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Using manipulatives in the foundations of arithmetic

Examples for teachers

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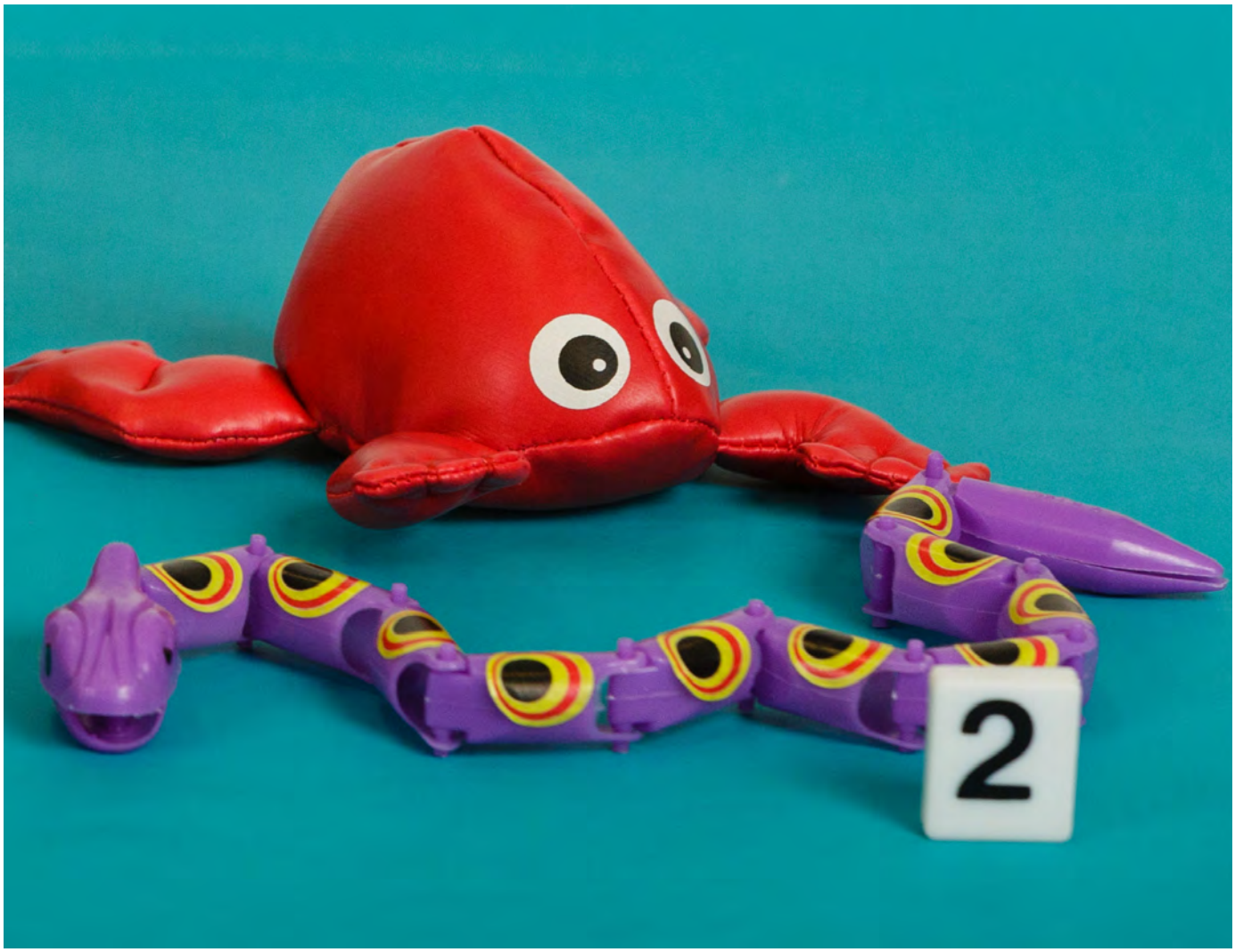

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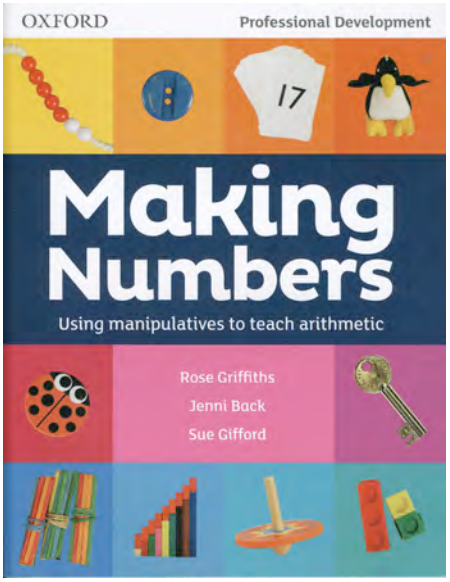
Introduction

During the last two years, we have worked on a Nuffield Foundation funded research and development project, exploring the use of manipulatives in the learning and teaching of arithmetic. An account of that research is published as Using manipulatives in the foundations of arithmetic: main report.

