cannot. There is a clear lesson here for education: number understanding should be based on logic, not on perception alone, and teaching should be designed to guide children to think about the logic of rational numbers.

The research that we reviewed shows that the partitioning scheme develops over a long period of time. This has led some researchers to develop ways to avoid asking the children to partition quantities by providing them with pre-divided shapes or with computer tools that do the partitioning for the children. The use of these resources has positive effects, but these positive effects seem to be obtained only after large amounts of instruction.

In some studies, the students had difficulties with the idea of improper fractions even after prolonged instruction. For example, one student argued with the researcher during instruction that you cannot have eight sevenths if you divided a whole into seven parts.

In contrast to the difficulties that children have with partitioning, children as young as five or six years in age are quite good at using correspondences in division, and do so without having to carry out the actual partitioning. Some children seem to understand even before receiving any instruction on fractions that, for example, two chocolates shared among four children and four chocolates shared among eight children will give the children in the two groups equivalent shares of chocolate; they demonstrate this equivalence in action by showing that in both cases there is one chocolate to be shared by two children.

Children's understanding of quantities smaller than one is often ahead of their knowledge of fractional representations when they solve problems using the correspondence scheme. This is true of understanding equivalence and even more so of understanding order. Most children at the age of eight or so realise that dividing 1 chocolate among three children will give bigger pieces than dividing one chocolate among four children. This insight that they have about quantities is not necessarily connected with their understanding of ordering fractions by magnitude: the same children might say that 1/3 is less than 1/4 because three is less than four. So we find in the domain of rational numbers the same distinction found in the domain of whole numbers between what children know about quantities and what they know about the numbers used to represent quantities.

Research shows that it is possible to help children connect their understanding of quantities with their understanding of fractions and thus make progress in rational number knowledge. Schools could make use of children's informal knowledge of fractional quantities and work with problems about situations, without requiring them to use formal representations, to help them consolidate this reasoning and prepare them for formalization.

Reflecting about these two schemes of action and drawing insights from them places children in different paths for understanding rational number. When children use the correspondence scheme, they can achieve some insight into the equivalence of fractions by thinking that, if there are twice as many things to be shared and twice as many recipients, then each one's share is the same. This involves thinking about a direct relation between the quantities. The partitioning scheme leads to understanding equivalence in a different way: if a whole is cut into twice as many parts, the size of each part will be halved. This involves thinking about an inverse relation between the quantities in the problem. Research consistently shows that children understand direct relations better than inverse relations and this may also be true of rational number knowledge.

The arguments children use when stating that fractional quantities resulting from sharing are or are not equivalent have been described in one study in the United Kingdom. These arguments include the use of correspondences (e.g. sharing four chocolates among eight children can be shown by a diagram to be equivalent to sharing two chocolates among four children because each chocolate is shared among two children), scalar arguments (twice the number of children and twice the number of chocolates means that they all get the same), and an understanding of the inverse relation between the number of parts and the size of the parts (i.e. twice the number of pieces means that each piece is halved in size). It would be important to investigate whether increasing teachers' awareness of children's own arguments would help teachers guide children's learning in this domain of numbers more effectively.

Some researchers have argued that a better starting point for teaching children about fractions is the use of situations where children can use correspondence reasoning than the use of situations where the scheme of partitioning is the relevant one. Our review of children's understanding of the equivalence and order of fractions supports this claim. However, there are no intervention studies comparing the outcomes of these two ways of introducing children to the use of fractions, and intervention studies would be crucial to solve this issue: one thing is children's informal knowledge but the outcomes of its formalization through instruction might be quite another. There is now considerably more information regarding children's informal strategies to allow for new teaching programmes to be designed and assessed. There is also considerable work on curriculum development in the domain of teaching fractions in primary school. Research that compares the different forms of teaching (based on partitioning or based on correspondences) and the introduction of different representations (decimal or ordinary) is now much more feasible than in the past. Intervention research, which could be carried out in the classroom, is urgently needed. The available evidence suggests that testing this hypothesis appropriately could result in more successful teaching and learning of rational numbers.

In the United Kingdom ordinary fractions continue to play an important role in primary school instruction whereas in some countries greater attention is given to decimal representation than to ordinary fractions in primary school. Two reasons are proposed to justify the teaching of decimals before ordinary fractions . First, decimals are common in metric measurement systems and thus their understanding is critical for learning other topics, such as measurement, in mathematics and science. Second, decimals should be easier than ordinary fractions to understand because decimals can be taught as an extension of place value representation; operations with decimals should also be easier and taught as extensions of place value representation.

It is certainly true that decimals are used in measurement and thus learning decimals is necessary but ordinary fractions often appear in algebraic expressions; so it is not clear *a priori* whether one form of representation is more useful than the other for learning other aspects of mathematics. However, the second argument, that decimals are easier than ordinary fractions, is not supported in surveys of students' performance: students find it difficult to make judgements of equivalence and order as much with decimals as with ordinary fractions. Students aged 9 to 11 years have limited success when comparing decimals written with different numbers of digits after the decimal point (e.g. 0.5 and 0.36): the rate of correct responses varied between 36% and 52% in the three different countries that participated in the study, even though all the children have been taught about decimals.

Some researchers (e.g. Nunes, 1997; Tall, 1992; Vergnaud, 1997) argue that different representations shed light on the same concepts from different perspectives. This would suggest that a way to strengthen students' learning of rational numbers is to help them connect both representations. Case studies of students who received instruction that aimed at helping students connect the two forms of representation show encouraging results. However, the investigation did not include the appropriate controls and so it does not allow for establishing firmer conclusions.

Students can learn procedures for comparing, adding and subtracting fractions without connecting these procedures with their understanding of equivalence and order of fractional quantities, independently of whether they are taught with ordinary or decimal fractions representation. This is not a desired outcome of instruction, but seems to be a quite common one. Research that focuses on the use of children's informal knowledge suggests that it is possible to help students make connections between their informal knowledge and their learning of procedures but the evidence is limited and the consequences of this teaching have not been investigated systematically.

Research has also shown that students do not spontaneously connect their knowledge of fractions developed with extensive quantities smaller than the unit with their understanding of intensive quantities. Students who succeed in understanding that two chocolates divided among four children and four chocolates divided among 8 children yield the same size share do not necessarily understand that a paint mixture made with two litres of white and two of blue paint will be the same shade as one made with four litres of white and four of blue paint.

