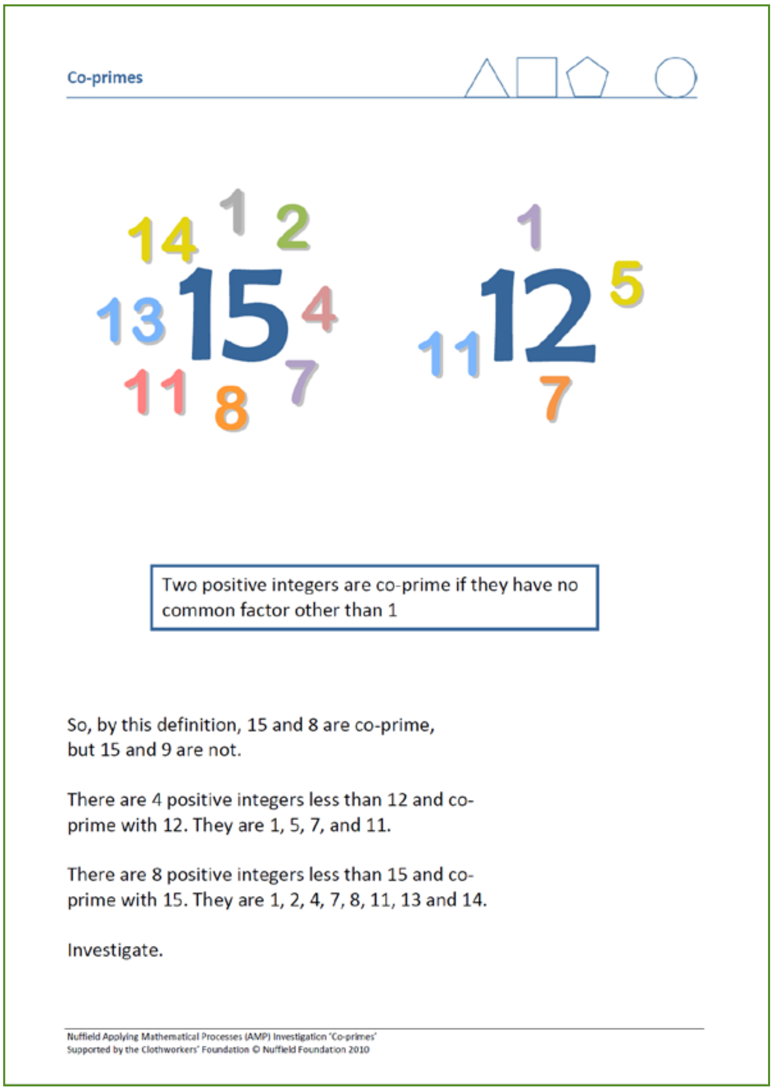
Co-Primes Teacher notes

Activity description

Starting with a definition of what it means for integers to be co-prime, pupils investigate how many positive integers are less than and co-prime to any given positive integer.



Suitability

Pupils working at all levels; individuals or pairs

Time

1 hour upwards

AMP resources

Slide and pupil stimulus sheet

Equipment

Calculator or computer
Computer spreadsheet

Key mathematical language

Factor, common factor, highest common factor
Prime, odd/even, integer, co-prime is also known as relatively prime

Key processes

Representing Choosing to move the situation from numerical examples to more abstract symbolism.

Analysing Working with a logical approach, generating accurate results.

Interpreting Identifying patterns and developing generalisations; considering the findings and relating answers to the original task.

Communicating and reflecting Providing evidence of the methods used, presenting the results obtained and the conclusions drawn from them.

Teacher guidance

This activity requires some confidence with the co-prime property, division and divisibility tests. It would also be useful for pupils to be aware of prime factorisation.

If pupils are not already confident with the idea of co-primality, this activity can be treated as having two stages. In the first part the definition of co-prime integers is introduced and illustrated. Pupils can be asked to pair numbers they consider to be co-prime, and asked to suggest numbers which are co-prime to a given number and explain why.

In the second part of the activity pupils investigate *how many* positive integers *less* than a given number are co-prime to that number. Within this activity, for the sake of brevity, we will refer to these numbers as "the Co-primes of …", so for example the Co-primes of 18 are 1,5,7,11,13,17 and the Co-primes of 10 are 1,3,7,9.

Please bear in mind that the use of "*the Co-primes of …*" to mean only those positive integers that are *less than and co-prime to* a given positive integer is not universal.

