Activity description

Pupils choose two numbers X and Y and find their mean. Then they replace X with Y and replace Y with the mean. They find the mean of these two numbers and then, using an iterative process, investigate what happens.

It is intended that this investigation be done using a calculator or computer.

Suitability

Pupils working at all levels; individuals or pairs

Time

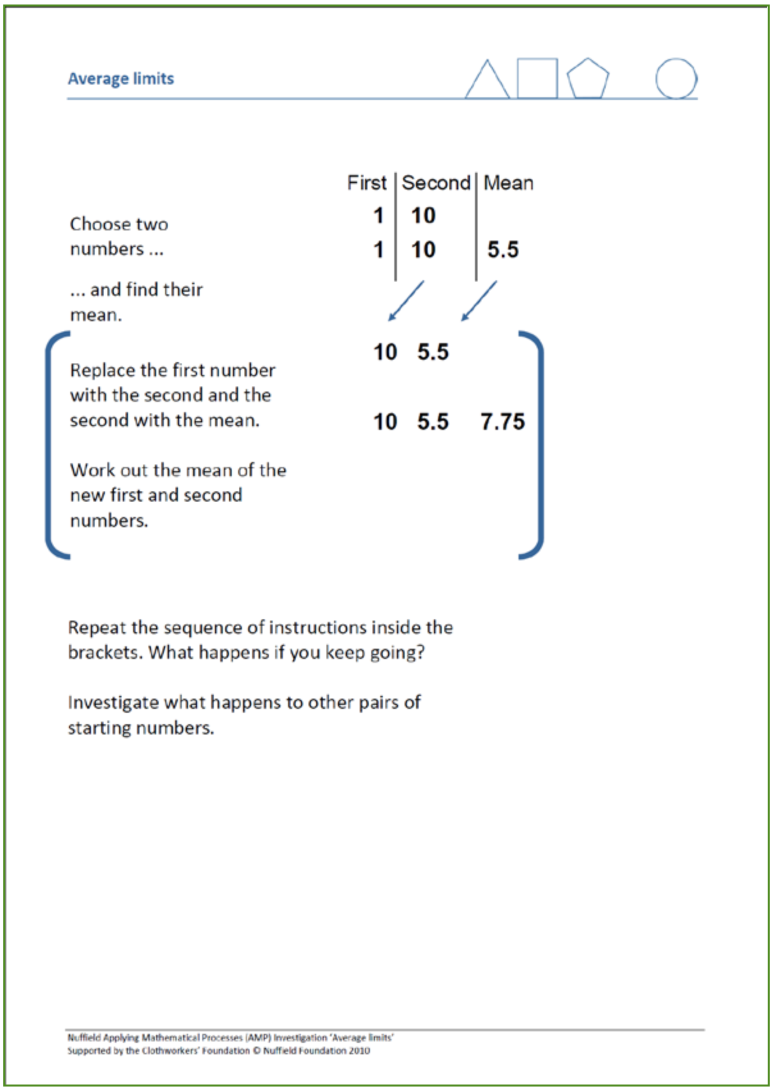
1 hour upwards

AMP resources

Pupil stimulus, spreadsheet

Equipment

Calculator or computer
Spreadsheet for introduction and/or to work through examples as required



Average limits

	First	Second	Mean
Choose two numbers ...	1	10	
... and find their mean.	1	10	5.5
Replace the first number with the second and the second with the mean.	10	5.5	
Work out the mean of the new first and second numbers.	10	5.5	7.75

Repeat the sequence of instructions inside the brackets. What happens if you keep going?

Investigate what happens to other pairs of starting numbers.

Nuffield Applying Mathematical Processes (AMP) Investigation 'Average limits'
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Key mathematical language

Mean, iterate, converge, limiting value, conjecture, decimal places, rounding, variables, algebraic expression

Key processes

Representing Moving from generation of a few results to systematic exploration of the relationship between starting numbers and limit.

Analysing Working systematically, controlling variables to identify patterns leading to generalisations.

Interpreting and evaluating Exploring and verifying generalisations about the limit with a view to expressing patterns algebraically; justifying results and generalisations.

Communicating and reflecting Explaining the approach taken and the outcomes; describing conclusions.

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Teacher guidance

It is intended that this task is carried out using a spreadsheet, but for the first explanatory example pupils should work out the means, using a calculator if needed. You can use the '**columns sheet**' on the provided spreadsheet to introduce the activity. Ask pupils for two numbers and their mean, and input these into the spreadsheet. The mean can be checked using the spreadsheet.