Researchers have for some time distinguished between different situations where fractions are used and argued that connections that seem obvious to an adult are not necessarily obvious to children. There is now evidence that this is so. There is a clear educational implication of this result: if teaching children about fractions in the domain of extensive quantities smaller than the unit does not spontaneously transfer to their understanding of intensive quantities, a complete fractions curriculum should include intensive quantities in the programme.

Finally, this review opens the way for a fresh research agenda in the teaching and learning of fractions. The source for the new research questions is the finding that children achieve insights into relations between fractional quantities before knowing how to represent them. It is possible to envisage a research agenda that would not focus on children's misconceptions about fractions, but on children's possibilities of success when teaching starts from thinking about quantities rather than from learning fractional representations.

Understanding relations and their graphical representation

Children form concepts about quantities from their everyday experiences and can use their schemas of action with diverse representations of the quantities (iconic, numerical) to solve problems. They often develop sufficient awareness of quantities to discuss their equivalence and order as well as how quantities are changed by operations. It is significantly more difficult for them to become aware of the relations between quantities and operate on relations.

The difficulty of understanding relations is clear both with additive and multiplicative relations between quantities. Children aged about eight to ten years can easily say, for example, how many marbles a boy will have in the end if he started a game with six marbles, won five in the first game, lost three in the second game, and won two in the third game. However, if they are not told how many marbles the boy had at the start and are asked how many more or fewer marbles this boy will have after playing the three games, they find this second problem considerable harder, particularly if the first game involves a loss.

Even if the children are taught how to represent relations and recognise that winning five in the first game does not mean having five marbles, they often interpret the results of operations on relations as if they were quantities. Children find both additive and multiplicative relations significantly more difficult than understanding quantities.

There is little evidence that the design of mathematics curricula has so far taken into account the importance of helping students become aware of the difference between quantities and relations. Some researchers have carried out experimental teaching studies which suggest that it is possible to promote students' awareness of additive relations as different from quantities; this was not an easy task but the instruction seemed to have positive results (but note that there were no control groups). Further research must be carried out to analyse how this knowledge affects mathematics learning: longitudinal and intervention studies would be crucial to clarify this. If positive results are found, there will be imperative policy implications.

The first teaching that children receive in school about multiplicative relations is about proportions. Initial studies on students' understanding of proportions previously led to the conclusion that students' problems with proportional reasoning stemmed from their difficulties with multiplicative reasoning. However, there is presently much evidence to show that, from a relatively early age (about five to six years in the United Kingdom), many children (our estimate is about two-thirds) already have informal knowledge that allows them to solve multiplicative reasoning problems.

Multiplicative reasoning problems are defined by the fact that they involve two (or more) measures linked by a fixed ratio. Students' informal knowledge of multiplicative reasoning stems from the schema of one-to-many correspondence, which they use both in multiplication and division problems. When the product is unknown, children set the elements in the two measures in correspondence (e.g. one sweet costs 4p) and figure out the product (how much five sweets will cost) by counting or adding. When the correspondence is unknown (e.g. if you pay 20p for five sweets, how much does each sweet cost), the children share out the elements (20p shared in five groups) to find what the correspondence is.

This informal knowledge is currently ignored in U.K. schools, probably due to the theory that multiplication is essentially repeated addition and division is repeated subtraction. However, the connections between addition and multiplication on the one hand, and subtraction and division on the other hand, are procedural and not conceptual. So students' informal knowledge of multiplicative reasoning could be developed in school from an earlier age.

Even after being taught other methods to solve proportions problems in school, students continue to use one-to-many correspondences reasoning to solve proportions problems; these solutions have been called building up methods. For example, if a recipe for four people is to be adapted to serve six people, students figure out that six people is the same as four people plus two people; so they figure out what half the ingredients will be and add this to the quantity required for four people. Building up methods have been documented in many different countries and also among people with low levels of schooling. A careful analysis of the reasoning in building-up methods suggests that the students focus on the quantities as they solve these problems, and find it difficult to focus on the relations between the quantities.

Research carried out independently in different countries has shown that students sometimes use additive reasoning about relations when the appropriate model is a multiplicative one. Some recent research has shown that students also use multiplicative reasoning in situations where the appropriate model is additive. These results suggest that children use additive and multiplicative models implicitly and do not make conscious decisions regarding which model is appropriate in a specific situation. We concluded from our review that students' problems with proportional reasoning stems from their difficulties in becoming explicitly aware of relations between quantities. Greater awareness of the models implicit in their solutions would help them distinguish between situations that involve different types of relations: additive, proportional or quadratic, for example.

The educational implication from these findings is that schools should take up the task of helping students become more aware of the models that they use implicitly and of ways of testing their appropriateness to particular situations. The differences between additive and multiplicative situations rests on the relations between quantities; so it is likely that the critical move here is to help students become aware of the relations between quantities implicit in the procedure they use to solve problems.

Two radically different approaches to teaching proportions and linear functions in schools can be identified in the literature. These constitute pragmatic theories, which can guide teachers, but have as yet not been tested systematically. The first, described as functional and human in focus, is based on the notion that students' schemas of action should be the starting point for this teaching. Through instruction, they should become progressively more aware of the relations between quantities that can be identified in such problems. Diagrams, tables and graphs are seen as tools that could help students understand the models of situations that they are using and make them into models for other situations later.

The second, described in the literature as algebraic, proposes that there should be a sharp separation between students' intuitive knowledge, in which physical and mathematical knowledge are intertwined, and mathematical knowledge. Students should be led to formalisations early on in instruction and re-establish the connections between mathematical structures and physical knowledge at a later point. Representations using ordinary and decimal fractions and the number line are seen as the tools that can allow students to abstract early on from the physical situations. Students should learn early on to represent equivalences between ordinary fractions (e.g. 2/4 = 4/8), a representation that would provide insight into proportions, and also equivalences between ordinary and decimal fractions (2/4 = 0.5), which would provide insight into the ordering and equivalence of fractions marked on the number line.

