Nuffield Foundation

8

Key understandings in mathematics learning

Paper 8: Methodological appendix

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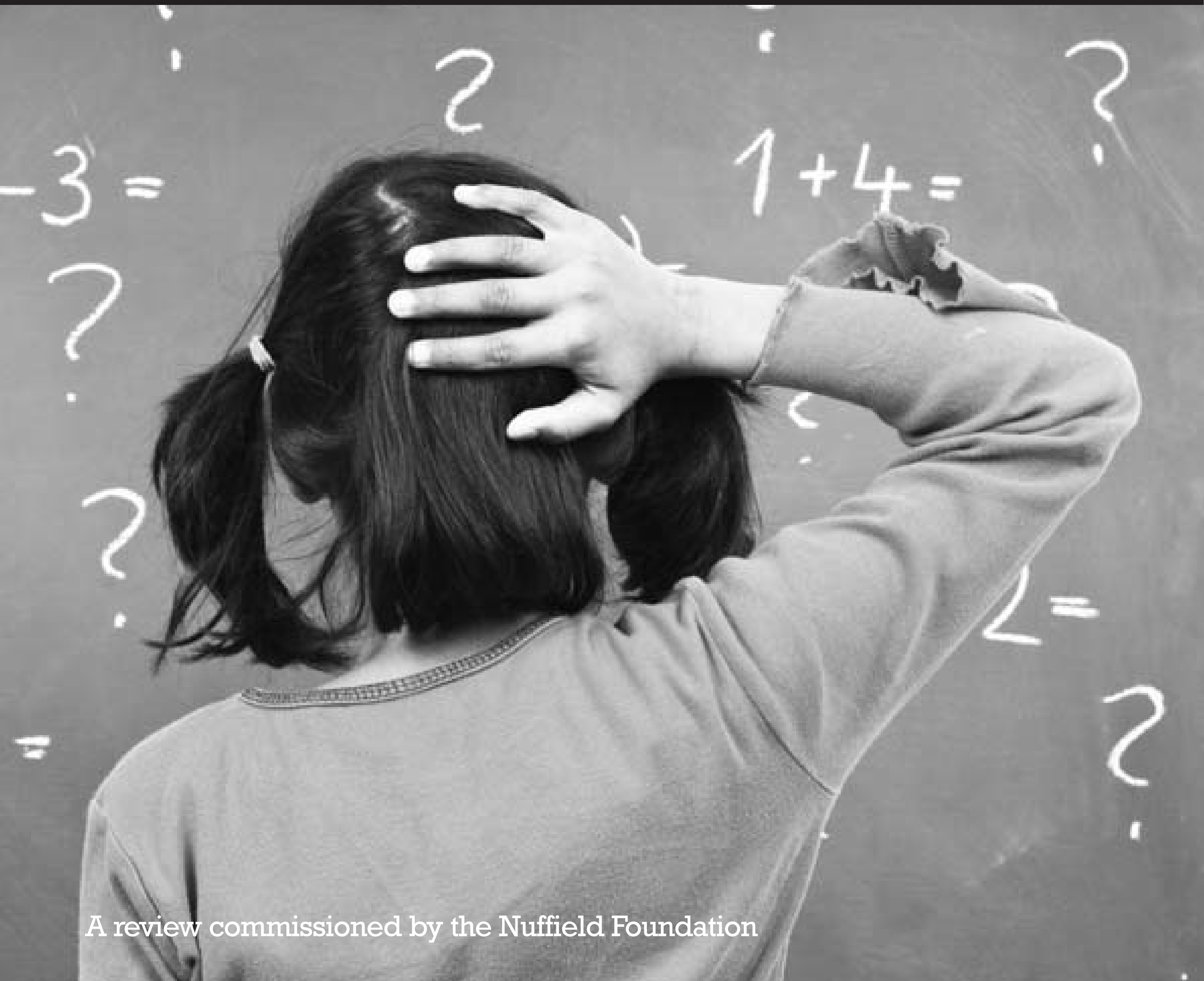

A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

Contents

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Appendix to papers 1 to 7

This review was conceived as standing between a research synthesis and a theoretical review. 'Research syntheses focus on empirical studies and seek to summarize past research by drawing overall conclusions from many separate investigations that address related or identical hypotheses. The research synthesis hopes to present the state of knowledge concerning the relation(s) of interest and to highlight important issues that research has left unresolved' (Cooper, 1998, p. 3). In a theoretical review, the aim is to present theories offered to explain a particular phenomenon and to compare them in breadth, internal consistency, and the empirical support that they find in empirical studies. 'Theoretical reviews will typically contain descriptions of critical experiments already conducted or suggested, assessments of which theory is most powerful and consistent with known relations, and sometimes reformulations or interactions or both of abstract notions from different theories.' (Cooper, 1998, p. 4).