Ask pupils what the next two numbers will be and their mean. Emphasise which numbers are being replaced each time. Continue to model the process for the first 4 or 5 rows, waiting for pupils to find the mean before using the spreadsheet to confirm each value. Compare and discuss any discrepancy between the results from their calculators and those given by the spreadsheet.

Pupils should continue with their chosen example until they notice something happening with successive results. They can then explore the limiting process with different pairs of starting numbers.

You may wish to model the process with more than one pair of starting numbers.

The '**rows sheet**' of the spreadsheet can be used to input results from several pairs of starting numbers for analysis.

During the activity

Ensure that pupils can use a computer spreadsheet, a programmable calculator or computer, or a calculator capable of recursive computation. If possible, encourage pupils to program their own formulae or macros to generate results.

Ensure that pupils do not give up too soon and that they continue with a pair of numbers until iterates are almost identical. If inputting numbers by hand, discourage pupils from rounding their results to too few decimal places.

Allow pupils to choose their own starting pairs of values. As the work progresses, emphasise that the purpose is to find a link between the two starting numbers and the limiting mean.

For those considering working on the problem using variables, encourage them to do so and to explore any patterns that emerge.

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Probing questions and feedback

AMP activities are well suited to formative assessment, enabling pupils to discuss their understanding and decide how to move forward. See www.nuffieldfoundation.org/whyAMP for related reading.

- If you continue generating results for a pair of numbers, what do you think will happen?
- Will the average limit be the same if the starting numbers are used in reverse order?
- How could you investigate how each of your starting numbers contributes to the average limit?
- Can you express any relationships you have found using algebra?
- How could you tell when you have found the average limit?
- If you have a conjecture for a general rule, how can you justify your belief that it will work in all cases?
- Are there any pairs of numbers for which the process does not converge to a limit? How can you justify your answer?
- How many pairs of starting numbers will give an average limit of 15?

Extensions

- Consider the problem for a string of three numbers (replacing the first number by the second, the second by the third, and the third by the mean of the three numbers, and then iterating the process). Consider longer strings.
- Use the geometric mean instead of the arithmetic mean.

  Choose an integer. Subtract half the number then add 1. Repeat with your result, and continue in the same way. What happens? Can you show why?
- Suppose that instead subtracting half, you subtracted one third, or one quarter, or … of your number, then added 1. What would happen then?

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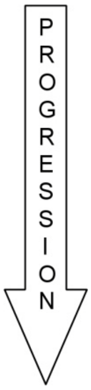
Progression table

The table below can be used for:

- sharing with pupils the aims of their work
- self- and peer-assessment
- helping pupils review their work and improve on it.

The table supports formative assessment but does not provide a procedure for summative assessment. It also does not address the rich overlap between the processes, nor the interplay of processes and activity-specific content. Please edit the table as necessary.

Representing *Uses a suitable mathematical approach*	**Analysing** *Systematic generation of results; accuracy of results*	**Interpreting and Evaluating** *Identifying patterns and generalising*	**Communicating and reflecting** *Quality of discussion and elegance of final solution*
Shows understanding of the task by producing some correct results Pupil A	Finds correct results for one pair of numbers for a few iterations Pupil A	Makes a simple observation on the limiting value Pupil B	Results shown clearly Pupil A
Uses efficient methods to generate results leading to limiting value Pupil B	Applies the rule until limiting value is found Pupil B	Makes valid observations, relating starting numbers to limiting value Pupils C, D, E	Shares sufficient results to show a limiting value Pupils B, C, D
Chooses to keep one variable constant and systematically changes the other variable Pupils C, D	Generates results with one variable constant while systematically varying the other Pupils C, D	Verifies or has supporting evidence for accurate relationships between limiting value and starting numbers Pupil F	Describes clearly and succinctly the systematic approach taken and reflects on how conclusions were arrived at
Chooses algebraic or function-based approach to represent the relationship between starting numbers and the limiting value Pupils E, F	Explores general relationships, algebraically or graphically Pupil F	Justifies functional relation(s) found	Communicates accurate functional relation(s) found, algebraically or graphically Pupil F



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Sample responses

Pupil A

Starting with 1 and 10, Pupil A has applied the procedure several times and listed the averages produced.

Further results are recorded in a table.

No observations are made about the numbers produced.

5.5
7.75
6.625
7.1815
6.90625
7.048875
6.976563
7.011719
6.994141
7.00293.

7.011719	6.994141	7.00293
6.994141	7.00293	6.998535
7.00293	6.99855	7.0007325
6.99855	7.00073275	6.99964125
7.00073275	6.99964125	7.000186913
6.99964125	7.000186913	6.999914081
7.000186913	6.999914081	7.000050497
6.999914081	7.000050497	6.999982289
7.000050497	6.999982289	
6.999982289		

Probing questions

- What do you notice about the numbers in the third column?
- Do you need to keep repeating the process to know what will happen?