Manipulatives are items that you can handle and move. These range from everyday items such as buttons and shells, to resources designed specifically for teaching arithmetic, such as tens and ones or coloured number rods.

Our intention here is to give classroom teachers a brief introduction to some of the key findings from our study and to provide examples of the kinds of activities we recommend. We want children to enjoy working with practical objects in a purposeful way – to be able to explore problems, to look for patterns in the number system and to become more fluent in their work in number.

Our complete guidance for teachers is published by Oxford University Press: *Making Numbers*: using manipulatives to teach arithmetic. OUP's free website for teachers, OxfordOwl, hosts four short *Making Numbers* films giving further advice for teachers and four lively animations for use in the classroom with children: Five Friends Counting, Ten Fishes in the Sea, Half a Hundred Hedgehogs, and Ten Tens and a Dragon.



There are many ways of using manipulatives, and some approaches work better than others. Children do not gain much from just being shown how to use manipulatives to follow a pencil and paper procedure. Instead we suggest a more effective and creative approach that builds children's understanding and confidence.

Rearranging objects helps children to try out new ideas. They can use a range of strategies alongside each other, including using manipulatives, talking, drawing and writing, and gradually moving towards a confident use of more abstract ways of working.

Manipulatives work because they:

- Help children make sense of arithmetic
- Help teachers see what children understand
- Increase children's engagement and enjoyment
- Develop visual images and understanding
- Help children to work together and share ideas
- Are tools to help children solve problems; investigate patterns and relationships; demonstrate and explain results and reasoning
- Provide a bridge to abstract thinking.

There are several things that can make children's use of manipulatives more effective. These include careful matching of both manipulatives and activities to the mathematical focus, and thinking about children's prerequisite knowledge.

Our aim is to develop children's 'number sense'

– and we decided it was helpful to think of three aspects: counting, comparing and composition.

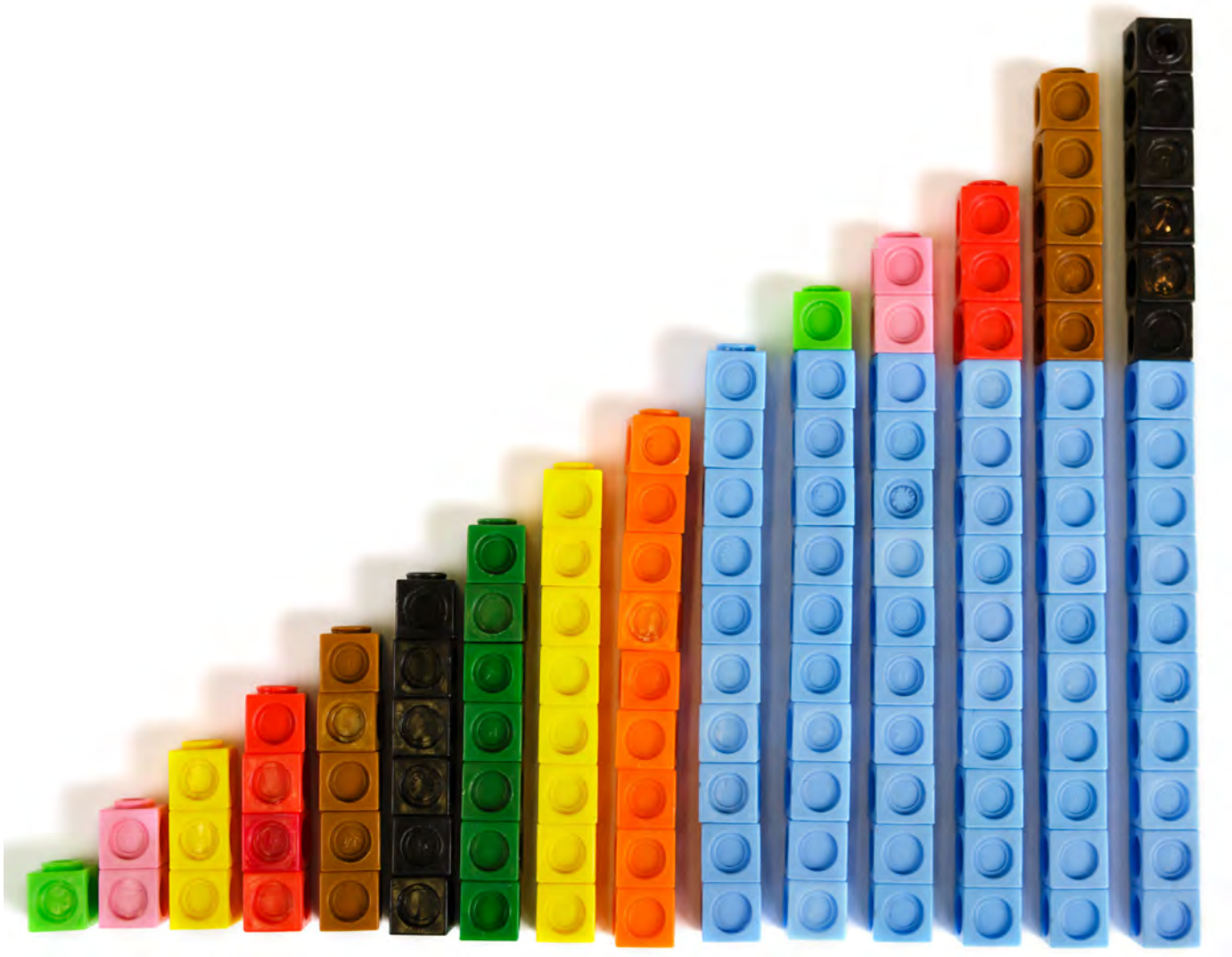
- **Counting:** knowing the number names in order, forwards and backwards; understanding how to count objects, events or actions in ones, and also in twos, fives, tens, tens and ones, and so on.
- **Comparing**: having a feel for the relative sizes of numbers; putting numbers in order; estimating.
- **Composition**: understanding how each number can be made up in different ways by addition, subtraction, multiplication and division; knowing how our number system uses groups of hundreds, tens and ones.

