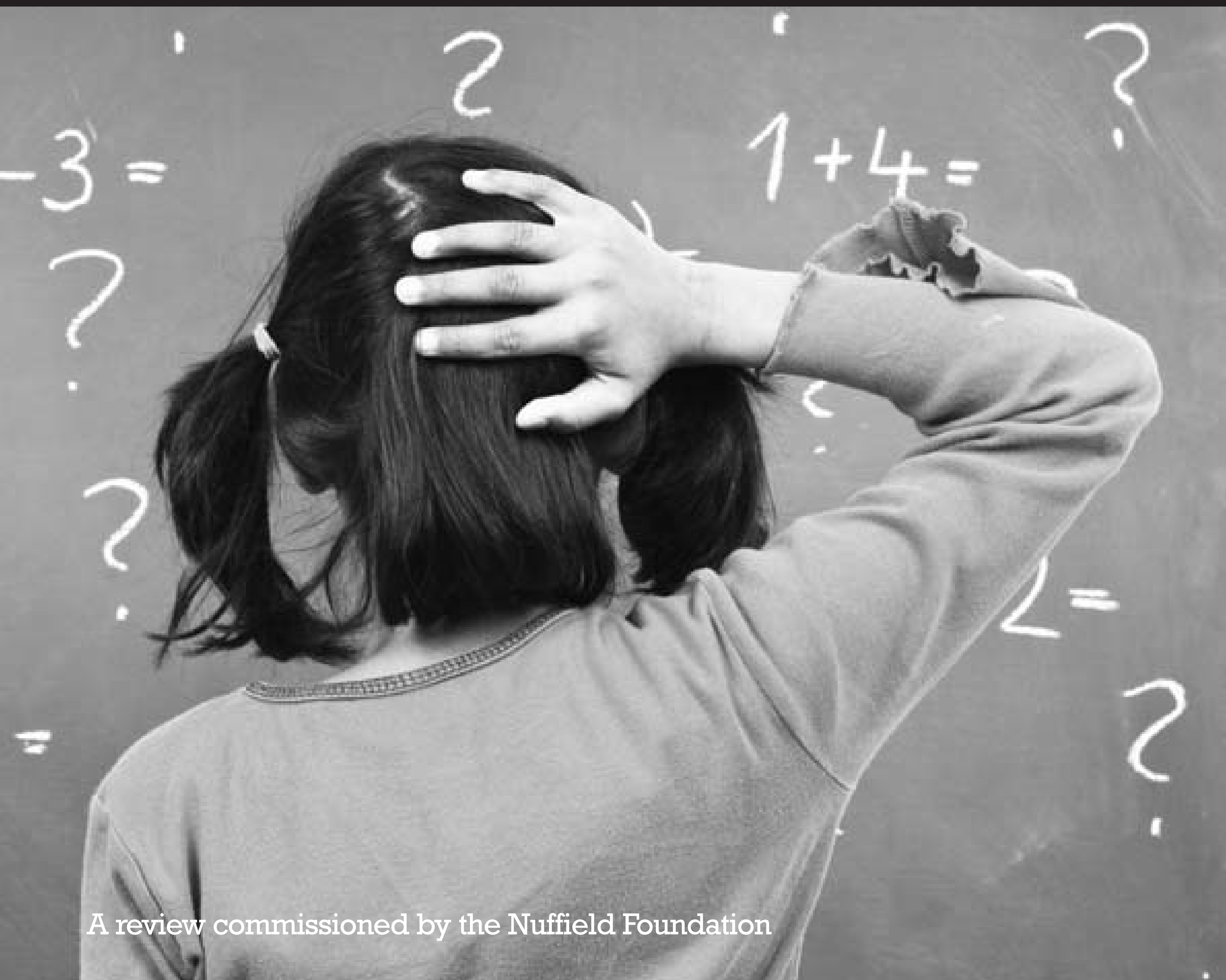
Nuffield Foundation

6

Key understandings in
mathematics learning

Paper 6: Algebraic reasoning

By Anne Watson, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

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About the author

Anne Watson is Professor of Mathematics Education at the University of Oxford.

About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 6: Algebraic reasoning

Headlines

- Algebra is the way we express generalisations about numbers, quantities, relations and functions. For this reason, good understanding of connections between numbers, quantities and relations is related to success in using algebra. In particular, students need to understand that addition and subtraction are inverses, and so are multiplication and division.

- To understand algebraic symbolisation, students have to (a) understand the underlying operations and (b) become fluent with the notational rules. These two kinds of learning, the meaning and the symbol, seem to be most successful when students know what is being expressed and have time to become fluent at using the notation.

- Students have to learn to recognise the different nature and roles of letters as: unknowns, variables, constants and parameters, and also the meanings of equality and equivalence. These meanings are not always distinct in algebra and do not relate unambiguously to arithmetical understandings, Mapping symbols to meanings is not learnt in one-off experiences.

- Students often get confused, misapply, or misremember rules for transforming expressions and solving equations. They often try to apply arithmetical meanings to algebraic expressions inappropriately. This is associated with over-emphasis on notational manipulation, or on 'generalised arithmetic', in which they may try to get concise answers.

Understanding symbolisation

The conventional symbol system is not merely an expression of generalised arithmetic; to understand it students have to understand the meanings of arithmetical operations, rather than just be able to carry them out. Students have to understand 'inverse' and know that addition and subtraction are inverses, and that division is the inverse of multiplication. Algebraic representations of relations between quantities, such as difference and ratio, encapsulate this idea of inverse. Using familiarity with symbolic expressions of these connections, rather than thinking in terms of generalising four arithmetical operations, gives students tools with which to understand commutativity and distributivity, methods of solving equations, and manipulations such as simplifying and expanding expressions.

The precise use of notation has to be learnt as well, of course, and many aspects of algebraic notation are inherently confusing (e.g. $2r$ and r^2). Over-reliance on substitution as a method of doing this can lead students to get stuck with arithmetical meanings and rules, rather than being able to recognise algebraic structures. For example, students who have been taught to see expressions such as:

$97 - 49 + 49$

as structures based on relationships between numbers, avoiding calculation, identifying variation, and having a sense of limits of variability, are able to reason with relationships more securely and at a younger age than those who have focused only on calculation. An expression such as $3x + 4$ is both the answer to a question, an object in itself, and also an algorithm or process for calculating a particular value. This has parallels in arithmetic: the answer to $3 \div 5$ is 3/5.

Time spent relating algebraic expressions to arithmetical structures, as opposed to calculations, can make a difference to students' understanding. This is especially important when understanding that apparently different expressions can be equivalent, and that the processes of manipulation (often the main focus of algebra lessons) are actually transformations between equivalent forms.

Meanings of letters and signs

Large studies of students' interpretation and use of letters have shown a well-defined set of possible actions. Learners may, according to the task and context:

- try to evaluate them using irrelevant information
- ignore them
- used as shorthand for objects, e.g. a = apple
- treat them as objects
- use a letter as a specific unknown
- use a letter as a generalised number
- use a letter as a variable.