From the first part of this activity, pupils may be aware that there can be infinitely many integers that are co-prime to any given positive integer. You may use this to discuss why this activity focuses on the co-primes smaller than a given positive integer, and not all the co-primes of that number.

Emphasise the need to be careful in the use of the term 'number' when distinguishing between a chosen example and the resulting number of Co-primes.

Probing questions and feedback

AMP activities are well suited to formative assessment, enabling pupils to discuss their understanding and decide how to move forward. See www.nuffieldfoundation.org/whyAMP for related reading.

- How can you be sure you have found all the Co-primes of that number?
- How have you selected the numbers for which you are finding the Co-primes?
- What generalisations can you make from the numbers you have chosen – either in words or using symbols?
- Which numbers could you use to test your theories, and why?

Extensions

- Devise a way of finding the Co-primes of any number using a spreadsheet.
- Consider co-primes larger than a given positive integer. Explore relationships between these and co-primes smaller than the given number.

Additional Information

The following information can be used to check the accuracy of pupils' results and generalisations:

For any prime p

p has $p-1$ Co-primes

p^2 has $p(p-1)$ Co-primes

p^3 has $p^2(p-1)$ Co-primes

$\vdots$

p^k has $p^{k-1}(p-1)$ Co-primes

In particular, if n is a power of 2, there are $n/2$ Co-primes of n and if n is a power of 3, there are $\frac{2}{3}n$ Co-primes of n.

We'll use $p=3$ to illustrate the result for the general prime power. Consider all positive integers less than or equal to 3^k and note that only every third one (3, 6, 9, etc) is not co-prime to 3^k. Thus two-thirds of the positive integers less than or equal to 3^k will be Co-prime to 3^k: [1, 2, ~~3~~, 4, 5, ~~6~~, ... , p^k-2, p^k-1, $\cancel{p^k}$]

In general, $\frac{(p-1)}{p}p^k$ positive integers will be Co-prime to p^k.

Note that $\frac{(p-1)}{p}p^k$ 'simplifies' to $p^{k-1}(p-1)$.

By extending this idea, it is possible to deduce the number of Co-primes of a general positive integer from its prime factorisation.

For a number whose prime factorisation is $p_1^{a_1}p_2^{a_2}\cdots\ p_k^{a_k}$, the number of Co-primes is

$p_1^{a_1-1}p_2^{a_2-1}\cdots p_k^{a_k-1}(p_1-1)(p_2-1)\cdots(p_k-1)$.

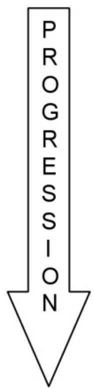
Progression table

The table below can be used for:

- sharing with pupils the aims of their work
- self- and peer-assessment
- helping pupils review their work and improve on it.

The table supports formative assessment but does not provide a procedure for summative assessment. It also does not address the rich overlap between the processes, nor the interplay of processes and activity-specific content. Please edit the table as necessary.

Representing *Choices about what to investigate and how to represent the information*	**Analysing** *Logic of approach, accuracy of results*	**Interpreting and evaluating** *Identification of patterns and generalisations*	**Communicating and reflecting** *Quality of the descriptions of both methods and outcomes*
Represents the activity with simple examples of pairs of numbers that are co-prime	Randomly finds further examples of numbers that are co-prime	Makes simple observations about findings **Pupil A**	Presents results so they can be understood by any reader **Pupil A**
Moves from considering pairs of co-prime numbers to considering the set of numbers co-prime to a given number **Pupils C, D**	Begins to develop a systematic approach to finding Co-primes of different numbers **Pupil C**	Identifies simple mathematical patterns relating to the original task **Pupils B, C**	Collates information in a suitable form and considers the results, e.g. simple example/s and statement **Pupils B, C**
Chooses to investigate a specific family of numbers, e.g. the Co-primes of even numbers **Pupil E**	Adopts a systematic approach, e.g. classifies results or organises results in a table **Pupil D**	Interprets results including those expressed in algebraic form and relates them back to the original task **Pupil E**	Presents results in organised form, expresses generalisation **Pupil E**
Considers prime factorisation **Pupil F**	Organises their investigation so that several different aspects can be explored efficiently Uses prime factorisation; uses algebraic symbols accurately	Identifies and considers the significance of a general rule generated from numerical results	Gives concise solutions using appropriate words, diagrams and symbols Describes different approaches to developing solutions **Pupil F**



Sample responses

Pupil A

Pupil A found pairs of co-prime integers and made simple observations which are clearly communicated.