Pupil B

Pupil B has generated results for two chosen numbers and makes the observation that the sum of the final two numbers is 14, and that the average of the numbers has arrived at a limit of 7.

There is no attempt to gather results from other starting numbers, and therefore insufficient evidence to make a general statement.

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A1 | fx

	A	B	C	D	E	F
3	Step	First	Second	Average		
4	1	1.0000000	10.0000000	5.5000000		
5	2	10.0000000	5.5000000	7.7500000		
6	3	5.5000000	7.7500000	6.6250000		
7	4	7.7500000	6.6250000	7.1875000		
8	5	6.6250000	7.1875000	6.9062500		
9	6	7.1875000	6.9062500	7.0468750		
10	7	6.9062500	7.0468750	6.9765625		
11	8	7.0468750	6.9765625	7.0117188		
12	9	6.9765625	7.0117188	6.9941406		
13	10	7.0117188	6.9941406	7.0029297		
14	11	6.9941406	7.0029297	6.9985352		
15	12	7.0029297	6.9985352	7.0007324		
16	13	6.9985352	7.0007324	6.9996338		
17	14	7.0007324	6.9996338	7.0001831		
18	15	6.9996338	7.0001831	6.9999084		
19	16	7.0001831	6.9999084	7.0000458		
20	17	6.9999084	7.0000458	6.9999771		
21	18	7.0000458	6.9999771	7.0000114		
22	19	6.9999771	7.0000114	6.9999943		
23	20	7.0000114	6.9999943	7.0000029		
24	21	6.9999943	7.0000029	6.9999986		
25	22	7.0000029	6.9999986	7.0000007		
26	23	6.9999986	7.0000007	6.9999996		
27	24	7.0000007	6.9999996	7.0000002		
28	25	6.9999996	7.0000002	6.9999999		
29	26	7.0000002	6.9999999	7.0000000		
30						
31	This shows that the average eventually comes to 7 when the					
32	sum of the First and Second is 14 at the end.					
33						

Probing questions

- If you were to choose different starting numbers, what would happen to the final limit?
- If you reversed the order of your starting numbers would the final limit be the same?

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Pupil C

1st	2nd	AVERAGE
1	11	8.66667
1	10	7
1	9	6.3r
1	8	5.66667
1	7	5
1	6	4.3r
1	5	3.66667
1	4	3

1	11	7.666667
1	12	8.33333
1	13	9
1	14	9.666667
1	15	10.33333
1	16	11
1	17	11.666667
1	18	12.33333
1	19	13
1	20	13.666667

Here is what I noticed:

1. When the 1st number was 1, and the second number was any number more than one, the averages went up in $\frac{2}{3}$.

Pupil C has fixed the first number as 1 and repeatedly increased the second number by 1, making the valid observation that the average increased by two-thirds each time. There is no evidence of work with any other starting number. The observation refers to the results in the accompanying table, but is not sufficiently clearly expressed to stand alone.

Probing questions

- What happens if the first number isn't 1?

  If you kept the second number constant, and increased the first number by 1 each time, how much would the result change by?

- For any two starting numbers, could you predict what the average limit would be?

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Pupil D

When the average ended in a .33334 = $\frac{1}{3}$.
They all had this pattern.

.66667 = $\frac{2}{3}$.
.33333 = $\frac{1}{3}$.
.99999 = (whole no.)

1st Number	2nd Number	Average	
1	1.	1.	1
1	2.	1.66667	$1\frac{2}{3}$
1	3.	2.33333	$2\frac{1}{3}$
1	4.	3	3
1	5.	3.66667	$3\frac{2}{3}$
1	6.	4.33334	$4\frac{1}{3}$
1	7.	5.	5
1	8	5.66666	$5\frac{2}{3}$
1	9.	6.33333	$6\frac{1}{3}$
1	10.	6.99999	7
1	11.	7.66666	$7\frac{2}{3}$
1	12.	8.33334.	$8\frac{1}{3}$
1	13.	9.	8
1	14.	9.66667.	$8\frac{2}{3}$
1	15.	10.3333.	$9\frac{1}{3}$
1	16.	11.	10
1	17.	11.6667.	$10\frac{2}{3}$
1	18.	12.3333.	$11\frac{1}{3}$
1	19.	13.	12
1	20.	13.6667.	$12\frac{2}{3}$
2	1.	1.66667.	$1\frac{2}{3}$
2	2.	2,	2
2	3.	2.66667.	$2\frac{2}{3}$
2	4.	3.33333.	$3\frac{1}{3}$
2	5.	4.	4
2	6.	4.66667.	$4\frac{2}{3}$
2	7.	5.33334.	$5\frac{1}{3}$
2	8.	6.	6
2	9.	6.66666.	$6\frac{2}{3}$
2	10.	7.33333.	$7\frac{1}{3}$
2	11	7.99999.	8
2	12.	8.66666.	$8\frac{2}{3}$
2	13.	9.33334.	$9\frac{1}{3}$
2	14.	10.	10
2	15.	10.6667.	$10\frac{2}{3}$
2	16.	11.3333.	$11\frac{1}{3}$
2	17.	12.	12
2	18.	12.6667.	$12\frac{2}{3}$
2	19.	13.3333.	$13\frac{1}{3}$
2	20.	14.	14.