It was quite clear to us that a review that aims to answer the question 'how children learn mathematics, ages 5 to 16' could not be treated as a straightforward research synthesis. The aim of a research synthesis is usually more restricted than this. For example, a research synthesis in education might try to examine the effect of one variable on another (e.g. the effect of reading aloud on children's literacy learning; Blok, 1999; Bus, van IJzendoorn, and Pellegrini, 1995) or the conditions under which a particular educational practice can be said to work (e.g. the effect of phonological or morphological instruction on literacy learning; Bus, and van IJzendoorn, (1999); Ehri, Nunes, Stahl, and Willows, 2001; Reed, 2008). Such searches start from previously defined variables, the incorporation of which in a study can easily be identified in a search through the literature. A review of the literature that starts with a much broader question cannot use the same conception of how the literature search will be carried out. The variables to be analysed are not conceived from the start and one of the aims of addressing such a broad question is in fact to clarify how mathematics learning could be conceptualised.

Theoretical syntheses have broader aims, which are in some ways similar to the aims adopted in this synthesis, but the current conception of theoretical syntheses can only be partially adopted in this review. Although there are occasionally alternative views of how a particular aspect of children's mathematics learning can be explained, the notion of critical experiments to assess which theory is more powerful cannot easily be met when we try to understand how children learn mathematics. The very conception of what it is that one is trying to explain varies even when the same words are used to describe the focus of the research. In the second paper in this review, we try to show exactly this. There are two alternative theories about children's understanding of number in developmental psychology but the phenomenon that they are trying to explain is not the same: Piaget's theory focuses on children's understanding of relations between quantities and Gelman's theory on children's counting skills. For older children, the problem becomes even more complex because there are alternative views of the nature and content of mathematical learning, and the role of pedagogy makes the notion of critical experiment either impossible or inapplicable. This is true of all research into secondary mathematics and reflects a change from seeing mathematics as the formalisation and extension of children's quantitative and spatial development to seeing learning mathematics as coming to understand abstract tools which can provide new formal and analytical perspectives on the world.

We did not approach this synthesis as a systematic review but as an attempt to summarise and develop some of the main ideas that are part of research and theory about how children learn mathematics. Within this perspective, we defined some inclusion and exclusion criteria from the outset.

Inclusion criteria

1 Theoretical explanations regarding how children learn mathematics which have been supported by research. There are theoretical explanations in the domain of mathematics learning which were proposed without their authors providing systematic empirical evidence. We did not consider these latter theories in the review except as frameworks to structure the approach in the absence of other explanations.

2 Research about children's mathematics learning in the age range 5 to 11 was considered when it focused on the four domains defined as the focus of this research: children's understanding of natural and rational numbers, relations between quantities and functions, and space and its representation. These were considered the cornerstones for further mathematics learning in the domains of algebra, modelling and applications to higher mathematical concepts; the focus of these two papers was on students aged 12 to 16. For algebra the available research on learning focuses on identifying typical errors, hence showing critical aspects of successful learning but not how that learning might take place. Further than this we looked at teaching experiments showing how students respond to different pedagogical approaches designed to overcome these typical difficulties. For modelling we intended to follow a similar approach but little was available except small-scale teaching experiments.

3 Research published in books and book chapters, journals and refereed conference proceedings which aim at understanding how children learn mathematics. Considering the constraints of time, the search in journals was limited to those available electronically and otherwise in the University of Oxford. A list of journals and their aims and scope is appended. The refereed conference proceedings of the International Group for the Study of the Psychology of Mathematics Education will be the only proceedings included in the review.

Exclusion criteria

1 There are domains of research, such as history of mathematics, mathematics teacher development, neuropsychological studies of adults with brain damage who have developed mathematics difficulties, and studies of mathematical abilities in animals and infants, which have not been so far connected to a theory of how children learn mathematics between 5 and 16 years. These domains of research are excluded.

2 Research that focused on learning how to use specific technologies rather than on how technologies are used by students to learn mathematics. There is a relatively large number of publications on how students learn to use particular tools that are relevant to mathematics (e.g. calculators, number line, spreadsheets, LOGO and Cabri). Considering our aim of understanding how children learn mathematics, we will only refer to research that uses these tools when the focus is on mathematics learning (e.g. using spreadsheets to help students understand the concept of variable).