Each of these approaches makes assumptions about the significance of students' informal knowledge at the start of the teaching programme. The functional approach assumes that students' informal knowledge can be formalised through instruction and that this will be beneficial to learning. The algebraic approach assumes that students' informal knowledge is an obstacle to students' mathematics learning. There is evidence from a combination of longitudinal and intervention methods, albeit with younger children, that shows that students' knowledge of informal multiplicative reasoning is a causal and positive factor in mathematics learning. Children who scored higher in multiplicative reasoning problems at the start of their first year in school performed significantly better in the government designed and school administered mathematics achievement test than those whose scores were lower. This longitudinal relationship remained significant after the appropriate controls were taken into account. The intervention study provides results that are less clear because the children were taught not only about multiplicative reasoning but also about other concepts considered key to mathematics learning. Nevertheless, children who were at risk for mathematics learning and received teaching that included multiplicative reasoning, along with two other concepts, showed average achievement in the standardised mathematics achievement tests whereas the control group remained in the bottom 20% of the distribution, as predicted by their assessment at the start of school. So, in terms of the assumptions regarding the role of informal knowledge, the functional approach seems to have the edge over the algebraic approach.

These two approaches to instruction also differ in respect to what students need to know to benefit from teaching and what they learn during the course of instruction. Within the functional approach, the tools used in teaching are diagrams, tables, and graphs so it is clear that students need to learn to read graphs in order to be able to use them as tools for thinking about relations between quantities and functions. Research has shown that students have ideas about how to read graphs before instruction and these ideas should be taken into account when graphs are used in the classroom. It is possible to teach students to read graphs and to use them in order to think about relations in the course of instruction about proportions, but much more research is needed to show how students' thinking changes if they do learn to use graphs to analyse the type of relation relevant in specific situations. Within the algebraic approach, it is assumed that students understand the equivalence of fractions without reference to situations. Our review of students' understanding of fractions, summarised in the previous section, shows that this is not trivial so it is necessary to show that students can, in the course of this teaching, learn both about fraction equivalence and proportional relations.

There is no evidence to show how either of these approaches to teaching works in promoting students' progress nor that one of them is more successful than the other. Research that can clarify this issue is urgently needed and could have a major impact in promoting better learning by U.K. students. This is particularly important in view of findings from the international comparisons that show that U.K. students do relatively well in additive reasoning items but comparatively poorly in multiplicative reasoning items.

Understanding space and its representation in mathematics

When children begin to be taught about geometry, they already know a great deal about space, shape, size, distance and orientation, which are the basic subject matter of geometry. They are also quite capable of drawing logical inferences about spatial matters. In fact, their spatial knowledge is so impressive and so sophisticated that one might expect geometry to be an easy subject for them. Why should they have any difficulty at all with geometry if the subject just involves learning how to express this spatial knowledge mathematically?

However, many children do find geometry hard and some children continue to make basic mistakes right through their time at school. There are two main reasons for these well-documented difficulties. One reason is that many of the spatial relations that children must think about and learn to analyse mathematically in geometry classes are different from the spatial relations that they learn about in their pre-school years. The second is that geometry makes great demands on children's spatial imagination. In order to measure length or area or angle, for example, we have to imagine spaces divided into equal units and this turns out to be quite hard for children to learn to do systematically.

Nevertheless, pre-school children's spatial knowledge and spatial experiences are undoubtedly relevant to the geometry that they must learn about later, and it is important for teachers and researchers alike to recognise this. From a very early age children are able to distinguish and remember different shapes, including basic geometrical shapes. Children are able to co-ordinate visual information about size and distance to recognise objects by their actual size, and also to co-ordinate visual shape and orientation information to recognise objects by their actual shapes. In social situations, children quite easily work out what someone else is looking at by extrapolating that person's line of sight often across quite large distances, which is an impressive feat of spatial imagination. Finally, they are highly sensitive not just to the orientation of lines and of objects in their environments, but also to the relation between orientations: for example, young children can, sometimes at least, recognise when a line in the foreground is parallel to a stable background feature.

These impressive spatial achievements must help children in their efforts to understand the geometry that they are taught about at school, but there is little direct research on the links between children's existing informal knowledge about space and the progress that they make when they are eventually taught about geometry. This is a worrying gap, because research of this sort would help teachers to make an effective connection between what their pupils know already and what they have to learn in their initial geometry classes. It would also give us a better understanding of the obstacles that children encounter when they are first taught about geometry.

Some of these obstacles are immediately apparent when children learn about measurement, first of length and then of area. In order to learn how to measure length, children must grasp the underlying logic of measurement and also the role of iterated (i.e. repeated) measurement units, e.g. the unit of 1 cm repeated on a ruler. Using a ruler also involves an active form of one-to-one correspondence, since the child must imagine and impose on the line being measured the same units that are explicit and obvious on the ruler. Research suggests that children do have a reasonable understanding of the underlying logic of measurement by the time that they begin to learn about geometry, but that many have a great deal of difficulty in grasping how to imagine one-to-one correspondence between the iterated units on the ruler and imagined equivalent units on the line that they are measuring. One common mistake is to set the 1 cm rather than the 0 cm point at one end of the line. The evidence suggests that many children apply a poorly understood procedure when they measure length and are not thinking, as they should, of one-to-one correspondence between the units on the ruler and the length being measured. There is no doubt that teachers should think about how to promote children's reflection on measurement procedures. Nunes, Light and Mason (1993), for example, showed that using a broken ruler was one way to promote this.

Measurement of area presents additional problems. One is that area is often calculated from lengths, rather than measured. So, although the measurement is in one kind of unit, e.g. centimetres, the final calculation is in another, e.g. square centimetres. This is what Vergnaud calls a 'product of measures' calculation. Another potential problem is that most calculations of area are multiplicative: with rectangles and parallelograms, one has to multiply the figure's base by height, and with triangles one must calculate base by height and then halve it. There is evidence that many children attempt to calculate area by adding parts of the perimeter, rather than by multiplying. One consequence of the multiplicative nature of area calculations is that doubling a figure's dimensions more than doubles its area. Think of a rectangle with a base of 10 cm and a height of 4 cm, its area is 40 cm^2: if you enlarge the figure by doubling its base and height (20 cm x 8 cm), you quadruple its area (160 cm^2). This set of relations is hard for pupils, and for many adults too, to understand.

The measurement of area also raises the question of relations between shapes. For example the proof that the same base by height rule for measuring rectangles applies to parallelograms as well rests on the demonstration that a rectangle can be transformed into a parallelogram with the same height and base without changing its area. In turn the rule for finding the area of triangles, A = ½ (base x height), is justified by the fact every triangle can be transformed into a parallelogram with the same base and height by doubling that triangle. Thus, rules for measuring area rest heavily on the relations between geometric shapes. Although Wertheimer did some ingenious studies on how children were able to use of the relations between shapes to help them measure the area of some of these shapes, very little research has been done since then on their understanding of this centrally important aspect of geometry.