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Pupil D *(see previous page)*

Pupil D has held the first variable constant, while systematically varying the second. An observation on successive averages ending in 2/3, 1/3, and 1 is noted, but no further observations are made.

The fraction equivalents have a transcription error on the first sequence, starting when the average is 9. An incorrect assumption is made at the beginning of the second sequence, that (2, 1) will give the same result as (1, 2).

Probing questions

- What was the effect of changing the first number from 1 to 2?
- What would happen if you fixed the second number and let the first number change?
- How did you calculate the limiting value for (2, 1)? Do you get the same result no matter which order the starting numbers are in?

Pupil E

The difference between A and B multiplied by two thirds and taken away from the larger number

or in algebra

Adverage Adverage $= A - \frac{2}{3}(A-B)$

Pupil E has established a general relationship between the results and the values of A and B, expressed using algebra. A and B are not defined as the first or second numbers, and the pupil has not justified the algebraic result.

Probing questions

- You say that two-thirds of the difference is subtracted from the larger number. Does it matter whether the larger number is the first or second number?
- Does your result work for any values of A and B, such as fractions or negative numbers?
- Why is this the formula for the average limit?

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Pupil F

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D27 | f_x =(B27+C27)/2

Average Limit Calculator

1st Number	2nd Number	Mean	
1	10	5.5	Mean
10	5.5	7.75	Row 2
5.5	7.75	6.625	Row 3
7.75	6.625	7.1875	Row 4
6.625	7.1875	6.90625	Row 5
7.18750	6.90625	7.04688	Row 6
6.90625	7.04688	6.97656	Row 7
7.04688	6.97656	7.01172	Row 8
6.97656	7.01172	6.99414	All others
7.01172	6.99414	7.00293	Clear
6.99414	7.00293	6.99854	
7.00293	6.99854	7.00073	
6.99854	7.00073	6.99963	
7.00073	6.99963	7.00018	
6.99963	7.00018	6.99991	
7.00018	6.99991	7.00005	
6.99991	7.00005	6.99998	
7.00005	6.99998	7.00001	
6.99998	7.00001	6.99999	
7.00001	6.99999	7.00000	
6.99999	7.00000	7.00000	
7.00000	7.00000	7.00000	

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A31 | f_x

	A	B	C	D	E	F	G	H
1	First	Second	Limit		First	Second	Limit	
2	1	10	7		4	1	2	
3	1	9	6.33		4	2	2.67	
4	1	8	5.67		4	3	3.33	
5	1	7	5		4	4	4	
6	1	6	4.33		4	5	4.67	
7	1	5	3.67		4	6	5.33	
8	1	4	3		4	7	6	
9	1	3	2.33		4	8	6.67	
10	1	2	1.67		4	9	7.33	
11	1	1	1		4	10	8	
12								
13	1	11	7.67					
14	1	12	8.33					
15								
16	For 1 as 1st no.			First (m)	Second (n	Limit		
17	Limit = (n - 1) x 0.67 + 1			5	10	8.33		
18				6	10	8.67		
19	For 2 as 1st no.			7	10	9		
20	Limit = (n - 2) x 0.67 + 2			8	10	9.33		
21				9	10	9.67		
22	For 3 as 1st no.			10	10	10		
23	Limit = (n - 3) x 0.67 + 3			11	10	10.33		
24								
25								
26	Limit = (n-m) x 0.67 + m							
27								
28								

Pupil F has investigated the problem by systematically varying the first and second number in turn. Patterns have been spotted and interpreted algebraically, first for one variable and then for two, although the pupil has not attempted any further justification of the algebraic results.

Probing questions

- Have you tested your conjecture for other numbers?
- Describe how you arrived at your formula.
- How would you explain or prove why this is the formula for the average limit?