We did not use methodological criteria in the choice of papers. Descriptive as well as experimental research, qualitative or quantitative studies were considered when we went through the search. In view of the brevity of the period dedicated to this synthesis, we did exclude materials that could neither be obtained by electronic means or in the libraries of the University of Oxford. There is, therefore, a bias towards papers published in English language journals, even though we could have read publications in three other languages.

The search process was systematic. We used the British Educational Index as a starting point for the search of papers in the four chapters about children in the age range 5 to 11. Three searches were carried out, one for natural and rational numbers, one for geometry and one for understanding relations and functions. We included in these searches three sets of key-words, the first defining the domain of research (mathematics education and other key words from the thesaurus), the second defining the topic area (e.g. natural number, rational number and other options from the thesaurus), and the third defining the age parameters (through schooling levels). Theses and one-page abstracts were excluded from the output list of references at this point. The references were then checked for availability and to see whether they reported research results and excluded if they were not available or did not report any research results. We repeated this search process using Psych-info, a data base which includes psychological research, which had been poorly represented in the previous data base. Finally, this initial search was complemented by a journal by journal search of the titles listed at the end of this note. This search seemed to yield mostly repeated references so we considered this the end of the process of search. We also consulted books and book chapters of works that are recognised in the literature and previous syntheses presented in the Handbook of Research on Mathematics Teaching and Learning. Two the Task Group Reports of the National Mathematics Advisory Panel, USA, were also consulted: the reports on learning processes and on conceptual knowledge. These were used as sources of references rather than for their conclusions. In the end, approximately 200 papers were downloaded and read by the authors. However, not all of these papers are cited in the chapters. The references used are those which did contribute to the development of the concepts and empirical results used in the synthesis.

For algebra, we conducted a systematic search in electronic journals in English for refereed research articles using algebra as the keyword. Journals are listed below. We did not define an age range since we were interested in how algebraic understanding develops throughout school, although this happens mainly in secondary education. We also used refereed innovation studies, which show what it is possible for learners to do, given particular kinds of teaching or technology; this tells us about possibilities. We restricted our use of these to studies for which the learning aims clearly relate to a broad view of algebra given above. For example, we did not include self-referential studies in which, for example, it is assumed that pattern-spotting is an important aspect of algebra, so teaching and learning pattern-spotting is researched, but we would for example include a study of teaching pattern-spotting where students' ability to use pattern-spotting for a higher level algebraic purpose was discussed as an outcome. We also used refereed studies of students' typical errors and methods (see below). These tell us what needs to be learnt and hence describe the development of algebraic understanding, but not how successful students learn it. We also drew on significant overviews and compilations of research on algebra. These reviews were used as gateways to other research literature. We excluded studies which focus only on short-term fluent performance of algebraic procedures in familiar situations unless this was linked specifically to the development of algebraic reasoning. Most of the studies we used base their claims to success on the complementary needs both to act fluently with symbolic expressions and to understand them. We accessed 174 papers plus 78 references in books in addition to the reviews and studies mentioned above. Of these, about 95 were read but not all are included as reference. Some of these overlapped in their conclusions, or added nothing or only a little to the main references.

For *Paper 7, Modelling, problem-solving and integrating concepts*, an initial search using U.S. and U.K. spellings gave very few relevant results. We therefore broadened the search to include: modelling, problem-solving, realistic, real-life, variable and word problems. This process was iterative as the search for explanations for what could be inferred about students' learning led us into other related areas. Later we did further searches on some other terms which emerged as important: linearity, linear assumption, equation. Finally, we searched for papers which addressed how students learned combinations of concepts which build on elementary concepts, such as trigonometry. In all we located over 3200 references using British Education Index, ERIC and other sources. Fortunately many of these were not research-based, or used the terms in irrelevant ways, or addressed the focus in limited ways related to young children. The final relevant list consisted of 125 papers and a journal special issue. We used these papers to point to other sources. Most of these papers were reports of teaching experiments. Teaching experiments usually have a particular commitment to the nature of an aspect of mathematics and how it is best learnt. The experiment is constructed to see if students will be able to do X in certain circumstances, and X is measured as an outcome but in this process knowledge of how X is learnt, and what can go wrong, can be found. In reading this literature we found an overall coherence about students' learning of higher mathematics and the final version of the paper was constructed to show these similarities. A list of journals accessed is included in this appendix. There were only four reviews of research used, two meta-analyses by Hembree, (1986; 1992) used as summaries of literature and the U.S. Task Panel (NMAP 2008) was used as a gateway to other sources.