In the *Making Numbers* book, we have organized activities in chapters that cover particular ranges of numbers: 0 to 12; 9 to 20; 15 to 50; and 25 to 200 and beyond. The ranges overlap to emphasize the need for children to be completely confident with more difficult transitions – for example, many children find the string 9, 10, 11, 12 quite challenging, so they need time to revise and gain fluency with those numbers. In this report, though, we are using the ideas of counting, comparing and composition to structure our ideas.

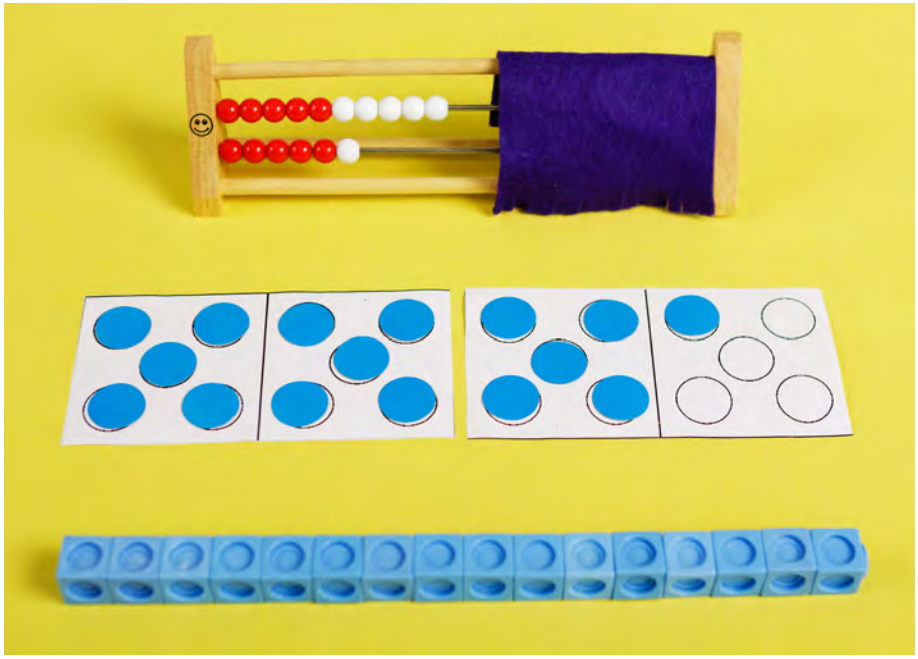
Counting

- Counting is more than just saying the number names in order. Children gradually learn that we count one number to each object, not missing any out or counting anything twice, and we can count things in any order and should still get the same total. Counting involves understanding how to count objects, events and actions in ones and also in twos, fives, tens and so on. Examples include counting 16 beads, 5 claps or 8 stairs; counting down as 5 buns are eaten; and counting money in 10p pieces.
- Children need plenty of opportunities to count collections of increasing magnitude. Using engaging items that children want to count is important. Activities based on counting collections of items are important in helping children develop number sense. They need to start with enumerating small numbers of items such as individual toys, and progress through counting increasingly large collections including a hundred and more.
- Counting is quite straightforward when the numbers of items involved are small but once we reach the teen numbers it is much harder to see the quantity in a collection, and organizing the items in groups becomes helpful. The key notion of counting in tens and ones underpins our number system. We examined approaches to organizing in groups and it seemed as though organizing in tens was presented in many cases as a fait accompli for which the purpose remained unclear to the children. It was clear that seeing a group of ten items as one 'ten', or *unitizing* can be very challenging.
- Initially, through focusing on numbers to 20, first experiences of numbers as 'a ten and some ones' can be consolidated to develop a sense of ten as a useful group with which to count. The teen numbers are tricky in their structure and word pattern but difficulties with them can be overcome through a range of strategies. Stern (1953) emphasised the value of activities that use counting to build up understanding of the structure of the number system. Building 'teens' staircases offers one such powerful model. Associating the numbers with both the word and symbol helps to support pattern spotting, including the repeating sequence and structure, and the increment of one.