Teachers have to understand that students may use any one of these approaches and students need to learn when these are appropriate or inappropriate. There are conventions and uses of letters throughout mathematics that have to be understood in context, and the statement 'letters stand for numbers' is too simplistic and can lead to confusion. For example:

- it is not always true that different letters have different values
- a letter can have different values in the same problem if it stands for a variable
- the same letter does not have to have the same value in different problems.

A critical shift is from seeing a letter as representing an unknown, or 'hidden', number defined within a number sentence such as:

$3 + x = 8$

to seeing it as a variable, as in $y = 3 + x$, or $3 = y - x$. Understanding x as some kind of generalized number which can take a range of values is seen by some researchers to provide a bridge from the idea of unknown to that of variables. The use of boxes to indicate unknown numbers in simple 'missing number' statements is sometimes helpful, but can also lead to confusion when used for variables, or for more than one hidden number in a statement.

Expressions linked by the 'equals' sign might be not just numerically equal, but also equivalent, yet students need to retain the 'unknown' concept when setting up and solving equations which have finite solutions. For example, $10x - 5 = 5(2x - 1)$ is a statement about equivalence, and x is a variable, but $10x - 5 = 2x + 1$ defines a value of the variable for which this equality is true. Thus x in the second case can be seen as an unknown to be found, but in the first case is a variable. Use of graphical software can show the difference visually and powerfully because the first situation is represented by one line, and the second by two intersecting lines, i.e. one point.

Misuse of rules

Students who rely only on remembered rules often misapply them, or misremember them, or do not think about the meaning of the situations in which they might be successfully applied. Many students will use guess-and-check as a first resort when solving equations, particularly when numbers are small enough to reason about 'hidden numbers' instead of 'undoing' within the algebraic structure. Although this is sometimes a successful strategy, particularly when used in conjunction with graphs, or reasoning about spatial structures, or practical situations, over-reliance can obstruct the development of algebraic understanding and more universally applicable techniques.

Large-scale studies of U.K. school children show that, despite being taught the BIDMAS rule and its equivalents, most do not know how to decide on the order of operations represented in an algebraic expression. Some researchers believe this to be due to not fully understanding the underlying operations, others that it may be due to misinterpretation of expressions. There is evidence from Australia and the United Kingdom that students who are taught to use flow diagrams, and inverse flow diagrams, to construct and reorganise expressions are better able to decide on the order implied by expressions involving combinations of operations. However, it is not known whether students taught this way can successfully apply their knowledge of order in situations in which flow diagrams are inappropriate, such as with polynomial equations, those involving the unknown on 'both sides', and those with more than one variable. To use algebra effectively, decisions about order have to be fluent and accurate.

Misapplying arithmetical meanings to algebraic expressions

Analysis of children's algebra in clinical studies with 12- to 13-year-olds found that the main problems in moving from arithmetic to algebra arose because:

- the focus of algebra is on relations rather than calculations; the relation $a + b = c$ represents three unknown quantities in an additive relationship
- students have to understand inverses as well as operations, so that a hidden value can be found even if the answer is not obvious from knowing number bonds or multiplication facts; $7 + b = 4$ can be solved using knowledge of addition, but $c + 63 = 197$ is more easily solved if subtraction is used as the inverse of addition
- some situations have to be expressed algebraically first in order to solve them. 'My brother is two years older than me, my sister is five years younger than me; she is 12, how old will my brother be in three years' time?' requires an analysis and representation of the relationships before solution. 'Algebra' in this situation means constructing a method for keeping track of the unknown as various operations act upon it.
- letters and numbers are used together, so that numbers may have to be treated as symbols in a structure, and not evaluated. For example, the structure $2(3+b)$ is different from the structure of $6 + 2b$ although they are equivalent in computational terms. Learners have to understand that sometimes it is best to leave number as an element in an algebraic structure rather than 'work it out'.
- the equals sign has an expanded meaning; in arithmetic it is often taken to mean 'calculate' but in algebra it usually means 'is equal to' or 'is equivalent to'. It takes many experiences to recognise that an algebraic equation or equivalence is a statement about relations between quantities, or between combinations of operations on quantities. Students tend to want 'closure' by compressing algebraic expressions into one term instead of understanding what is being expressed.

Expressing generalisations

In several studies it has been found that students understand how to use algebra if they have focused on generalizing with numerical and spatial representations in which counting is not an option. Attempts to introduce symbols to very young students as tools to be used when they have a *need* to express known general relationships, have been successful both for aiding their understanding of symbol use, and understanding the underlying quantitative relations being expressed. For example, some year 1 children first compare and discuss quantities of liquid in different vessels, and soon become able to use letters to stand for unknown amounts in relationships, such as $a > b$; $d = e$; and so on. In another example, older primary children could generalise the well-known questions of how many people can sit round a line of tables, given that there can be two on each side of each table and one at each of the extreme ends. The ways in which students count differ, so the forms of the general statement also differ and can be compared, such as: 'multiply the number of tables by 4 and add 2 or 'it is two times one more than the number of tables'.

The use of algebra to express *known* arithmetical generalities is successful with students who have developed advanced mental strategies for dealing with additive, multiplicative and proportional operations (e.g. compensation as in $82 - 17 = 87 - 17 - 5$). When students are allowed to use their own methods of calculation they often find algebraic structures for themselves. For example, expressing 13×7 as $10 \times 7 + 3 \times 7$, or as $2 \times 7^2 - 7$, are enactments of distributivity and learners can represent these symbolically once they know that letters can stand for numbers, though this is not trivial and needs several experiences. Explaining a general result, or structure, in words is often a helpful precursor to algebraic representation.

Fortunately, generalising from experience is a natural human propensity, but the everyday inductive reasoning we do in other contexts is not always appropriate for mathematics. Deconstruction of diagrams and physical situations, and identification of relationships between variables, have been found to be more successful methods of developing a formula than pattern-generalisation from number sequences alone. The use of verbal descriptions has been shown to enable students to bridge between observing relations and writing them algebraically.

Further aspects of algebra arise in the companion summaries, and also in the main body of *Paper 6: Algebraic reasoning*.

Recommendations

Research about mathematical learning	Recommendations for teaching
The bases for using algebraic symbolisation successfully are (a) understanding the underlying operations and relations and (b) being able to use symbolism correctly.	Emphasis should be given to reading numerical and algebraic expressions relationally, rather than computationally. For algebraic thinking, it is more important to understand how operations combine and relate to each other than how they are performed. Teachers should avoid emphasising symbolism without understanding the relations it represents.
Children interpret 'letter stands for number' in a variety of ways, according to the task. Mathematically, letters have several meanings according to context: unknown, variable, parameter, constant.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of children's possible interpretations of letters, and also that when correctly used, letters can have a range of meanings. Teachers should avoid using materials that oversimplify this variety. Hands-on ICT can provide powerful new ways to understand these differences in several representations.
Children interpret '=' to mean 'calculate'; but mathematically '=' means either 'equal to' or 'equivalent to'.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of the difficulties about the '=' sign and use multiple contexts and explicit language. Hands-on ICT can provide powerful new ways to understand these differences in several representations.
Students often forget, misremember, misinterpret situations and misapply rules.	Developers of the curriculum, advisory schemes of work and teaching methods need to take into account that algebraic understanding takes time, multiple experiences, and clarity of purpose. Teachers should emphasise situations in which generalisations can be identified and described to provide meaningful contexts for the use of algebraic expressions. Use of software which carries out algebraic manipulations should be explored.
Everyone uses 'guess-&-check' if answers are immediately obvious, once algebraic notation is understood.	Algebra is meaningful in situations for which specific arithmetic cannot be easily used, as an expression of relationships. Focusing on algebra as 'generalised arithmetic', e.g. with substitution exercises, does not give students reasons for using it.
Even very young students can use letters to represent unknowns and variables in situations where they have reasoned a general relationship by relating properties. Research on inductive generalisation from pattern sequences to develop algebra shows that moving from expressing simple additive patterns to relating properties has to be explicitly supported.	Algebraic expressions of relations should be a commonplace in mathematics lessons, particularly to express relations and equivalences. Students need to have multiple experiences of algebraic expressions of general relations based in properties, such as arithmetical rules, logical relations, and so on as well as the well-known inductive reasoning from sequences.

