Maxima and Minima

To find a maximum or minimum:

- Find an expression for the quantity you are trying to maximise/minimise (y say) in terms of one other variable (x).
- Find an expression for $\frac{dy}{dx}$ and put it equal to 0.
- Solve the resulting equation to find any x values that give a maximum or minimum.
- Find $\frac{d^2y}{dx^2}$ and substitute each value of x. A ***negative*** value implies a ***maximum***.
  A ***positive*** value implies a ***minimum***.
- Calculate the maximum/minimum value.

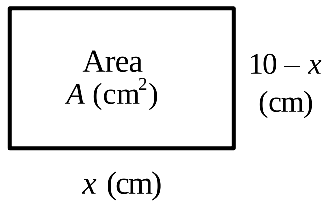
Example A piece of wire 20 cm long is bent into the shape of a rectangle.
Find the maximum area it can enclose.

Let the length be x, then the width will be $10 - x$ (long side + short side = half of 20)

$$A = x(10 - x) = 10x - x^2$$

$$\frac{dA}{dx} = 10 - 2x = 0 \text{ gives } x = 5$$

$$\frac{d^2A}{dx^2} = -2 \text{ implying a maximum.}$$



The area is maximum when each side is 5 cm.
The maximum area is **25cm^2**

(Note the area is a maximum when the shape is a square.)

Example The velocity of a car, v ms^{-1} between two road junctions is modelled by

$$v = 3t - 0.2t^2 \qquad \text{for } 0 \le t \le 15$$

where t is the time in seconds after it sets off from the first junction.
Find the maximum speed.

For maximum speed $\frac{dv}{dt} = 3 - 0.4t = 0$

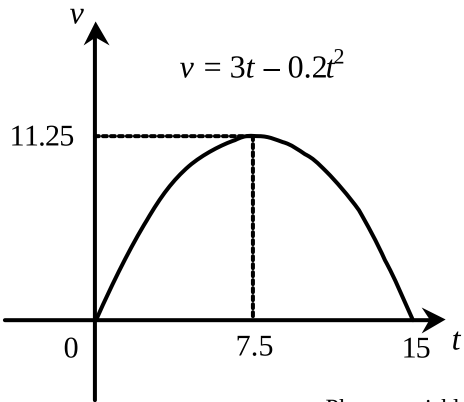
Note $\frac{dv}{dt}$ = acceleration
When the speed reaches a maximum, the acceleration is zero.

$$0.4t = 3$$

$$t = \frac{3}{0.4} = 7.5 \text{ (seconds)}$$

$$\frac{d^2v}{dt^2} = -0.4 \text{ implying a maximum.}$$

Maximum speed $= 3t - 0.2t^2 = 3 \times 7.5 - 0.2 \times 7.5^2$
$= \mathbf{11.25\ ms^{-1}}$



Example The function $p = x^3 - 18x^2 + 105x - 88$ models the way the profit per item made, p pence, depends on x, the number produced in thousands.
Find the maximum and minimum values of p and sketch a graph of p against x.

For max/min: $\frac{dp}{dx} = 3x^2 - 36x + 105 = 0$

Simplify the equation: $x^2 - 12x + 35 = 0$

then solve it $(x-5)(x-7) = 0$

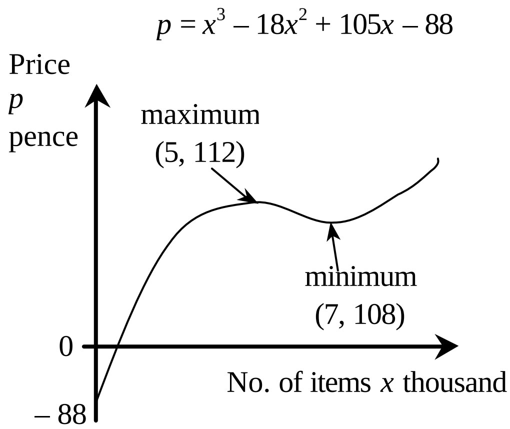
$x = 5$ or 7

$\frac{d^2p}{dx^2} = 6x - 36$ is negative when $x = 5$ (maximum)

and positive when $x = 7$ (minimum)

When $x = 5$, $p = 5^3 - 18 \times 5^2 + 105 \times 5 - 88 = 112$

When $x = 7$, $p = 7^3 - 18 \times 7^2 + 105 \times 7 - 88 = 108$



There is a **maximum point** at **(5, 112)** and a **minimum point** at **(7, 108)**.