The staircase here has one number missing and children can be challenged to find it:

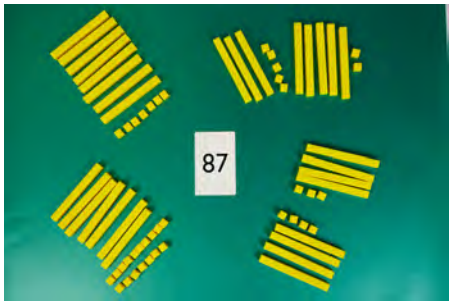


- There is much more to the business of counting than we might initially think and watching children use manipulatives to express mathematical ideas can offer insight into their understanding of the counting process. Encouraging children to represent the teen numbers with a variety of manipulatives can be helpful in supporting them to explore the structure of the numbers so that they see them as odd or even, involving groups of 5, or a ten and some ones. For example, children can 'Show me 16' with different manipulatives:



- It takes a while for children to appreciate the power of counting in tens and its connections with the structure of our number system. Five can also be a natural grouping for counting: five fingers on a hand is a group we see often and our ability to see 5 instantly without counting can help support counting in groups.
  Children enjoy counting in tens because they can make quick progress to large numbers. Exploring different ways to show 10 is a powerful tool to help children to recognize ten in different contexts:

- Making sets of tens and ones equipment for oneself is an important precursor to using commercially produced tens and ones equipment (Denvir and Brown 1986) and we suggest using a number of activities including having a race to count out a given number with a mixture of pre-grouped or individual items. For example, seeing how much quicker it is to collect a set of 52 items if some are pre-grouped in tens, than it is to do so with single items, can be a powerful experience in helping children to see the importance of using tens.
- This can be developed further by getting children to make their own bags or bundles of ten items. They can then take them apart and put them together again to examine the way in which larger numbers of items can be made. Making and breaking their own tens and hundreds will help children to understand the structure of commercially produced resources and to use them as 'tools' to work with rather than 'crutches' to support the following of an arithmetic procedure (Moscardini, 2009).
- Another powerful activity is taking a number and considering how it might be shown using similar equipment in different ways. See the example of 87 below:



As well as showing that 87 can be made from 8 tens and 7 ones, it could also be made from 7 tens and 17 ones. Counting in tens and more than ten ones is more difficult than you might think!
We can also consider 87 as composed of a range of other partitions such as 44 and 43 (near doubles) or 52 and 35 (groups of 5 tens or ones in each part) shown here. Looking at what is the same and what is different about these is useful.

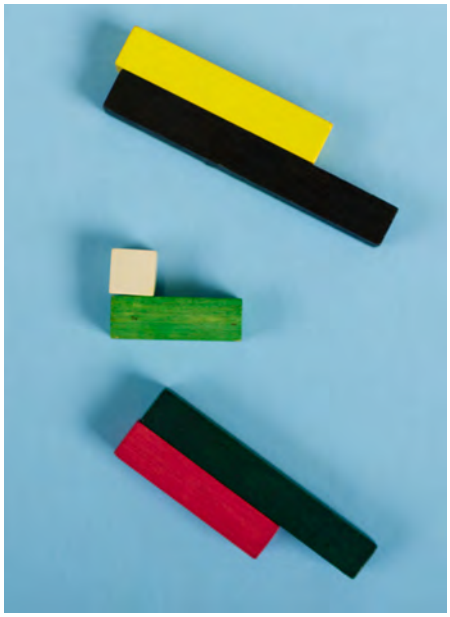
- Base 10 apparatus like that shown above is in common use in English classrooms. The manipulatives offer a vehicle through which children can articulate, formalise and symbolize their existing understandings. The objects bring their intuitive understandings into the world in a way that can be talked about, explored, exemplified, extended and represented. Whatever manipulatives are used, it is vital that children develop an understanding of how they represent the structures of the base 10 number system, before they are used to model arithmetic operations.

Comparing

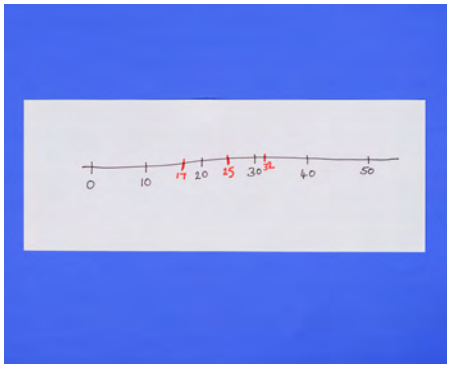
Comparing numbers involves having a feel for their relative size, being able to put numbers in order, and estimating. Comparing numbers can involve knowing that 6 is smaller than 8 but bigger than 2, being able to see a group of objects as 'about 20' and ordering the numbers from 35, 3, 127, 53 from largest to smallest. All three of these aspects of comparison are sometimes overlooked in English classrooms and we have sought to address this issue with some suggestions.