Recommendations for research

- The main body of *Paper 6: Algebraic reasoning* includes a number of areas for which further research would be valuable, including the following.
- How does explicit work on understanding relations between quantities enable students to move successfully from arithmetical to algebraic thinking?
- What kinds of explicit work on expressing generality enable students to use algebra?
- What are the longer-term comparative effects of different teaching approaches to early algebra on students' later use of algebraic notation and thinking?
- How do learners' synthesise their knowledge of elementary algebra to understand polynomial functions, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations?
- What useful kinds of algebraic expertise could be developed through the use of computer algebra systems in school?

Algebraic reasoning

In this review of how students learn algebra we try to balance an approach which focuses on what learners can do and how their generalising and use of symbols develop (a 'bottom up' developmental approach), and a view which states what is required in order to do higher mathematics (a 'top down' hierarchical approach). The 'top down' view often frames school algebra as a list of techniques which need to be fluent. This is manifested in research which focuses on errors made by learners in the curriculum and small-scale studies designed to ameliorate these. This research tells us about development of understanding by identifying the obstacles which have to be overcome, and also reveals how learners think. It therefore makes sense to start by outlining the different aspects of algebra. However, this is not suggesting that all mathematics teaching and learning should be directed towards preparation for higher mathematics.

By contrast a 'bottom up' view usually focuses on algebraic thinking, taken to mean the expression and use of general statements about relationships between variables. Lins (1990) sought a definition of algebraic thinking which encompassed the different kinds of engagement with algebra that run through mathematics. He concluded that algebraic thinking was an intentional shift from context (which could be 'real', or a particular mathematical case) to structure. Thus 'algebraic thinking arises when people are detecting and expressing structure, whether in the context of problem solving concerning numbers or some modelled situation, whether in the context of resolving a class of problems, or whether in the context of studying structure more generally' (Lins, 1990). Thus a complementary 'bottom up' view includes consideration of the development of students' natural ability to discern patterns and generalise them, and their growing competence in understanding and using symbols; however this would not take us very far in considering all the aspects of school algebra. The content of school algebra as the development of algebraic reasoning is expressed by Thomas and Tall (2001) as the shifts between procedure, process/concept, generalised arithmetic, expressions as evaluation processes, manipulation, towards axiomatic algebra. In this perspective it helps to see manipulation as the generation and transformation of equivalent expressions, and the identification of specific values for variables within them.

At school level, algebra can be described as:
- manipulation and transformation of symbolic statements
- generalisations of laws about numbers and patterns
- the study of structures and systems abstracted from computations and relations
- rules for transforming and solving equations
- learning about variables, functions and expressing change and relationships
- modelling the mathematical structures of situations within and outside mathematics.

Bell (1996) and Kaput (1998; 1999) emphasise the process of symbolisation, and the need to operate with symbolic statements and the use them within and outside algebra, but algebra is much more than the acquisition of a sign system with which to express known concepts. Vergnaud (1998) identifies new concepts that students will meet in algebra as: equations, formulae, functions, variables and parameters. What makes them new is that symbols are higher order objects than numbers and become mathematical objects in their own right; arithmetic has to work in algebraic systems, but symbol systems are not merely expressions of general arithmetic. Furthermore, 'the words and symbols we use to communicate do not refer directly to reality but to represented entities: objects, properties, relationships, processes, actions, and constructs, about which there is no automatic agreement' (p.167).

In this paper I draw on the research evidence about the first five of the aspects above. In the next paper I shall tackle modelling and associated issues, and their relation to mathematical reasoning and application more generally at school level.

It would be naïve to write about algebraic reasoning without reporting the considerable difficulties that students have with adopting the conventions of algebra, so the first part of this review addresses the relationship between arithmetic and algebra, and the obstacles that have to be overcome to understand the meaning of letters and expressions and to use them. The second part looks at difficulties which are evident in three approaches used to develop algebraic reasoning: expressing generalities; solving equations; and working with functions. The third part summarises the findings and makes recommendations for practice and research.

Part 1: arithmetic, algebra, letters, operations, expressions

Relationships between arithmetic and algebra

In the United States, there is a strong commitment to arithmetic, particularly fluency with fractions, to be seen as an essential precursor for algebra: 'Proficiency with whole numbers, fractions, and particular aspects of geometry and measurement are the Critical Foundation of Algebra. … The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in Algebra can be expected.' (NMAP, 2008). While number sense precedes formal algebra in age-related developmental terms, this one-way relationship is far from obvious in mathematical terms. In the United Kingdom where secondary algebra is not taught separately from other mathematics, integration across mathematics makes a two-way relationship possible, seeing arithmetic as particular instances of algebraic structures which have the added feature that they can be calculated. For example, rather than knowing the procedures of fractions so that they can be generalised with letters and hence make algebraic fractions, it is possible for fraction calculations to be seen as enactments of relationships between rational structures, those generalised enactments being expressed as algorithms.

For this review we see number sense as preceding formal algebra in students' learning, but to imagine that algebraic understanding is merely a generalisation of arithmetic, or grows directly from it, is a misleading over-simplification.

Kieran's extensive work (e.g. 1981, 1989, 1992) involving clinical studies with ten 12- to 13-year-olds leads her to identify five inherent difficulties in making a direct shifts between arithmetic and algebra.

- The focus of algebra is on relations rather than calculations; the relation $a + b = c$ represents two unknown numbers in an additive relation, and while $3 + 5 = 8$ is such a relation it is more usually seen as a representation of 8, so that $3 + 5$ can be calculated whereas $a + b$ cannot.

- Students have to understand inverses as well as operations, so that finding a hidden number can be done even if the answer is not obvious from knowing number bonds or multiplication facts; $7 + b = 4$ can be done using knowledge of addition, but $c + 63 = 197$ is more easily done if subtraction is used as the inverse of addition[1]. Some writers claim that understanding this structure is algebraic, while others say that doing arithmetic to find an unknown is arithmetical reasoning, not algebraic reasoning.