The model predicts a peak on the graph when 5000 are produced, the profit per item then being £1.12. The profit per item falls to £1.08 when 7000 are produced before rising again.

Example A cylindrical hot water tank is to have a capacity of 4 m^3.
Find the radius and height that would have the least surface area.

Substitute for h to find a formula for the surface area in terms of just one variable, r:

$$S = 2pr^2 + 2pr \times \frac{4}{pr^2}$$

Simplify: $S = 2pr^2 + 8r^{-1}$

Differentiate: $\frac{dS}{dr} = 4pr - 8r^{-2} = 0$ for max/min

Solve the equation: $4pr = \frac{8}{r^2}$

$$r^3 = \frac{8}{4p} = 0.6366...$$

$$r = \sqrt[3]{0.6366...} = 0.860..$$

$$\frac{d^2S}{dr^2} = 4p + 16r^{-3} = 4p + \frac{16}{r^3} \text{ is positive}$$

This implies minimum surface area.

Also $h = \frac{4}{pr^2} = \frac{4}{p \times 0.860...^2} = 1.72...$

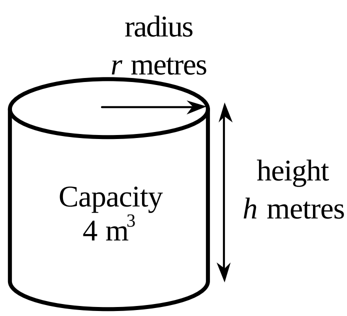
The tank with minimum area has **radius 0.86 m** and **height 1.72 m** (to 2 dp).

Formulae for a cylinder:

Surface Area $S = 2pr^2 + 2prh$

Volume $V = pr^2h = 4$

gives $h = \frac{4}{pr^2}$



The minimum surface area can be found from the formula above:

$$S = 2p \times 0.86..^2 + 2p \times 0.86... \times 1.72...$$

The minimum surface area is **13.9 m^2** (to 1 dp)

Use differentiation to solve the following problems.

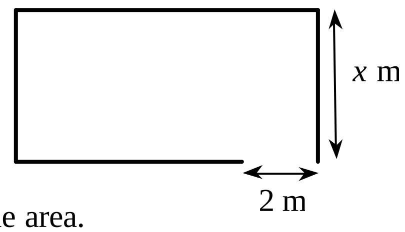
1 The velocity of a car, v ms^{-1} as it travels over a level crossing is modelled by $v = t^2 - 4t + 12$ for $0 \le t \le 4$ where t is the time in seconds after it reaches the crossing.
Find the car's minimum speed.

2 When a ball is thrown vertically upwards, its height h metres after t seconds is modelled by $h = 20t - 5t^2$. Find the maximum height it reaches.

3 A plane initially flying at a height of 240 m dives to deliver some supplies. Its height after t seconds is $h = 8t^2 - 80t + 240$ (m). Find the plane's minimum height during the manoeuvre.

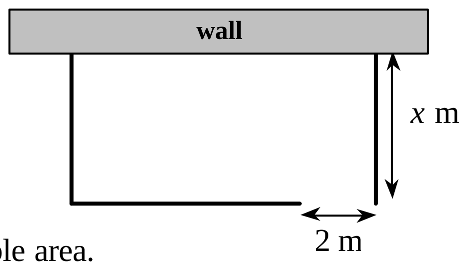
4 The closing price of a company's shares in pence is $p = 2x^3 - 12x^2 + 18x + 45$ for $0 \le x \le 5$ where x is the number of days after the shares are released.
Find the maximum and minimum values of p and sketch a graph of p against x.

5 A farmer has 100 metres of fencing to use to make a rectangular enclosure for sheep as shown. He will leave an opening of 2 metres for a gate.



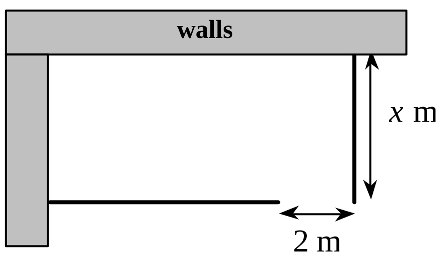
a) Show that the area of the enclosure is given by: $A = 51x - x^2$
b) Find the value of x that will give the maximum possible area.
c) Calculate the maximum possible area.

6 A farmer has 100 metres of fencing to use to make a rectangular enclosure for sheep as shown. He will use an existing wall for one side of the enclosure and leave an opening of 2 metres for a gate.