- Young children have an intuitive feel for amounts which are more or less but we need to build on this with them. First steps in comparison involve comparing two numbers. When the gap between two numbers is large comparison is easy, e.g. 3 and 15, but when it is small this is much more difficult. It is easier to see that 3 is smaller than 8 than to compare 6 and 8.
- Teachers of young children told us that they found it difficult to create opportunities for children to compare numbers so we developed some activities for them such as the 'Who has most?' game. In this, children throw a dice to see how many counters each toy animal is given and then compare the numbers in the dishes:

- As we have already pointed out, the teen numbers can be tricky for children and estimating how many there are in a child's handful of small items can be a good way to support this. Once children have estimated and counted various handfuls, they can then label them with the appropriate number and arrange them in order of size from smallest to largest or vice versa.
- Comparing quantities represented in different ways can offer children further insights into the number system and its structure. Goutard's work (1964) offers a number of such activities, such as finding pairs of Cuisenaire rods with the same difference. Here are some pairs of rods with a difference of 2: in each pair one rod is 2 smaller than the other. Children can be encouraged to consider what is the same and what is different.



- These ideas can be developed further once children have begun to develop understanding of numbers being represented on the number line. Working with manipulatives to show different numbers, labelling them with the written number and then locating them on the number line before listing them in order can be valuable in helping children to understand the abstract representation of number that the number line offers. The sequence of images below shows this clearly as well as exemplifying the processes of doing, talking and recording in a range of forms:



Composition

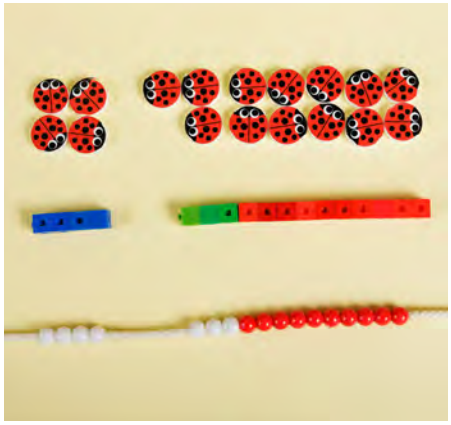
Composition of number involves the understanding that each number can be made up in many different ways by addition, subtraction, multiplication and division. It also involves understanding how our number system uses groups of hundreds, tens and ones. Examples might include recognizing that 6 can be made with a 4 and a 2, or is the same as 2 x 3 or 5 + 1 or 7 – 1; and understanding that 125 can be made up from one 100, two 10s and five 1s or five 25s or in many other ways.

By introducing these ideas with small numbers, children's confidence and creativity can be developed so that they are able to bring the deep understandings they have developed to numbers of increasing magnitude. Modeling the composition of numbers with manipulatives can help support this by offering children a range of possible ways in which to visualize numbers.

- With very small numbers we suggest using toys as manipulatives and telling stories about what is happening to them. For example in this image three of the toys have carrots and the children would be able to see clearly how many, in the set of five toys, do not.

This can eventually lead to relating the image to the equations: 3 + 2 = 5 or 5 – 2 = 3 or 2 = 5 – 3. Small sets can also be split into more than two parts: 8 can be seen as 2 and 3 and 3.

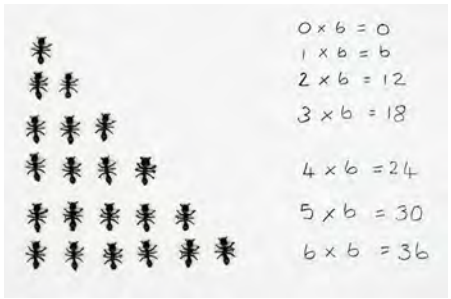
- Partitioning groups into two parts and representing them using different manipulatives is another way of helping children to visualize numerical relationships.



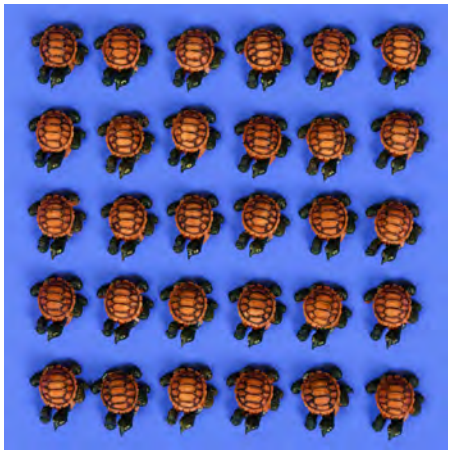
In the photograph above, 17 is partitioned into 4 and 13 and the three different representations have some similarities and some differences. Spotting them can help to emphasise different aspects such as groups of 5 and 10 or odd or even numbers.

- Calculators can help with this too and making the teen numbers by adding 1, 2, 3, 4… onto 10 reinforces the order of the digits in teen numbers and the repeated pattern in the digits of 1, 2, 3, 4 … as each successive number is added.
- Coloured rods such as Cuisenaire are useful manipulatives in that they represent whole numbers without their composition in ones being immediately apparent. This means that once children see the pink rod as being equivalent to 4 white rods, they may more readily work with 4 as a unit in itself without counting up to 4 every time. The coloured rods can offer a very useful way of visualising the partitioning of the numbers to 10 into two. See for example the image below showing how 9 can be split into 2 parts:

- Coloured rods can be particularly powerful in exploring the multiplicative relationships in numbers: a ten rod can be made with five 2 rods or two 5 rods but you cannot make it with 3 rods or 4 rods.
- Manipulatives can also help children to build up the idea of the multiplication tables: here is an 'Ant Times Table' which was created by counting the legs on toy ants:

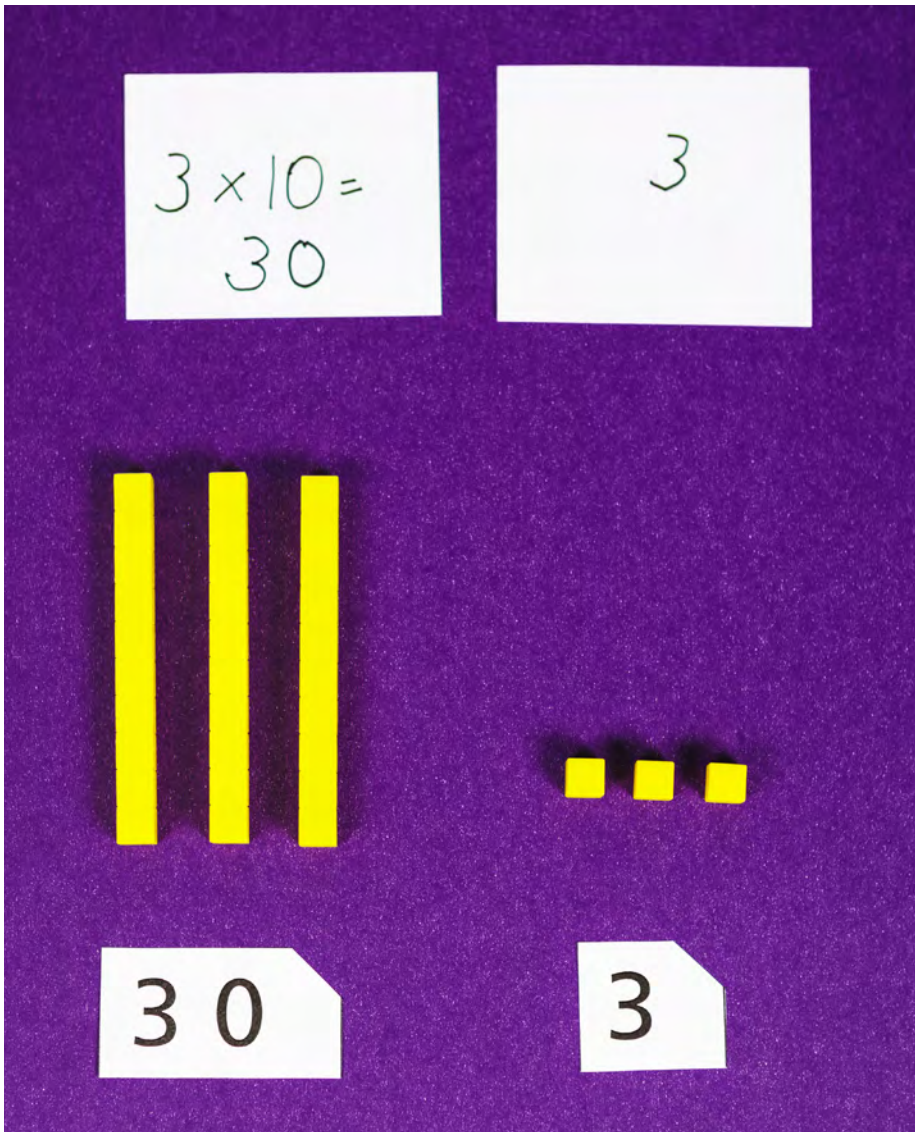


- Making arrays offers a powerful model for multiplication and manipulatives can help with this. Children have difficulties in recognizing and understanding arrays and need plenty of experience with constructing them in order to develop their awareness of their pattern and structure (Bobis, Mulligan and Lowrie, 2013). Here is an array of toy turtles:



This array shows 5 columns of 6 as well as 6 rows of 5 and offers an illustration for the commutativity of multiplication. If the children are able to make arrays on card so that they can turn them around, this awareness can be developed more easily.

- The composition of numbers in tens and ones (and eventually hundreds, tens and ones) is vital to the number system: offering children a range of experiences to explore this composition with manipulatives will help them to build the necessary concepts. It will also help them to develop the idea that each successive column in the number system is ten times as big as the one to its right. This idea can be built by swapping one unit for one ten as in the example below:



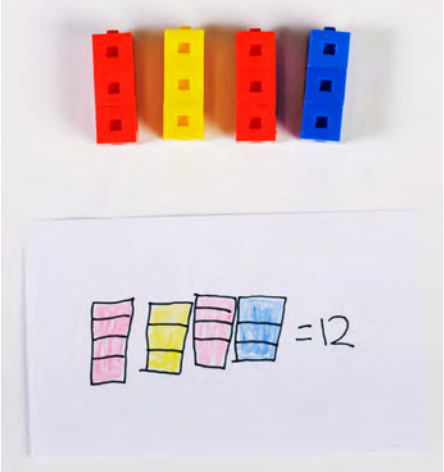
- Thompson (2000) discusses the different ways in which children think about numbers like 37 in terms of place value and introduces the ideas of quantity value (30 and 7) and column value (3 tens and 7 ones). Children will need a lot of practice with swapping tens for ones and vice versa before they are completely convinced that ten 1s is the same as one 10 and that ten 10s is the same as one 100. Multiplying single digit numbers by ten and then the teen numbers by ten using base 10 apparatus and making their own recording of what is happening can help with this.