- Some situations have to be expressed algebraically in order to solve them, rather than starting a solution straight away. 'I am 14 and my brother is 4 years older than me' can be solved by addition, but 'My brother is two years older than me, my sister is five years younger than me; she is 12, how old will my brother be in three years' time?' requires an analysis and representation of the relationships before solution. This could be with letters, so that the answer is obtained by finding k where $k - 5 = 12$ and substituting this value into $(k + 2) + 3$. Alternatively it could be done by mapping systems of points onto a numberline, or using other symbols for the unknowns. 'Algebra' in this situation means constructing a method for keeping track of the unknown as various operations act upon it.

- Letters and numbers are used together, so that numbers may have to be treated as symbols in a structure, and not evaluated. For example, the structure $2(a + b)$ is different from the structure of $2a + 2b$ although they are equivalent in computational terms.

- The equals sign has an expanded meaning; in arithmetic it often means 'calculate' but in algebra it more often means 'is equal to' or even 'is equivalent to'.

If algebra is seen solely as generalised arithmetic (we take this to mean the expression of general arithmetical rules using letters), many problems arise for learning and teaching. Some writers describe these difficulties as manifestations of a 'cognitive gap' between arithmetic and algebra (Filloy and Rojano, 1989; Herscovics and Linchevski, 1994). For example, Filloy and Rojano saw students dealing arithmetically with equations of the form $ax + b = c$, where a, b and c are numbers, using inverse operations on the numbers to complete the arithmetical statement. They saw this as 'arithmetical' because it depended only on using operations to find a 'hidden' number. The same students acted algebraically with equations such as $ax + b = cx + d$, treating each side as an expression of relationships and using direct operations not to 'undo' but to maintain the equation by manipulating the expressions and equality. If such a gap exists, we need to know if it is developmental or epistemological, i.e. do we have to wait till learners are ready, or could teaching make a difference? A bottom-up view would be that algebraic thinking is often counter-intuitive, requires good understanding of the symbol system, and abstract meanings which do not arise through normal engagement with phenomena. Nevertheless the shifts required to understand it are shifts the mind is able to make given sufficient experiences with new kinds of object and their representations. A top-down view would be that students' prior knowledge, conceptualisations and tendencies create errors in algebra. Carraher and colleagues (Carraher, Brizuela & Earnest, 2001; Carraher, Schliemann & Brizuela, 2001) show that the processes involved in shifting from an arithmetical view to an algebraic view, that is from quantifying expressions to expressing relations between variables, are repeated for new mathematical structures at higher levels of mathematics, and hence are characteristics of what it means to learn mathematics at every level rather than developmental stages of learners. This same point is made again and again by mathematics educators and philosophers who point out that such shifts are fundamental in mathematics, and that reification of new ideas, so that they can be treated as the elements for new levels of thought, is how mathematics develops both historically and cognitively. There is considerable agreement that these shifts require the action of teachers and teaching, since they all involve new ways of thinking that are unlikely to arise naturally in situations (Filloy and Sutherland, 1996).

Some of the differences reported in research rest on what is, and what is not, described as algebraic. For example, the equivalence class of fractions that represent the rational number 3/5 is all fractions of the form $3k/5k$ ($k \in N$)[2]. It is a curriculum decision, rather than a mathematical one, whether equivalent fractions are called 'arithmetic' or 'algebra' but whatever is decided, learners have to shift from seeing 3/5 as 'three cakes shared between five people' to a quantitative label for a general class of objects structured in a particular quantitative relationship. This is an example of the kind of shift learners have to make from calculating number expressions to seeing such expressions as meaningful structures.

Attempts to introduce symbols to very young students as tools to be used when they have a need to express general relationships, can be successful both for them understanding symbol use, and understanding the underlying quantitative relations being expressed (Dougherty, 1996; 2001). In Dougherty's work, students starting school mathematics first compare and discuss quantities of liquid in different vessels, and soon become able to use letters to stand for unknown amounts. Arcavi (1994) found that, with a range of students from middle school upwards over several years, symbols could be used as tools early on to express relationships in a situation. The example he uses is the well-known one of expressing how many people can sit round a line of tables, given that there can be two on each side and one at each of the extreme ends. The ways in which students count differ, so the forms of the general statement also differ, such as: 'multiply the number of tables by 4 and add two' or 'it is two times one more than the number of tables'. In Brown and Coles' work (e.g. 1999, 2001), several years of analysis of Coles' whole-class teaching showed that generalising by expressing structures was a powerful basis for students to *need* symbolic notation, which they could then use with meaning. For example, to express a number such that 'twice the number plus three' is 'three less' than 'add three and double the number' a student who has been in a class of 12-year-olds where expression of general relationships is a normal and frequent activity introduced *N* for himself without prompting when it is appropriate.

When students are allowed to use their own methods of calculation they often find algebraic structures for themselves. For example, expressing 13×7 as $10 \times 7 + 3 \times 7$, or as $2 \times 7^2 - 7$, are enactments of distributivity (and, implicitly, commutativity and associativity) and can be represented symbolically, though this shift is not trivial (Anghileri, Beishuizen and van Putten, 2002; Lampert, 1986). On the other hand, allowing students to develop a mindset in which any method that gives a right answer is as good as any other can lock learners into additive procedures where multiplicative ones would be more generalisable, multiplicative methods where exponential methods would be more powerful, and so on. But some number-specific arithmetical methods *do* exemplify algebraic structures, such as the transformation of 13×7 described above. This can be seen either as 'deriving new number facts from known number facts' or as an instance of algebraic reasoning.

The importance of a link between the kinds of transformations necessary for mental arithmetic and algebraic thinking is demonstrated in a three-year longitudinal teaching and testing programme of 116 students aged 12 to 14 (Britt and Irwin, 2007). Students who had developed advanced mental strategies, (e.g. compensation as in $82 - 17 = 87 - 17 - 5$) for dealing with additive, multiplicative and proportional operations, could use letters in conventional algebra once they knew that they 'stood for' numbers. Those who did best at algebra were those in schools where teachers had focused on generalizing with numerical and spatial representations in situations where counting was not a sensible option.

There are differences in the meaning of notation as one shifts between arithmetic and algebra. Wong (1997) tested and interviewed four classes of secondary students to see whether they could distinguish between similar notations used for arithmetic and algebra. For example, in arithmetic the expression $3(4 + 5)$ is both a structure of operations and an invitation to calculate, but in algebra $a(b + c)$ is only a structure of operations. Thus students get confused when given mixtures such as $3(b + 5)$ because they can assume this is an invitation to calculate. This tendency to confuse what is possible with numbers and letters is subtle and depends on the expression. For example, Wong found that expressions such as $(2a^m)^n$ are harder to simplify and substitute than $(hk)^n$, possibly because the second expression seems very clearly in the realm of algebra and rules about letters. Where Booth and Kieran claim that it is not the symbolic conventions alone that create difficulties but more often a lack of understanding of the underlying operations, Wong's work helpfully foregrounds some of the inevitable confusions possible in symbolic conventions. The student has to understand when to calculate, when to leave an expression as a statement about operations, what particular kind of number (unknown, general or variable) is being denoted, and what the structure looks like with numbers and letters in combination. As an example of the last difficulty, 2^X is found to be harder to deal with than x^2 although they are visually similar in form.

The question for this review is therefore not *whether* learners can make such shifts, or *when* they make them, but *what* are the shifts they have to make, and in what circumstances do they make them.