If You have An EVEN NUMBER ALL THE CO-PRIMES are odd

With Any Number the number before is always a co-prime

Probing questions

- Instead of even numbers, what other types of numbers could you start with?
- Do you know a shorter way to write any number and the number before it?

Pupil B

'All Co-primes of *n* add up in pairs to *n*.'

The first and last numbers add up to the number given and the second and second from last numbers add up to the number given, and so on into the middle of the set of numbers

=12

$\boxed{12 \cdot 1, 5, 7, 11}$

=12

Pupil B has provided the evidence upon which he/she has identified a simple pattern and put it in context – the value of *n*. The pupil has observed a property worth investigating further, and this can be helpful for the original problem.

Probing questions

- How can you use your property in the task of finding the number of Co-primes for a chosen integer?
- How can we determine whether the property you have found will be true for all positive integers?

Pupil C

Pupil C has started to consider how many Co-primes a prime number has.

The pupil worked systematically and has observed a relationship. This is communicated clearly and illustrated by a sequence of examples.

Primes numbers always have every number below them as Co-Primes. For example:
19 has 18 Co-primes.
17 has 16 Co-Primes.
13 has 12 Co-Primes.
11 has 10 Co-Primes.
notice this is Always one Less than the actual number.

Probing questions

- Why do prime numbers have this number of Co-primes?
- Do any other numbers have the property you have observed?
- How could you use this property if you extended your investigation?

Pupil D

Pupil D has been systematic, shown the methodology used, and presented the findings in an organised way.

'All numbers have an even number of Co-primes.'

less than + coprime with	no. of nos	nos themselves
10	4	1, 3, 7, 9
* 11	10	1, 2, 3, 4, 5, 6, 7, 8, 9, 10
12	4	1, 5, 7, 11
* 13	12	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
14	6	1, 3, 5, 9, 11, 13
- 15	8	1, 2, 4, 7, 8, 11, 13, 14
- 16	8	1, 3, 5, 7, 9, 11, 13, 15
* 17	16	14, 15, 16 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13,
18	6	1, 5, 7, 11, 13, 17
* 19	18	16, 17, 18 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15
- 20	8	1, 3, 7, 9, 11, 13, 17, 19

Another striking thing about the numbers in the second column of the table is that they are all even

Probing questions

- Do you know why your numbers have this number of Co-primes?
- Do any other numbers have six Co-primes?
- Is there a rule for the number of Co-primes for any particular number or group of numbers?

Pupil E

Pupil E has chosen to investigate cubes. After testing a conjecture and finding it to be false, a new observation about some numerical results is expressed algebraically.

'For p^3, where p is prime, the number of Co-primes is $(p-1)p^2$.'

Is this true for cube numbers?

8. Factors of 8 = 2, 4.

Co-primes = 1, 3, 5, 7.

$8 - 2 = 6 \neq$ no. of co-primes.

27. Factors of 27 = 9, 3,

Co-primes = 1, 2, 4, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 20, 22, 23, 25, 26. There are 18.

$27 - 3 = 24 \neq$ no. of co-prime

However it is noticed that

$n - n^{2/3} =$ no. of co-primes

e.g. $8 - 4 = 4$ ✓

$27 - 9 = 18$ ✓

(where the cube number has no other factors)

Probing questions

- You have spotted a rule for cubes of primes. Can you explain why you think this rule is correct?
- Can you predict how many Co-primes 63 has?
- Is there a general rule for the number of Co-primes for nm?

Pupil F

Pupil F has considered the relationship between the prime factorisation of a number and how many Co-primes it has.

Maximum no. of co-primes of 200 $= \frac{n}{2}$

$= 100$

5	100
5	20
2	4
2	2
	1

prime factors of 100 $= 5^2 \times 2^2$

∴ prime factors of 200 $= 5^2 \times 2^3$

Therefore all the multiples of 5 and 2 cannot be co-primes

There will be $200 \div 5 + \frac{200}{2}$ numbers like this

every second multiple will be shared by 2. → 2, 2 way

eliminated by this = 120 numbers eliminated,

~~please all the prime numbers then~~

∴ no. of co-primes of 200 = 80

Probing questions

- Would your method allow you to find the number of Co-primes of any number?
- Is there a general rule for how many Co-primes a number has?