List of journals consulted for Papers 2 to 5

British Journal of Developmental Psychology
British Journal of Educational Psychology
Child Development
Cognition and Instruction
Educational Studies in Mathematics
Eurasia Journal of Mathematics, Science and Technology Education
International Electronic Journal of Mathematics Education
International Journal for Mathematics and Learning
International Journal of Science and Mathematics Education
Journal for Research in Mathematics Education
Journal for Research in Mathematics Education Monograph
Journal of Educational Psychology
Learning and Instruction

List of journals consulted for Papers 6 and 7

British Journal of Developmental Psychology
British Journal of Educational Psychology
Child Development
Cognition and Instruction
Educational Studies in Mathematics
International Journal of Mathematical Education in Science and Technology
Journal for Research in Mathematics Education
Learning and Instruction
Mathematical Thinking and Learning
Proceedings of International Group for the Psychology of Mathematics Education

Reviews and collections used for algebra

Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. Kluwer, Dordrecht

Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*. University of Melbourne, Australia, Dec 9-14, 2001.

Greenes, C. and Rubenstein, R. (eds.) *Algebra and Algebraic Thinking in School Mathematics.* 70th Yearbook. Reston, VA: NCTM.

Kaput, J., Carraher, D. and Blanton, M. (eds.) *Algebra in the early grades.* New York: Erlbaum

Mason, J. and Sutherland, R. (2002), *Key Aspects of Teaching Algebra in Schools*, QCA, London

Nickson, M. (2000) *Teaching and learning mathematics: a teacher's guide to recent research and its applications.* London: Cassell

NMAP, National Mathematics Advisory Panel (2008) downloaded April 2008 from: http://www.ed.gov/about/bdscomm/list/mathpanel/index.html

ZDM The International Journal of Mathematics Education volume 40 (2008)

Large-scale studies used for Papers 6 and 7

Concepts in Secondary Mathematics and Science Project (CSMS) (see Hart *et al.,* 1981)
Diagnostic tests derived from clinical interviews with 30 children age 11 to 16. In these interviews the test items were trialled and revised, and students' own methods and typical errors were observed. Common errors and methods were found across schools which were not teacher-taught but had arisen through students' own reasoning. The sample for testing was from urban, rural and city areas across England. It was selected from volunteer schools according to IQ distributions in order to represent the country as a whole. About 3000 students took the Algebra test.

Strategies and Errors in Secondary Mathematics Project (SESM) focused on a small number of errors arising in the CSMS study. There used a large number of individual interviews and some teaching experiments involving several classes of students.

Ryan and Williams
Ryan and Williams randomly-sampled 13 000 English school children from ages 4 to 15 using diagnostic tests designed to reveal typical errors and child-methods, as CSMS, but with the express purpose of identifying progress made by students in mathematics. They found little progress made between ages 11 to 14, and that many errors were similar to those found by Hart *et al.* 20 years earlier. See Ryan, J. and Williams, J. (2007) *Children's Mathematics 4-15: learning from errors and misconceptions,* Maidenhead: Open University Press. More details of the tests can be found in *Mathematics Assessment for Learning and Teaching,* (2005) London: Hodder and Stoughton.

Mollie MacGregor and Kaye Stacey
A series of pencil and paper tests were administered to 2000 students from a representative sample of volunteer schools in Years 7–10 (ages 11 –15) in 24 Australian secondary schools.

Assessment of Performance Unit (APU) test results of 1979 (Foxman *et al*., 1981). These tests involved a cohort of 12 500 students age 11 to 15 and were designed to track development of mathematical understanding by sampling across schools and regions.

Children's Mathematical Frameworks study (CMF) (Johnson, 1989), 25 classes in 21 schools in the United Kingdom were tested to find out why and how students between 8 and 13 cling to guess-and-check and number-fact methods rather than new formal methods offered by teachers.

References for appendix

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Bus, A. G., van IJzendoorn, M. H., and Pellegrini, A. D. (1995). Joint book reading makes for success in learning to read: A meta-analysis on intergenerational transmission of literacy. *Review of Educational Research, 65*, 1-21.

Blok, H. (1999). Reading to young children in educational settings: A meta-analysis of recent research. *Language Learning, 49*, 343-371.

Cooper, H. M. (1998). *Synthesizing research : a guide for literature reviews*. Thousand Oaks, CA: Sage Publications.

Ehri, L. C., Nunes, S. R., Stahl, S. A., and Willows, D. M. (2001). Systematic Phonics Instruction Helps Students Learn to Read: Evidence from the National Reading Panel's Meta-Analysis. *Review of Educational Research, 71*, 393-447.

Reed, D. K. (2008). A Synthesis of Morphology Interventions and Effects on Reading Outcomes for Students in Grades K–12. *Learning Disabilities Research and Practice, 23*, 36-49.

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