Summary

- Algebra is not just generalised arithmetic; there are significant differences between arithmetical and algebraic approaches.

- The shifts from arithmetic to algebra are the kinds of shifts of perception made throughout mathematics, e.g. from quantifying to relationships between quantities; from operations to structures of operations.

- Mental strategies can provide a basis for understanding algebraic structures.

- Students will accept letters and symbols standing for numbers when they have quantitative relationships to express; they seem to be able to use letters to stand for 'hidden' numbers and also for 'any' number.

- Students are confused by expressions that combine numbers and letters, and by expressions in which their previous experience of combinations are reversed. They have to learn to 'read' expressions structurally even when numbers are involved.

Meaning of letters

Students' understanding of the meaning of letters in algebra, and how they use letters to express mathematical relationships, are at the root of algebraic development. Kuchemann (1981) identified several different ways adolescent students used letters in the Chelsea diagnostic test instrument (Hart, 1981). His research is based on test papers of 2900 students between 12 and 16 (see Appendix 1).

Letters were:
- evaluated in some way, e.g. $a = 1$
- ignored, e.g. $3a$ taken to be 3
- used as shorthand for objects, e.g. a = apple
- treated as objects
- used as a specific unknown
- used as a generalised number
- used as a variable.

Within his categorisation there were correct and incorrect uses, such as students who ascribed a value to a letter based on idiosyncratic decisions or past experience, e.g. $x = 4$ because it was 4 in the previous question. These interpretations appear to be task dependent, so learners had developed a sense of what sorts of question were treated in what kinds of ways, i.e. generalising (sometimes idiosyncratically) about question-types through familiarity and prior experience.

Booth (1984) interviewed 50 students aged 13 to 15 years, following up with 17 further case study students. She took a subset of Kuchemann's meanings, 'letters stand for numbers', and further unpicked it to reveal problems based on students' test answers and follow-up interviews. She identified the following issues which, for us, identify more about what students have to learn.

- It is not always true that different letters have different values; for example one solution to $3x + 5y = 8$ is that $x = y = 1$.

- A letter can have different values in the same problem, but not at the same time, if it stands for a variable (such as an equation having multiple roots, or questions such as 'find the value of $y = x^2 + x + 2$ when $x = 1, 2, 3\ldots$')

- The same letter does not have to have the same value in different problems.

- Values are not related to the alphabet ($a = 1$, $b = 2 \ldots$; or $y > p$ because of relative alphabetic position).

- Letters do not stand for objects (a for apples) except where the objects are units (such as m for metres).

- Letters do not have to be presented in alphabetical order in algebraic expressions, although there are times when this is useful[3].

- Different symbolic rules apply in algebra and arithmetic, e.g.: '2 lots of x' is written '$2x$' but two lots of 7 are not written '27'.

As well as in Booth's study, paper and pencil tests that were administered to 2000 students in aged 11 to 15 in 24 Australian secondary schools in 1992 demonstrated all the above confusions (MacGregor and Stacey, 1997).

These problems are not resolved easily, because letters are used in mathematics in varying ways. There is no single correct way to use them. They are used as labels for objects that have no numerical value, such as vertices of shapes, or for objects that do have numerical value but are treated as general, such as lengths of sides of shapes. They denote fixed constants such as g, e or π, also non-numerical constants such as i, and also they represent unknowns which have to be found, and variables. Distinguishing between these meanings is usually not taught explicitly, and this lack of instruction might cause students some difficulty. On the other hand it is very hard to explain how to know the difference between a parameter, a constant and a variable (e.g. when asked to 'vary the constant' to explore a structure), and successful students may learn this only when it is necessary to make such distinctions in particular usage. It is particularly hard to explain that the O and E in O + E = O (to indicate odd and even numbers) are not algebraic, even though they do refer to numbers. Interpretation is therefore related to whether students understand the algebraic context, expression, equation, equivalence, function or other relation. It is not surprising that Furinghetti and Paola (1994) found that only 20 out of 199 students aged 12 to 17 could explain the difference between parameters and variables and unknowns (see also Bloedy-Vinner, 1994). Bills' (2007) longitudinal study of algebra learning in upper secondary students noticed that the letters x and y have a special status, so that these letters trigger certain kinds of behaviour (e.g. these are the variables; or (x,y) denotes the general point). Although any letter can stand for any kind of number, in practice there are conventions, such as x being an unknown; x, y, z being variables; a, b, c being parameters/coefficient or generalised lengths, and so on.

A critical shift is from seeing a letter as representing an unknown, or 'hidden', number defined within a number sentence such as:

$3 + x = 8$

to seeing it as a variable, as in $y = 3 + x$, or $3 = y – x$. While there is research to show how quasi-variables such as boxes can help students understand the use of letters in relational statements (see Carpenter and Levi, 2000) the shift from unknown to variable when similar letters are used to have different functions is not well-researched. Understanding x as some kind of generalised number which can take a range of values is seen by some researchers to provide a bridge from the idea of unknown to that of variables (Bednarz, Kieran and Lee, 1996).

The algebra of unknowns is about using solution methods to find mystery numbers; the algebra of variables is about expressing and transforming relations between numbers. These different lines of thought develop throughout school algebra. The 'variable' view depends on the idea that the expressions linked by the 'equals' sign might be not just numerically equal, but also equivalent, yet students need to retain the 'unknown' concept when setting up and solving equations which have finite solutions. For example, $10x – 5 = 5(2x – 1)$ is a statement about equivalence, and x is a variable, but $10x – 5 = 2x + 1$ defines a value of the variable for which this equality is true. Thus x in the second case can be seen as an unknown to be found.

It is possible to address some of the problems by giving particular tasks which force students to sort out the difference between parameters and variables (Drijvers, 2001). A parameter is a value that defines the structure of a relation. For example, in $y = mx + c$ the variables are x and y, while m and c define the relationship and have to be fixed before we can consider the covariation of x and y. In the United Kingdom this is dealt with implicitly, and finding the gradient and intercept in the case just described is seen as a special kind of task. At A-level, however, students have to find coefficients for partial fractions, or the coefficients of polynomials which have given roots, and after many years of 'finding x' they can find it hard to use particular values for x to identify parameters instead. By that time only those who have chosen to do mathematics need to deal with it, and those who earlier could only find the m and c in $y=mx + c$ by using formulae without comprehension may have given up maths. Fortunately, the dynamic possibilities of ICT offer tools to fully explore the variability of x and y within the constant behaviour of m and c and it is possible that more extensive use of ICT and modelling approaches might develop the notion of variable further.

Summary

- Letters standing for numbers can have many meanings.

- The ways in which operations and relationships are written in arithmetic and algebra differ.

- Learners tend to fall into well-known habits and assumptions about the use of letters.

- A particular difficulty is the difference between unknowns, variables, parameters and constants, unless these have meaning.

- Difficulties in algebra are not merely about using letters, but about understanding the underlying operations and structures.

- Students need to learn that there are different uses for different letters in mathematical conventions; for example, a, b and c are often used as parameters, or generalised lengths in geometry, and x, y and z are often used as variables.