In contrast, there is a great deal of research on children's understanding of angles. This research shows that children have very little understanding of angles before they are taught about geometry. The knowledge that they do have tends to be quite disconnected because children often fail to see the connection between angles in dissimilar contexts, like the steepness of a slope and how much a person has to turn at a corner. There is evidence that children begin to connect what they know about angles as they grow older: they acquire, in the end, a fairly abstract understanding of angle. There is also evidence, mostly from studies with the programming language Logo, that children learn about angle relatively well in the context of movement.

Children's initial uncertainties with angles contrast sharply to the relative ease with which they adopt the Cartesian framework for plotting positions in any two-dimensional space. This framework requires them to be able to extrapolate imaginary pairs of straight lines, one of which is perpendicular to the vertical axis and the other to the horizontal axis, and then to work out where these imaginary lines will meet, in order to plot specific positions in space. At first sight this might seem an extraordinarily sophisticated achievement, but research suggests that it presents no intellectual obstacle at all to most children. Their success in extrapolating imaginary straight lines and working out their meeting point may stem for their early experiences in social interactions of extrapolating such lines when working out what other people are looking at, but we need longitudinal research to establish whether this is so. Some further research suggests that, although children can usually work out specific spatial positions on the basis of Cartesian co-ordinates, they often find it hard to use these co-ordinates to work out the relation between two or more different positions in space.

We also need research on another possible connection between children's early informal spatial knowledge and how well they learn about geometry later on. We know that very young children tell shapes apart, even abstract geometric shapes, extremely well, but it is also clear that when children begin to learn about geometry they often find it hard to decompose complicated shapes into several simpler component shapes. This is a worrying difficulty because the decomposition of shapes plays an important part in learning about measurement of area and also of angles. More research is needed on how children learn that particular shapes can be broken down into other component shapes.

Overall, research suggests that the relation between the informal knowledge that children build up before they go to school and the progress that they make at school in geometry is a crucial one. Yet, it is a relation on which there is very little research indeed and there are few theories about this possible link as well. The theoretical frameworks that do exist tend to be pragmatic ones. For example, the Institute Freudenthal group assume a strong link between children's preschool spatial knowledge and the progress that they make in learning about geometry later on, and argue that improving children's early understanding of space will have a beneficial effect on their learning about geometry. Yet, there is no good empirical evidence for either of these two important claims.

Algebraic reasoning

Research on learning algebra has considered a range of new ideas that have to be understood in school mathematics: the use and meaning of letters and expressions to represent numbers and variables; operations and their properties; relations, functions, equations and inequalities; manipulation and transformation of symbolic statements. Young children are capable of understanding the use of a letter to take the place of an unknown number, and are also able to construct statements about comparisons between unknown quantities, but algebra is much more than the substitution of letters for numbers and numbers for letters. Letters are used in mathematics in varying ways. They are used as labels for objects that have no numerical value, such as vertices of shapes or for objects that do have numerical value, such as lengths of sides of shapes. They denote fixed constants such as *g*, *e* or π and also non-numerical constants such as *I* and they represent unknowns and variables. Distinguishing between these meanings is usually not taught explicitly, and this lack of instruction might cause children some difficulty: g, for example, can indicate grams, acceleration due to gravity, an unknown in an equation, or a variable in an expression.

Within common algebraic usage, Küchemann (1981) identified six different ways adolescents used letters in the Chelsea diagnostic test instrument (Hart *et al*., 1984). Letters could be evaluated in some way, ignored, used as shorthand for objects or treated as objects used as a specific unknown, as a generalised number, or as a variable. These interpretations appear to be task-dependent, so learners had developed a sense of what sorts of question were treated in what kinds of ways, i.e. generalising (sometimes idiosyncratically) about question-types through familiarity and prior experience.

The early experiences students have in algebra are therefore very important, and if algebra is presented as 'arithmetic with letters' there are many possible confusions. Algebraic statements are about relationships between variables, constructed using operations; they cannot be calculated to find an answer until numbers are substituted, and the same relationship can often be represented in many different ways. The concept of equivalent expressions is at the heart of algebraic manipulation, simplification, and expansion, but this is not always apparent to students. Students who do not understand this try to act on algebraic expressions and equations in ways which have worked in arithmetical contexts, such as trial-and-error, or trying to calculate when they see the equals sign, or rely on learnt rules such as 'BODMAS' which can be misapplied.

Students' prior experience of equations is often associated with finding hidden numbers using arithmetical facts, such as 'what number, times by 4, gives 24?' being expressed as $4p = 24$. An algebraic approach depends on understanding operations or functions and their inverses, so that addition and subtraction are understood as a pair, and multiplication and division are understood as a pair. This was discussed in an earlier section. Later on, roots, exponents and logarithms also need to be seen as related along with other functions and their inverses. Algebraic understanding also depends on understanding an equation as equating two expressions, and solving them as finding out for what values of the variable they are equal. New technologies such as graph-plotters and spreadsheets have made multiple representations available and there is substantial evidence that students who have these tools available over time develop a stronger understanding of the meaning of expressions, and equations, and their solutions, than equivalent students who have used only formal pencil-and-paper techniques.

Students have to learn that whereas the mathematical objects they have understood in primary school can often be modelled with material objects they now have to deal with objects that cannot always be easily related to their understanding of the material world, or to their out-of-school language use. The use of concrete models such as rods of 'unknown' related lengths, tiles of 'unknown' related areas, equations seen as balances, and other diagrammatical methods can provide bridges between students' past experience and abstract relationships and can enable them to make the shift to seeing relations rather than number as the main focus of mathematics. All these metaphors have limitations and eventually, particularly with the introduction of negative numbers, the metaphors they provide break down. Indeed it was this realisation that led to the invention of algebraic notation.

Students have many perceptions and cognitive tendencies that can be harnessed to help them learn algebra. They naturally try to relate what they are offered to what they already know. While this can be a problem if students refer to computational arithmetic, or alphabetic meaning of letters (e.g. a = apples), it can also be useful if they refer to their understanding of relations between quantities and operations and inverses. For example, when students devise their own methods for mental calculation they often use relations between numbers and the concepts of distributivity and associativity.