Moving between concrete and symbolic representations

The literature review and our work with teachers identified the need for guidance in helping children to link the use of manipulatives with spoken, drawn or imagined representations which become increasingly abstract.

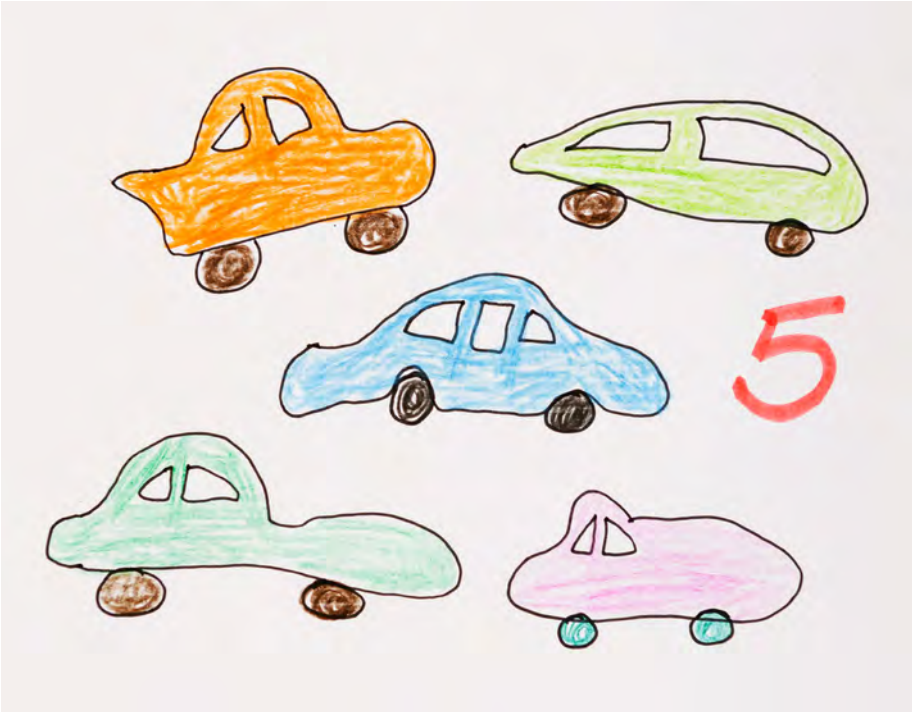
We have become increasingly aware of the sophisticated relationship between children doing activities with manipulatives, talking about what they are doing, recording what they have found out and explaining their practical work and their recording (and sometimes checking whether their recording says what they mean it to say).

Our research suggests that there is no single way of using manipulatives: there are a range of approaches that encourage children to express what they are thinking, to experiment and develop their ideas, and this is where teachers need guidance on strategies.



- Manipulatives offer children a vehicle through which to express their mathematical ideas because they can move things about as they explain what they are thinking.
- Moving backwards and forwards through Mason and Johnston-Wilder's (2006) three worlds of experience, between manipulable objects, mental imagery and drawing and abstract symbols helps children and adults to develop mathematical thinking and reasoning. The talk accompanying every move between these worlds offers another layer in the complex process that also allows the teacher to hear what the children are thinking.
- Children's recording may start with simple drawings. Here is an example of a child's representation of 5 cars; the teacher added the number 5 after the child had counted the cars out loud. The child may then go on to making recordings of groups of items with blobs rather than more accurate drawing. This iconic stage is important in moving towards the fluent use of symbols.
- Using manipulatives and drawing helps develop children's accuracy. In the example of recording shown above right, the teacher could see what the children had made as well as the image they had created of it:

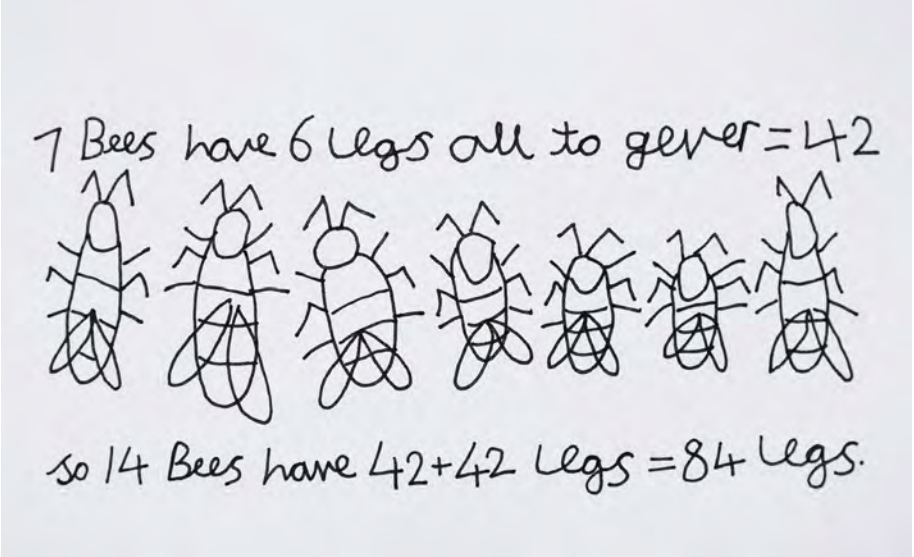
The teacher was able to find out whether the children had really meant to show 4 + 3 + 4 + 3 or whether this was meant to be four 3s. When asked to explain their drawing, they immediately saw the mistake they had made.