Recognising operations

In several intervention studies and textbooks students are expected to use algebraic methods for problems for which an answer is required, and for which *ad hoc* methods work perfectly well. This arises when solving equations with one unknown on one side where the answer is a positive integer (such as $3x + 2 = 14$); in word problems which can be enacted or represented diagrammatically (such as 'I have 15 fence posts and 42 metres of wire; how far apart must the fence posts be to use all the wire and all the posts to make a straight fence?'); and in these and other situations in which trial-and-adjustment work easily. Students' choice to use non-algebraic methods in these contexts cannot be taken as evidence of problems with algebra.

In a teaching experiment with 135 students age 12 to 13, Bednarz and Janvier (1996) found that a mathematical analysis of the operations required for solution accurately predicted what students would find difficult, and they concluded that problems where one could start from what is known and work towards what is not known, as one does in arithmetical calculations, were significantly easier than problems in which there was no obvious bridge between knowns and the unknown, and the relationship had to be worked out and expressed before any calculations could be made. Many students tried to work arithmetically with these latter kinds of problem, starting with a fictional number and working forwards, generating a structure by trial and error rather than identifying what would be appropriate. This study is one of many which indicate that understanding the meaning of arithmetical operations, rather than merely being able to carry them out, is an essential precursor not only to deciding what operation is the right one to do, but also to expressing and understanding structures of relations among operations (e.g. Booth, 1984). The impact of weak arithmetical understanding is also observed at a higher level, when students can confuse the kinds of proportionality expressed in $y = k/x$ and $y = kx$, thinking the former must be linear because it involves a ratio (Baker, Hemenway and Trigueros, 2001). The ratio of k to x in the first case is specific for each value of x, but the ratio of y to x in the second case is invariant and this indicates a proportional relationship.

Booth (1984) selected 50 students from four schools to identify their most common errors and to interview those who made certain kinds of error. This led her to identify more closely how their weakness with arithmetic limited their progress with algebra. The methods they used to solve word problems were bound by context, and depended on counting, adding, and reasoning with whole and half numbers. They were unable to express how to solve problems in terms of arithmetical operations, so that algebraic expressions of such operations were of little use, being unrelated to their own methods. Similarly, their methods of recording were not conducive to algebraic expression, because the roles of different numbers and signs were not clear in the layout. For example, if students calculate as they go along, rather than maintaining the arithmetical structure of a question, much information is lost. For example, $4^2 - 2^2$ becomes $16 - 4$ and the 'difference between two squares' is lost; similarly, turning rational or irrational numbers into decimal fractions can lose both accuracy and structure.

In Booth's work it was not the use of letters that is difficult, but the underlying arithmetical understanding. This again supports the view that it is not until *ad hoc*, *number fact* and *guess-and-test* methods fail that students are likely to see a need for algebraic methods, and in a curriculum based on expressions and equations this is likely occur when solving equations with non-integer answers, where a full understanding of division expressed as fractions would be needed, and when working with the unknown on both sides of an equation. Alternatively, if students are trying to express general relationships, use of letters is essential once they realise that particular examples, while illustrating relationships, do not fully represent them. Nevertheless students' invented methods give insight into what they might know already that is formalisable, as in the 13×7 example given above.

Others have also observed the persistence of arithmetic (Kieran, 1992; Vergnaud, 1998). Vergnaud compares two student protocols in solving a distance/time problem and comments that the additive approach chosen by one is not conceptually similar to the multiplicative chosen by the other, even though the answers are the same, and that this linear approach is more natural for students than the multiplicative. Kieran (1983) conducted clinical interviews with six 13-year-old students to find out why they had difficulty with equations. The students tended to see tasks as about 'getting answers' and could not accept an expression as meaningful in itself. This was also observed by Collis (1971) and more recently by Ryan and Williams in their large scale study of students' mathematical understanding, drawing on a sample of about 15 000 U.K. students

(2007). Stacey and Macgregor (2000, p. 159) talk of the 'compulsion to calculate' and comment that at every stage students' thinking in algebraic problems was dominated by arithmetical methods, which deflected them from using algebra. Furthermore, Bednarz and Janvier (1996) showed that even those who identified structure during interviews were likely to revert to arithmetical methods minutes later. It seemed as if testing particular numbers was an approach that not only overwhelmed any attempts to be more analytical, but also prevented development of a structural method.

This suggests that too much focus on substitution in early algebra, rather than developing understanding of how structure is expressed, might allow a 'calculation' approach to persist when working with algebraic expressions. If calculation does persist, then it is only where calculation breaks down that algebraic understanding becomes crucial, or, as in Bednarz and Janvier (1996), where word problems do not yield to straightforward application of operations. For a long time in Soviet education word problems formed the core of algebra instruction. Davydov (1990) was concerned that arithmetic does not necessarily lead to awareness of generality, because the approach degenerates into 'letter arithmetic' rather than the expression of generality. He developed the approach used by Dougherty (2001) in which young students have to express relationships before using algebra to generalize arithmetic. For example, students in the first year of school compare quantities of liquid ('do you have more milk than me?') and express the relationship as, say, $G < R$. They understand that adding the same amount to each does not make them equal, but that they have to add some to G to make them equal. They do not use numbers until relationships between quantities are established.

Substituting values can, however, help students to understand and verify relationships: it matters if this is for an unknown: $5 = 2x - 7$ where only one value will do; or for an equation where variables will be related: $y = 2x - 7$; or to demonstrate equivalence: e.g. does $5(x + y) - 3 = 5x + 5y - 3$ or $5x + y - 3$? But using substitution to understand what expressions *mean* is not helpful. Furthermore the choice of values offered in many textbooks can exacerbate misunderstandings about the values letters can have. They can reinforce the view that a letter can only take one value in one situation, and that different letters have to have different values, and even that $a = 1$, $b = 2$ etc.

Summary

- Learners use number facts and guess-and-check rather than algebraic methods if possible.
- Doing calculations, such as in substitution and guess-and-check methods, distracts from the development of algebraic understanding.
- Substitution can be useful in exploring equivalence of expressions.
- Word problems do not, on their own, scaffold a shift to algebraic reasoning.
- Learners have to understand operations and their inverses.
- Methods of recording arithmetic can scaffold a shift to understanding operations.

What shifts have to be made between arithmetic and algebra?

Changing focus slightly, we now turn to what the learner has to see differently in order to overcome the inherent problems discussed above. A key shift which has to be made is from focusing on answers obtained in any possible way, to focusing on structure. Kieran (1989, 1992), reflecting on her long-term work with middle school students, classifies 'structure' in algebra as (1) surface structure of expression: arrangement of symbols and signs; (2) systemic: operations within an expression and their actions, order, use of brackets etc.; (3) structure of an equation: equality of expressions and equivalence.