Students naturally try to generalise when they see repeated behaviour, and this ability has been used successfully in approaches to algebra that focus on expressing generalities which emerge in mathematical exploration. When learners need to express generality, the use of letters to do so makes sense to them, although they still have to learn the precise syntax of their use in order to communicate unambiguously. Students also respond to the visual impact of mathematics, and make inferences based on layout, graphical interpretation and patterns in text; their own mathematical jottings can be structured in ways that relate to underlying mathematical structure. Algebraic relationships represented by graphs, spreadsheets and diagrammatic forms are often easier to understand than when they are expressed in symbols. For example, students who use function machines are more likely to understand the order of operations in inverse functions.

The difficulties learners have with algebra in secondary school are nearly all due to their inability to shift from earlier understandings of arithmetic to the new possibilities afforded by algebraic notation.

- They make intuitive assumptions and apply pragmatic reasoning to a symbol system they do not yet understand.
- They need to grasp the idea that an algebraic expression is a statement about relationships between numbers and operations.
- They may confuse equality with equivalence and try to get answers rather than transform expressions.
- They get confused between using a letter to stand for something they know, and using it to stand for something they do not know, and using it to stand for a variable.
- They may not have a purpose for using algebra, such as expressing a generality or relationship, so cannot see the meaning of what they are doing.

New technologies offer immense possibilities for imbuing algebraic tasks with meaning, and for generating a need for algebraic expression.

The research synthesis sets these observations out in detail and focuses on detailed aspects of algebraic activity that manifest themselves in school mathematics. It also formulates recommendations for practice and research.

Modelling, problem-solving and integrating concepts

Older students' mathematical learning involves situations in which it is not immediately apparent what mathematics needs to be done or applied, nor how this new situation relates to previous knowledge. Learning mathematics includes learning when and how to adapt symbols and meanings to apply them in unfamiliar situations and also knowing when and how to adapt situations and representations so that familiar tools can be brought to bear on them. Students need to learn how to analyse complex situations in a variety of representations, identify variables and relationships, represent these and develop predictions or conclusions from working with representations of variables and relationships. These might be presented graphically, symbolically, diagrammatically or numerically.

In secondary mathematics, students possess not only intuitive knowledge from outside mathematics and outside school, but also a range of quasi-intuitive understandings within mathematics, derived from earlier teaching and generalisations, metaphors, images and strategies that have served them well in the past. In Tall and Vinner's pragmatic theory, (1981) these are called 'concept images', which are a ragbag of personal conceptual, quasi-conceptual, perceptual and other associations that relate to the language of the concept and are loosely connected by the language and observable artefacts associated with the concept. The difference between students' concept images and conventional definitions causes problems when they come to learn new concepts that combine different earlier concepts. They have to expand elementary meanings to understand new abstract concepts, and sometimes these concepts do not fit with the images and models that students know. For example, rules for combining quantities do not easily extend to negative numbers; multiplication as repeated addition does not easily extend to multiplying decimals.

There is little research and theoretical exploration regarding how combinations of concepts are understood by students in general. For example, it would be helpful to know if students who understand the use of letters, ratio, angle, functions, and geometrical facts well have the same difficulties in learning early trigonometry as those whose understanding is more tenuous. Similarly, it would be helpful to know if students whose algebraic manipulation skills are fluent understand quadratic functions more easily, or differently, from students who do not have this, but do understand transformation of graphs.

There is research about how students learn to use and apply their knowledge of functions, particularly in the context of modelling and problem-solving. Students not only have to learn to think about relationships (beyond linear relationships with which they are already familiar), but they also need to think about relations between relations. Our analysis (see Paper 4) suggested that curricula presently do not consider the important task of helping students become aware of the distinctions between quantities and relations; this task is left to the students themselves. It is possible that helping students make this distinction at an earlier age could have a positive impact on their later learning of algebra.

In the absence of specific instructions, students tend to repeat patterns of learning that have enabled them to succeed in other situations over time. Students tend to start on new problems with qualitative judgements based on a particular context, or the visual appearance of symbolic representations, then tend to use additive reasoning, then form relationships by pattern recognition or repeated addition, and then shift to proportional and relational thinking if necessary. The tendency to use addition as a first resort persists as an obstacle into secondary mathematics. Students also tend to check their arithmetic if answers conflict rather than adapting their reasoning by seeing if answers make sense or not, or by analysing what sorts of relations are important in the problem. Pedagogic intervention over time is needed to enable learners to look for underlying structure and, where multiple representations are available (graphs, data, formulae, spreadsheets), students can, over time, develop new habits that focus on covariation of variables. However, they need knowledge and experience of a range of functions to draw on. Students are unlikely to detect an exponential relationship unless they have seen one before, but they can describe changes between nearby values in additive terms. A shift to describing changes in multiplicative terms does not happen naturally.

We hoped to find evidence about how students learn to use mathematics to solve problems when it is not immediately clear what mathematics they should be using. Some evidence in elementary situations has been described in an earlier section, but at secondary level there is only evidence of successful strategies, and not about how students come to have these strategies. In modelling and some other problem-solving situations successful students know how to identify variables and how to form an image of simultaneous variation. Successful students know how to hold one variable still while the change in another is observed. They are also able to draw on a repertoire of known function-types to say more about how the changes in variables are related. It is more common to find secondary students treating each situation as *ad hoc* and using trial-and-adjustment methods which are arithmetically-based. Pedagogic intervention over time is needed to enable them to shift towards seeing relationships, and relations between relations, algebraically and using a range of representational tools to help them do so.

The tendencies described above are specific instances of a more general issue. 'Outside' experiential knowledge is seldom appropriate as a source for meaning in higher mathematics, and students need to learn how to distinguish between situations where earlier and 'outside' understandings are, and are not, going to be helpful. For example, is it helpful to use your 'outside' knowledge about cooking when solving a ratio problem about the size of cakes? In abstract mathematics the same is true: the word 'similar' means something rather vague in everyday speech, but has specific meaning in mathematics. Even within mathematics there are ambiguities. We have to understand, for example, that -40 is greater in magnitude than -4, but a smaller number.

All students generalise inductively from the examples they are given. Research evidence of secondary mathematics reveals many typical problems that arise because of generalising irrelevant features of examples, or over-generalising the domain of applicability of a method, but we found little systematic research to show instances where the ability to generalise contributes positively to learning difficult concepts, except to generate a need to learn the syntax of algebra.