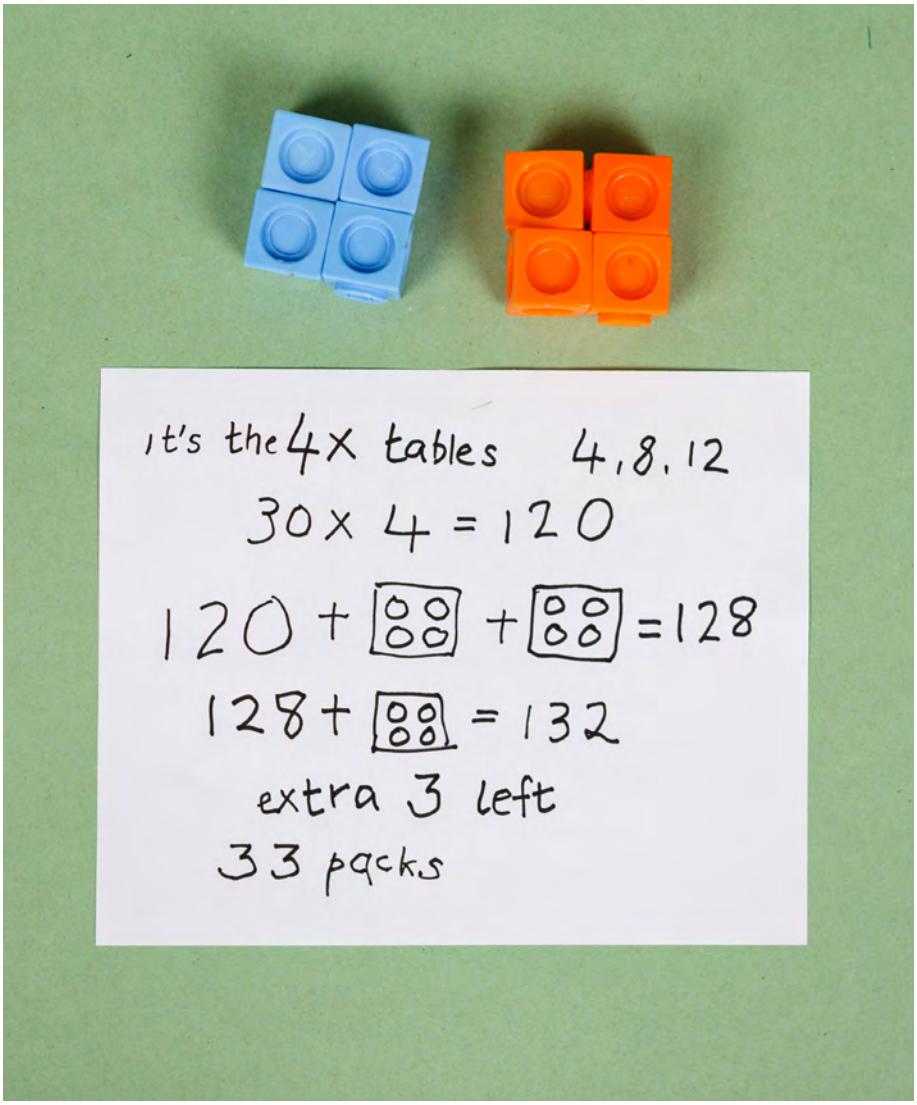


- Moving from concrete to abstract is not necessarily a one way process. A child working on creating times tables for 6, using plastic insects, referred backwards and forwards to the insects and produced the following piece of reasoning:
  The child had looked at the 7 plastic bees she had collected together and drawn carefully, worked out that there were 42 legs on 7 bees which meant there would be 84 legs on 14 bees. Good reasoning was firmly based on the manipulatives she had used.
- Children can combine the use of manipulatives, drawing and writing to explain their reasoning. For example, this child was showing how he calculated the number of packs of 4 yoghurts, needed for 129 children.
  He used his model cubes to work out how many yoghurts in 30 packs (120) and then added on two more packs. His recording was idiosyncratic but his reasoning was sound and his verbal explanation clear. This illustrates the power that the manipulatives gave him to solve the problem: he was able to indicate his model and explain his reasoning in a way that might have proved too abstract for him without the objects to refer to.

These examples suggest how teachers can foster mathematical reasoning and thinking, particularly emphasizing children's active ways of making sense of relationships through representations.

We are aiming to continue experimenting to help children gain a thorough understanding of the number system in a purposeful and interesting way. We hope our work so far will encourage you to try things out and to think about ideas you want to explore.





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