Boero (2001) identifies transformation and anticipation as key processes in algebraic problem solving, drawing on long-term research in authentic classrooms, reconstructing learners' meanings from what they do and say. He observed two kinds of transformation, firstly the contextual arithmetical, physical and geometric transformations students do to make the problem meaningful within their current knowledge (see also Filloy, Rojano and Robio,2001); secondly, the new kinds of transformation made available by the use of algebra. If students' anticipation is locked into arithmetical activity: finding answers, calculating, proceeding step-by-step from known to unknown (see also Dettori, Garutti, and Lemut, 2001), and if their main experience of algebra is to simplify expressions, then the shift to using the new kinds of transformation afforded by algebra is hindered. Thus typical secondary school algebraic behaviour includes reaching for a formula and substituting numbers into it (Arzarello, Bazzini and Chiappini, 1994), as is often demonstrated in students' meaningless approaches to finding areas and perimeters (Dickson, 1989 *a*). Typically students will multiply every available edge length to get area, and add everything to get perimeter. These approaches might also be manifestations of learners' difficulties in understanding area (see Paper 5, Understanding space and its representation in mathematics) which cause them to rely on methods rather than meaning.

The above evidence confirms that the relationship between arithmetic and algebra is not a direct conceptual hierarchy or necessarily helpful. Claims that arithmetical understanding has to precede the teaching of algebra only make sense if the focus is on the meaning of operations and on arithmetical structures, such as inverses and fractional equivalence, rather than in correct calculation. A focus on answers and *ad hoc* methods can be a distraction unless the underlying structures of the *ad hoc* methods are generalisable and expressed structurally. Booth (1984) found that inappropriate methods were sometimes transferred from arithmetic; students often did not understanding the purpose of conventions and notations, for example not seeing a need for brackets when there are multiple operations. The possibilities of new forms of expression and transformation have to be appreciated, and the visual format of algebraic symbolism is not always obviously connected to its meanings (Wertheimer, 1960; Kirschner, 1989). For example, the meaning of index notation has to be learnt, and while y^3 can be related to its meaning in some way, $y^{1/2}$ is rather harder to interpret without understanding abstract structure.

In the U.K. context of an integrated curriculum, a non-linear view of the shift between arithmetic and algebra can be considered. Many researchers have shown that middle-school students can develop algebraic reasoning through a focus on relationships, rather than calculations[4]. Coles, Dougherty and Arcavi have already been mentioned in this respect, and Blanton and Kaput (2005) showed in an intervention-and-observation study of cohort of 20 primary teachers, in particular one self-defined as 'not a maths person' in her second year of teaching, could integrate algebraic reasoning into their teaching successfully, particularly using ICT as a medium for providing bridges between numbers and structures. Fujii and Stephens (2001, 2008) examined the role of quasi-variables (signs indicating missing values in number sentences) as a precursor to understanding generalization. Brown and Coles (1999, 2001) develop a classroom environment in a U.K. secondary school in which relationships are developed which need to be expressed structurally, and algebraic reasoning becomes a tool to make new questions and transformations possible. These studies span ages 6 to lower secondary and provide school-based evidence that the development of algebraic reasoning can happen in deliberately-designed educational contexts. In all these contexts, calculation is deliberately avoided by focusing on, quantifiable but not quantified, relationships, and using Kieran's first level of structure, surface structure, to express phenomena at her third level, equality of expressions. A study with 105 11- and 12-year-olds suggests that explaining verbally what to do in general terms is a precursor to understanding algebraic structure (Kieran's third level) (Reggiani 1994). In this section I have shown that it is possible for students to make the necessary shifts given certain circumstances, and can identify necessary experiences which can support the move.

Summary of what has to be learnt to shift from arithmetic and algebra

- Students need to focus on relations and expressions, not calculations.
- Students need to understand the meaning of operations and inverses.
- Students need to represent general relations which are manifested in situations
- In algebra letters and numbers are used together; algebra is not just letters.
- The equals sign means 'has same value as' and 'is equivalent to' – not 'calculate'.
- Arithmetic can be seen as instances of general relationships between quantities.
- Division is a tool for constructing a rational expression.
- The value of a number is less important than its relation to other numbers in an expression.
- Guessing and checking, or using known number facts, has to be put aside for more general methods.

- A letter does not always stand for a particular unknown.

Without explicit attention to these issues, learners will use their natural and quasi-intuitive reasoning to:
- try to match their use of letters to the way they use numbers
- try to calculate expressions
- try to use '=' to mean 'calculate'
- focus on value rather than relationship
- try to give letters values, often based on alphabetical assumptions.

Understanding expressions

An expression such as $3x + 4$ is both the answer to a question, an object in itself, and also an algorithm or process for calculating a particular number. This is not a new way of thinking in mathematics that only appears with algebra: it is also true that the answer to $3 \div 5$ is 3/5, something that students are expected to understand when they learn about intensive quantities and fractions. Awareness of this kind of dual meaning has been called proceptual thinking (Gray and Tall, 1994), combining the process with its outcome in the same way as a multiple is a number in itself and also the outcome of multiplication. The notions of 'procept' and 'proceptual understanding' signify that there is a need for flexibility in how we act towards mathematical expressions.

Operational understanding

Many young students understand, at least under some circumstances, the inverse relation between addition and subtraction but it takes students longer to understand the inverse relation between multiplication and division[5]. This may be particularly difficult when the division is not symbolized by the division sign ÷ but by means of a fraction, as in 1/3. Understanding division when it is symbolically indicated as a fraction would require students to realise that a symbol such as 1/3 represents not only a quantity (e.g. the amount of pizza someone ate when the pizza was cut into three parts) but also as an operation. Kerslake (1986) has shown that older primary and younger secondary students in the United Kingdom rarely understand fractions as indicating a division. A further difficulty is that multiplication, seen as repeated addition, does not provide a ready image on which to build an understanding of the inverse operation. An array can be split up vertically or horizontally; a line of repeated quantities can only be split up into commensurate lengths. The language of division in schools is usually 'sharing' or 'shared by' rather than divide, thus triggering an assignment metaphor. This is a long way from the notion of number required in order to, for example, find y when $6y = 7$. There is evidence that students understand some properties of operations better in some contexts than in others (e.g. Nunes and Bryant, 1995).

As well as knowing about operations and their inverses, students need to know that only addition and multiplication are commutative in arithmetic, so that with subtraction and division it matters which way round the numbers go. Also in subtraction and multiplication it makes a difference if an unknown number or variable is not the number being acted on in the operation. For example, if $7 – p = 4$, then to find p the appropriate inverse operation is $7 – 4$. In other words 'subtract from n' is self-inverse. A similar issue arises with 'divide into n'.

We are unconvinced by the U.S. National Mathematics Advisory Panel's suggestion that fractions must be understood before algebra is taught (NMAP, 2008). Their argument is based on a 'top-down' curriculum view and not on research about how such ideas are learnt. The problems just described are algebraic, yet contribute to a full understanding of fractions as rational structures. There is a strong argument for seeing the mathematical structure of fractions as the unifying concept which draws together parts, wholes, divisions, ratio, scalings and multiplicative relationships, but it may only be in such situations as solving equations, algebraic fractions, and so on that students need to extend their view of division and fractions, and see these as related.

To understand algebraic notation requires an understanding that terms made up of additive, multiplicative and exponential operations, e.g. $(4a^3b – 8a)$, are variables rather than instruction to calculate, and have a structure and equivalent forms. It has been suggested that spending time relating algebraic terms to arithmetical structures can provide a bridge between arithmetic and algebra (Banerjee and Subramaniam, 2004). More research is needed, but working this way round, rather than introducing terms by reverting to substitution and calculation, seems to have potential.