Finally, we found considerable evidence that students do, given appropriate experiences over time, change the ways in which they approach unfamiliar mathematical situations and new concepts. We only found anecdotal evidence that these new ways to view situations are extended outside the mathematics classroom. There is considerable evidence from long-term curriculum studies that the procedures students have to learn in secondary mathematics are learnt more easily if they relate to less formal explorations they have already undertaken. There is evidence that discussion, verbalisation, and explicitness about learning can help students make these changes.

Five common themes across the topics reviewed

In our view, a set of coherent themes cuts across the rich, and at first sight heterogeneous, topics around which we have organised our outline. These themes rise naturally from the material that we have mentioned, and they do not include recent attempts to link brain studies with mathematical education. In our view, knowledge of brain functions is not yet sophisticated enough to account for assigning meaning, forming mathematical relationships or manipulating symbols, which we have concluded are the significant topics in studies of mathematical learning.

In this section, we summarise five themes that emerged as significant across the research on the different topics, summarised in the previous sections.

Number

Number is not a unitary idea that develops conceptually in a linear fashion. In learning, and in mathematical meaning, understanding of number develops in complementary strands, sometimes with discontinuities and changes of meaning. Emphasis on calculation and manipulation with numbers rather than on understanding the underlying relations and mathematical meanings can lead to over-reliance and misapplication of methods.

Most children start school with everyday understandings that can contribute to their early learning of number. They understand 'more' and 'less' without knowing actual quantities, and can compare discrete and continuous quantities of familiar objects. Whole number is the tool which enables them to be precise about comparisons and relations between quantities, once they understand cardinality.

Learning to count and understanding quantities are separate strands of development which have to be experienced alongside each other. This allows comparisons and combinations to be made that are expressed as relations. Counting on its own does not provide for these. Counting on its own also means that the shift from discrete to continuous number is a conceptual discontinuity rather than an extension of meaning.

Rational numbers (we have used 'fraction' and 'rational number' interchangeably in order to focus on their meaning for learners, rather than on their mathematical definitions) arise naturally for children from understanding division in sharing situations, rather than from partitioning wholes. Understanding rational numbers as a way of comparing quantities is fundamental to the development of multiplicative and proportional reasoning, and to applications in geometry, science, and everyday life. This is not the same as saying that children should do arithmetic with rational numbers. The decimal representation does not afford this connection (although it is relatively easy to do additive arithmetic with decimal fractions, as long as the same number of digits appears after the decimal point).

The connection between number and quantity becomes less obvious in higher mathematics, e.g. on the co-ordinate plane the numbers indicate scaled lengths from the axes, but are more usefully understood as values of the variables in a function. Students also have to extend the meaning of number to include negative numbers, infinitesimals, irrationals, and possibly complex numbers. Number has to be abstracted from images of quantity and used as a set of related, continuous, values which cannot all be expressed or depicted precisely. Students also have to be able to handle number-like entities in the form of algebraic terms, expressions and functions. In these contexts, the idea of number as a systematically related set (and subsets) is central to manipulation and transformation; they behave like numbers in relations, but are not defined quantities that can be enumerated. Ordinality of number also has a place in mathematics, in the domain of functions that generate sequences, and also in several statistical techniques.

Successful learning of mathematics includes understanding that number describes quantity; being able to make and use distinctions between different, but related, meanings of number; being able to use relations and meanings to inform application and calculation; being able to use number relations to move away from images of quantity and use number as a structured, abstract, concept.

Logical reasoning plays a crucial part in every branch of mathematical learning

The importance of logic in children's understanding and learning of mathematics is a central theme in our review. This idea is not a new one, since it was also the main claim that Piaget made about children's understanding of mathematics. However, Piaget's theory has fallen out of favour in recent years, and many leading researchers on mathematics learning either ignore or actively dismiss his and his colleagues' contribution to the subject. So, our conclusion about the importance of logic may seem a surprising one but, in our view, it is absolutely inescapable. We conclude that the evidence demonstrates beyond doubt that children rely on logic in learning mathematics and that many of their difficulties in solving mathematical problems are due to failures on their part to make the correct logical move which would have led them to the correct solution.

We have reviewed evidence that four different aspects of logic have a crucial role in learning about mathematics. Within each of these aspects we have been able to identify definite changes over time in children's understanding and use of the logic in question. The four aspects follow.

The logic of correspondence (one-to-one and one-to-many correspondence)

Children must understand one-to-one correspondence in order to learn about cardinal number. Initially they are much more adept at applying this kind of correspondence when they share than when they compare spatial arrays of items. The extension of the use of one-to-one correspondence from sharing to working out the numerical equivalence or non-equivalence of two or more spatial arrays is a vastly important step in early mathematical learning.

One–to-many correspondence, which itself is an extension of children's existing knowledge of one-to-one correspondences, plays an essential, but until recently largely ignored, part in children's learning about multiplication. Researchers and teachers have failed to consider that one-to-many correspondence is a possible basis for children's initial multiplicative reasoning because of a wide-spread assumption that this reasoning is based on children's additive knowledge. However, recent evidence on how to introduce children to multiplication shows that teaching them multiplication in terms of one-to-many correspondence is more effective than teaching them about multiplication as repeated addition.

The logic of inversion

The subject of inversion was also neglected until fairly recently, but it is now clear that understanding that the addition and subtraction of the same quantity leaves the quantity of a set unchanged is of great importance in children's additive reasoning. Longitudinal evidence also shows that this understanding is a strong predictor of children's mathematical progress. Experimental research demonstrates that a flexible understanding of inversion is an essential element in children's geometrical reasoning as well. It is highly likely that children's learning about the inverse relation between multiplication and division is an equally important a part of mathematical learning, but the right research still has to be done on this question. Despite this gap, there is a clear case for giving the concept of inversion a great deal more prominence than it has now in the school curriculum.