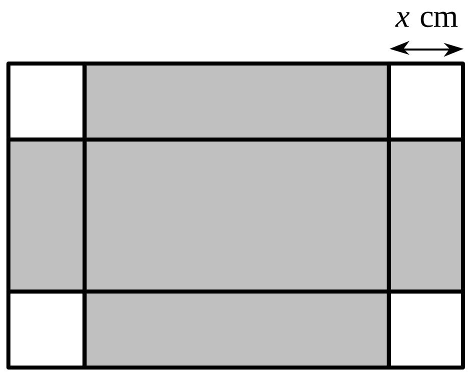


a) Show that the area of the enclosure is given by: $A = 102x - 2x^2$
b) Find the value of x that will give the maximum possible area.
c) Calculate the maximum possible area.

7 A farmer has 100 metres of fencing to use to make a rectangular enclosure for sheep as shown. He will existing walls for two sides of the enclosure and leave an opening of 2 metres for a gate.



a) Show that the area of the enclosure is given by: $A = 102x - x^2$
b) Find the value of x that will give the maximum possible area.
c) Calculate the maximum possible area.

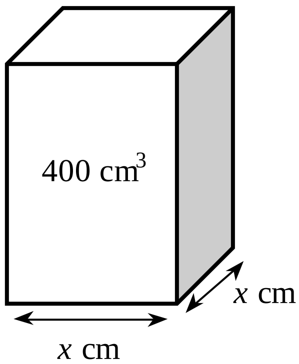


8 An open-topped box is to be made by removing squares from each corner of a rectangular piece of card and then folding up the sides.

a) Show that if the original rectangle of card measured 80 cm by 50 cm and the squares removed from the corners have sides x cm long, then the volume of the box is given by: $V = 4x^3 - 260x^2 + 4000x$

b) Find the value of x that will give the maximum possible volume.
c) Calculate the maximum possible volume.

9 Repeat question 8 starting with
a) a rectangular card measuring 160 cm by 100 cm
b) a rectangular card measuring 60 cm by 40 cm.

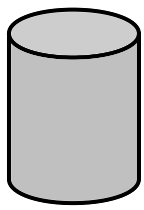


10 A **closed** tank is to have a square base and capacity 400 cm^3
a) Show that the total surface area of the container is given by:

$$S = 2x^2 + \frac{1600}{x}$$

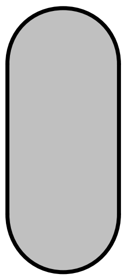
b) Find the value of x that will give the minimum surface area.
c) Calculate the minimum surface area.

11 Find the minimum surface area of an **open-topped** tank with a square base and capacity 400 cm^3



12 A soft drinks manufacturer wants to design a cylindrical can to hold half a litre (500 cm^3) of drink.
Find the minimum area of material that can be used to make the can.

13 Repeat question 20 for a can to hold 1 litre of drink.



14 An aircraft window consists of a central rectangle and two semi-circular ends as shown in the sketch.

A window is required to have an area of 1 m^2
Find the dimensions of the window that has the smallest possible perimeter.

Teacher Notes

Unit Advanced Level, Modelling with calculus

Skills used in this activity:
- solving maximum and minimum problems

Preparation
Students need to be able to:
- differentiate polynomials;
- solve linear and quadratic equations;
- sketch curves.

Notes on Activity
It is recommended that you use the Nuffield activity 'Stationary Points' before this activity.
A Powerpoint presentation, 'Maxima and Minima', includes the examples from pages 1 and 2.

Answers

1 8 ms^{-1}
2 20 m
3 40 m
4 Maximum 53 p after 1 day, minimum 45 p after 3 days
5 a) 25.5 m b) 650.25 m^2
6 a) 25.5 m b) 1300.5 m^2
7 a) 25.5 m b) 2601 m^2
8 b) 10 cm c) 18 000 cm^3
9 a) $x = 20$ giving 144000 cm^3 b) $x = 7.85$ giving 8450 cm^3 (3 sf)
10 b) 7.37 cm c) 326 cm^2 (3 sf)
11 $x = 9.28$ giving 259 cm^2 (3 sf)
12 $r = 4.30$ giving 349 cm^2 (3 sf)
13 $r = 5.42$ giving 554 cm^2 (3 sf)
14 $r = 0.564$ m, giving width 1.13 m, height 0 m i.e. minimum perimeter is when shape is a circle of radius 0.564 m with perimeter 3.54 m.