The logic of class inclusion and additive composition

Numbers consist of other numbers. One cannot understand what 6 means unless one also knows that sets of 6 are composed of 5 + 1 items, or 4 + 2 items etc. The logic that allows children to work out that every number is a set of combination of other numbers is known as class inclusion. This form of inclusion, which is also referred to as additive composition of number, is the basis of the understanding of ordinal number: every number in the number series is the same as the one that precedes it plus one. It is also the basis for learning about the decade structure: the number 4321 consists of four thousands, three hundreds two tens and one unit, and this can only be properly understood by a child who has thoroughly grasped the additive composition of number. This form of understanding also allows children to compare numbers (7 is 4 more than 3) and thus to understand numbers as a way of expressing relations as well as quantities. The evidence clearly shows that children's ability to use this form of inclusion in learning about number and in solving mathematical problems is at first rather weak, and needs some support.

The logic of transitivity

All ordered series, including number, and also forms of measurement involve transitivity ($a > c$ if $a > b$ and $b > c$: $a = c$ if $a = b$ and $b = c$). Empirical evidence shows that children as young as 5-years of age do to some extent grasp this set of relations, at any rate with continuous quantities like length. However, learning how to use transitive relations in numerical measurements (for example, of area) is an intricate and to some extent a difficult business. Research, including Piaget's initial research on measurement, shows that one powerful reason for children finding it difficult to apply transitive reasoning to measurement successfully is that they often do not grasp the importance of iteration (repeated units of measurement). These difficulties persist through primary school.

One of the reasons why Piaget's ideas about the importance of logic in children's mathematical understanding have been ignored recently is probably the nature of evidence that he offered for them. Although Piaget's main idea was a positive one (children's logical abilities determine their learning about mathematics), his empirical evidence for this idea was mainly negative: it was about children's difficulties with the four aspects of logic that we have just discussed. A constant theme in our review is that this is not the best way to test a causal theory about mathematical learning. We advocate instead a combination of longitudinal research with intervention studies. The results of this kind of research do strongly support the idea that children's logic plays a critical part in their mathematical learning.

Children should be encouraged to reflect on their implicit models and the nature of the mathematical tools

Children need to re-conceptualise their intuitive models about the world in order to access the mathematical models that have been developed in the discipline. Some of the intuitive models used by children lead them to appropriate mathematical problem solving, and yet they may not know why they succeeded. This was exemplified by students' use of one-to-many correspondence in the solution of proportions problems: this schema of action leads to success but students may not be aware of the invariance of the ratio between the variables when the scheme is used to solve problems. Increasing students' awareness of this invariant should improve their mathematical understanding of proportions.

Another example of implicit models that lead to success is the use of distributivity in oral calculation of multiplication and division. Students who know that they can, instead of multiplying a number by 15, multiply it by 10 and then add half of this to the product, can be credited with implicit knowledge of distributivity. It is possible that they would benefit later on, when learning algebra, from the awareness of their use of distribitivity in this context. This understanding of distributivity developed in a context where they could justify it could be used for later learning.

Other implicit models may lead students astray. Fischbein, Deri, Nello and Marino (1985) and Greer (1988) have shown that some implicit models interfere with students' problem solving. If, for example, they make the implicit assumption that in a division the dividend must always be larger than the divisor, they might shift the numbers around in implementing the division operation when the dividend is actually smaller than the divisor. So, when students have developed implicit models that lead them astray, they would also benefit from greater awareness of these implicit models.

The simple fact that students do use intuitive models when they are learning mathematics, whether the teacher recognises the models or not, is a reason for wanting to help students develop an awareness of the models they use. Instruction could and should play a crucial role in this process.

Finally, reflecting on implicit models can help students understand mathematics better and also link mathematics with reality and with other disciplines that they learn in school. Freudenthal (1971) argued that it would be difficult for teachers of other disciplines to tie the bonds of mathematics to reality if these have been cut by the mathematics teacher. In order to tie these bonds, mathematics lessons can explore models that students use intuitively and extend these models to scientific concepts that have been shown to be challenging for students. One of the examples explored in a mathematics lesson designed by Treffers (1991) focuses on the mathematics behind the concept of density. He tells students the number of bicycles owned by people in United Kingdom and in the Netherlands. He also tells them the population of these two countries. He then asks them in which country there are more bicycles. On the basis of their intuitive knowledge, students can easily engage in a discussion that leads to the concept of density: the number of bicycles should be considered in relation to the number of people. A similar discussion might help students understand the idea of population density and of density in physics, a concept that has been shown to be very difficult for students. The discussion of how one should decide which country has more bicycles draws on students' intuitive models; the concept of density in physics extends this model. Streefland and Van den Heuvel-Panhuizen (see Paper 4) suggested that a model of a situation that is understood intuitively can become a model for other situations, which might not be so accessible to intuition. Students' reflection about the mathematics encapsulated in one concept is termed by Treffers horizontal mathematising; looking across concepts and thinking about the mathematics tools themselves leads to vertical mathematising, i.e. a re-construction of the mathematical ideas at a higher level of abstraction. This pragmatic theory about how students' implicit models develop can be easily put to test and could have an impact on mathematics as well as science education.

Mathematical learning depends on children understanding systems of symbols

One of the most powerful contributions of recent research on mathematical learning has come from work on the relation of logic, which is universal, to mathematical symbols and systems of symbols, which are human inventions, and thus are cultural tools that have to be taught. This distinction plays a role in all branches of mathematical learning and has serious implications for teaching mathematics.

Children encounter mathematical symbols throughout their lives, outside school as well as in the classroom. They first encounter them in learning to count. Counting systems with a base provide children with a powerful way of representing numbers. These systems require the cognitive skills involved in generative learning. As it is impossible to memorise a very long sequence of words in a fixed order, counting systems with a base solve this problem: we learn only a few symbols (the labels for units, decades, hundred, thousand, million etc.) by memory and generate the other ones in a rule-based manner. The same is true for the Hindu-Arabic place value system for writing numbers: when we understand how it works, we do not need to memorise how each number is written.

Mathematical symbols are technologies in the sense that they are human-made tools that improve our ability to control and adapt to the environment. Each of these systems makes specific cognitive demands from the learner. In order to understand place-value representation, for example, students' must understand additive composition. If students have explicit knowledge of additive composition and how it works in place-value representation, they are better placed to learn column arithmetic, which should then enable students to calculate with very large numbers; this task is very taxing without written numbers. So the costs of learning to use these tools are worth paying: the tools enable students to do more than they can do without the tools. However, research shows that students should be helped to make connections between symbols and meanings: they can behave as if they understand how the symbols work while they do not understand them completely: they can learn routines for symbol manipulation that remain disconnected from meaning.

This is also true of rational numbers. Children can learn to use written fractions by counting the number of parts into which a whole was cut and writing this below a dash, and counting the number of parts painted and writing this above the dash. However, these symbols can remain disconnected from their logical thinking about division. These disconnections between symbols and meaning are not restricted to writing fractions: they are also observed when students learn to add and subtract fractions and also later when students learn algebraic symbols.

Plotting variables in the Cartesian plane is another use of symbol systems that can empower students: they can, for example, more easily analyse change by looking at graphs than they can by intuitive comparisons. Here, again, research has shown how reading graphs also depends on the interpretations that students assign to this system of symbols.

A recurrent theme in the review of research across the different topics was that the disconnection between symbols and meanings seems to explain many of the difficulties faced by primary school students in learning mathematics. The inevitable educational implication is that teaching aims should include promoting connections between symbols and meaning when symbols are introduced and used in the classroom.

This point is, of course, not new, but it is well worth reinforcing and, in particular, it is well worth remembering in the light of current findings. The history of mathematics education includes the development of pedagogical resources that were developed to help students attribute meaning to mathematical symbols. But some of these resources, like Dienes' blocks and Cuisinaire's rods, are only encountered by students in the classroom; the point we are making here is that students acquire informal knowledge in their everyday lives, which can be used to give meaning to mathematical symbols learned in the classroom. Research in mathematics education over the last five decades or so has helped describe the situations in which these meanings are learned and the way in which they are structured. Curriculum development work that takes this knowledge into account has already started (a major example is the research by members of the Freudenthal Institute) but it is not as widespread as one would expect given the discoveries from past research.

Children need to learn modes of enquiry associated with mathematics

We identify some important mathematical modes of enquiry that arise in the topics covered in this synthesis.

Comparison helps us make new distinctions and create new objects and relations

A cycle of creating and naming new objects through acting on simple objects pervades mathematics, and the new objects can then be related and compared to create higher-level objects. Making additive and multiplicative comparisons is an aspect of understanding relations between quantities and arithmetic. These comparisons are manifested precisely as difference and ratio. Thus difference and ratio arise as two new mathematical ideas, which become new mathematical objects of study and can be represented and manipulated. Comparisons are related to making distinctions, sorting and classifying based on perceptions, and students need to learn to make these distinctions based on mathematical relations and properties, rather than perceptual similarities.

Reasoning about properties and relations rather than perceptions

Many of the problems in mathematics that students find hard occur when immediate perceptions lead to misapplication of learnt methods or informal reasoning. Throughout mathematics, students have to learn to interpret representations before they think about how to respond. They need to think about the relations between different objects in the systems and schemes that are being represented.

Making and using representations

Conventional number symbols, algebraic syntax, coordinate geometry, and graphing methods, all afford manipulations that might otherwise be impossible. Coordinating different representations to explore and extend meaning is a fundamental mathematical skill that is implicit in the use of the number line to represent quantities, for example, the use of graphs to express functions. Equivalent representations, such as for number, algebraic relationships and functions, can provide new insights through comparison and isomorphic analogical reasoning.

Action and reflection-on-action

Learning takes place when we reflect on the effects of actions. In mathematics, actions may be physical manipulation, or symbolic rearrangement, or our observations of a dynamic image, or use of a tool. In all these contexts, we observe what changes and what stays the same as a result of actions, and make inferences about the connections between action and effect. In early mathematics such reflection is usually embedded in children's classroom activity, such as when using manipulatives to model changes in quantity. In later mathematics changes and invariance may be less obvious, particularly when change is implicit (as in a situation to be modelled) or useful variation is hard to identify (as in a quadratic function).

Direct and inverse relations

Direct and inverse relations are discussed in several of our papers. While it may sometimes be easier to reason in a direct manner that accords with action, it is important in all aspects of mathematics to be able to construct and use inverse reasoning. Addition and subtraction must be understood as a pair, and multiplication and division as a pair, rather than as a set of four binary operations. As well as enabling more understanding of relations between quantities, this also establishes the importance of reverse chains of reasoning throughout mathematical problem-solving, algebraic and geometrical reasoning. For example, using reverse reasoning makes it more likely that students will learn the dualism embedded in Cartesian representations; that all points on the graph fulfil the function, and the function generates all points on the graph.

Informal and formal reasoning

At first young children bring everyday understandings into school, and mathematics can allow them to formalise these and make them more precise. On the other hand, intuitions about continuity, approximation, dynamic actions and three-dimensional space might be over-ridden by early school mathematics – yet are needed later on. Mathematics also provides formal tools which do not describe everyday outside experience, but enable students to solve problems in mathematics and in the world which would be unnoticed without a mathematical perspective. In the area of word problems and realistic problems learning when and how to apply informal and formal reasoning is important. Later on, counter-intuitive ideas have to take the place of early beliefs, such as 'multiplication makes things bigger' and students have to be wary of informal, visual and immediate responses to mathematical stimuli.

A recurring issue in the papers is that students find it hard to coordinate *attention on local and global changes*. For example, young children confuse quantifying 'relations between relations' with the original quantities; older children who cannot identify covariation of functions might be able to talk about separate variation of variables; students readily see term-to-term patterns in sequences rather than the generating function; changes in areas are confused with changes in length.

Epilogue

Our aim has been to write a review that summarises our findings from the detailed analysis of a large amount of research. We sought to make it possible for educators and policy makers to take a fresh look at mathematics teaching and learning, starting from the results of research on key understandings, rather than from previous traditions in the organisation of the curriculum. We found it necessary to organise our review around ideas that are already core ideas in the curriculum, such as whole and rational number, algebra and problem solving, but also to focus on ideas that might not be identified so easily in the current curriculum organisation, such as students' understanding of relations between quantities and their understanding of space.

We have tried to make cogent and convincing recommendations about teaching and learning, and to make the reasoning behind these recommendations clear to educationalists. We have also recognised that there are weaknesses in research and gaps in current knowledge, some of which can be easily solved by research enabled by significant contributions of past research. Other gaps may not be so easily solved, and we have described some pragmatic theories that are, or can be, used by teachers when they design instruction. Classroom research, stemming from the exploration of these pragmatic theories, can provide new insights for further research in the future.

Endnotes

1 Details of the search process is provided in Appendix 1. This contains the list of data bases and journals consulted and the total number of papers read although not all of these can be cited in the six papers that comprise this review.

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