Summary

- Learners tend to persist in additive methods rather than using multiplicative and exponential where appropriate.
- It is hard for students to learn the nature of multiplication and division – both as inverse of multiplication and as the structure of fractions and rational numbers.
- Students have to learn that subtraction and division are non-commutative, and that their inverses are not necessarily addition and multiplication.
- Students have to learn that algebraic terms can have equivalent forms, and are not instructions to calculate. Matching terms to structures, rather than using them to practice substitution, might be useful.

Relational reasoning

Students may make shifts between arithmetic and algebra, and between operations and relations, naturally with enough experience, but research suggests that teaching can make a difference to the timing and robustness of the shift. Carpenter and Levi (2000) have worked substantially over decades to develop an approach to early algebra based on understanding equality, making generalisations explicit, representing generalisations in various ways including symbolically, and talking about justification and proof to validate generalities. Following this work, Stephens and others have demonstrated that students can be taught to see expressions such as:

97 – 49 + 49

as structures, in Kieran's second sense of relationships among operations (see also the paper on natural numbers). In international studies, students in upper primary in Japan generally tackled these relationally, that is they did not calculate all the operations but instead combined operations and inverses, at a younger age than Australian students made this shift. Chinese students generally appeared to be able to choose between rapid computation and relational thinking as appropriate, while 14-year-old English students varied between teachers in their treatment of these tasks (Fujii and Stephens, 2001, 2008; Jacobs, Franke, Carpenter, Levi and Battey, 2007). This 'seeing' relationally seems to depend on the ability to discern details (Piaget, 1969 p. xxv) and application of an intelligent sense of structure (Wertheimer, 1960) and also to know when and how to handle specifics and when to stay with structure. The power of such approaches is illustrated in the well-known story of the young Gauss' seeing a structural way to sum an arithmetic progression. In Fujii and Stephens' work, seeing patterns based on relationships between numbers, avoiding calculation, identifying variation, having a sense of limits of variability, were all found to be predictors of an ability to reason with relationships rather than numbers.

These are fundamental algebraic shifts. Seeing algebra as 'generalised arithmetic' is not achieved by inductive reasoning from special cases, but by developing a structural perspective on number sentences.

Summary

- Learners naturally generalise, they look for patterns and habits, and familiar objects.
- Inductive reasoning from several cases is a natural way to generalise, but it is often more important to look at expressions as a whole.
- Learners can shift from 'seeing' number expressions as instructions to calculate to seeing them as relationships.
- This shift can be scaffolded by teaching which encourages students not to calculate but to identify and use relations between numbers.
- Learners who are fluent in both ways of seeing expressions, as structures or as instructions to calculate, can choose which to use.

Combining operations

Problems arise when an expression contains more than one operation, as can be seen in our paper on functional relations where young children cannot understand the notion of relations between relations, such as differences of differences. In arithmetical and algebraic expressions, some relations between relations appear as combinations of operations, and learners have to decide what has to be 'done' first and how this is indicated in the notation. Carpenter and Levi (2000), Fujii and Stephens (2001, 2008), Jacobs *et al*. (2007), draw attention to this in their work on how students read number sentences. Linchevski and Herscovics (1996) studied how 12- and13-year-olds decided on the order of operations. They found that students tended to overgeneralise the order, usually giving addition priority over subtraction; or using operations in left to right order; they can show lack of awareness of possible internal cancellations; they can see brackets as merely another way to write expressions rather than an instruction to act first, for example: 926 – 167 – 167 and 926 – (167 + 167) yielded different answers (Nickson, 2000 p. 120); they also did not understand that signs were somehow attached to the following number.

Apart from flow diagrams, a common way to teach about order in the United Kingdom is to offer 'BODMAS[6]' and its variants as a rule. However, it is unclear whether such an approach adequately addresses typical errors made by students in their use of expressions.

The following expression errors were manifested in the APU tests (Foxman *et al.,* 1985). These tests involved a cohort of 12 500 students age 11 to 15 years. There is also evidence in more recent studies (see Ryan and Williams, 2007) that these are persistent, especially the first.

- Conjoining: e.g. $a + b = ab$

- Powers are interpreted as multiplication, an error made by 20% of 15-year-olds

- Not understanding that having no coefficient means the coefficient is 1

- Adding all three values when substituting in, say, $u + gt$

- Expressing the cost of a packet of sweets where x packets cost 90p as $x/90$

The most obvious explanation of the conjoining error is that conjoining is an attempt to express and 'answer' by constructing closure, or students may just not know that letters together in this notation mean 'multiply'.

Ryan and Williams (2007) found a significant number of 14-year-olds did not know what to do with an expression; they tried to 'solve' it as if it is an equation, again possibly a desire for an 'answer'. They also treated subtraction as if it is commutative, and ignored signs associated with numbers and letters. Both APU (Foxman *et al.,* 1985) and Hart, (1981) concluded that understanding operations was a greater problem than the use of symbols to indicate them, but it is clear from Ryan and Williams' study that interpretation is also significantly problematic. The prevalence of similar errors in studies 20 years apart is evidence that these are due to students' normal sense-making of algebra, given their previous experiences with arithmetic and the inherent non-obviousness of algebraic notation.

Summary

- Understanding operations and their inverses is a greater problem than understanding the use of symbols.

- Learners tend to use their rules for reading and other false priorities when combining operations, i.e. interpreting left to right, doing addition first, using language to construct expressions, etc. They need to develop new priorities.

- New rules, such as BODMAS (which can be misused), do not effectively and quickly replace old rules which are based on familiarity, habit, and arithmetic.

Equals sign

A significant body of research reports on difficulties about the meaning of the equals sign Sfard and Linchevski (1994) find that students who can do $7x + 157 = 248$ cannot do $112 = 12x + 247$, but these questions include two issues: the position and meaning of the equals sign and that algorithmic approaches lead to the temptation to subtract smaller from larger, erroneously, in the second example. They argue that the root problem is the failure to understand the inverse relation between addition and subtraction, but this research shows how conceptual difficulties, incomplete understandings and notations can combine to make multiple difficulties. If students are taught to make changes to both sides of an equation in order to solve it (i.e. transform the equation $y – 5 = 8$ into $y – 5 + 5 = 8 + 5$) and they do not see the need to maintain equivalence between the values in the two sides of the equation, then the method that they are being taught is mysterious to them, particularly as many of the cases they are offered at first can be easily solved by arithmetical methods. Booth (1984) shows that these errors combine problems with understanding operations and inverses and problems understanding equivalence.

There are two possible ways to tackle these problems: to identify all the separate problems, treat them separately, and expect learners to apply the relevant new understandings when combinations occur; or to treat algebraic statements holistically and semantically, so that the key feature of the above examples is equality. There is no research which shows conclusively that one approach is better than the other (a statement endorsed in NMAP's review (2008)).

There is semantic and syntactic confusion about the meaning of '=' that goes beyond learning a notation (Kieran, 1981; 1992). Sometimes, in algebra, it is used to mean that the two expressions are equal in a particular instance where their values are equal; other times it is used to mean that two expressions are equivalent and one can be substituted for another in every occurrence. Strictly speaking, the latter is equivalence and might be written as ' ' but we are not arguing for this to become a new 'must do' for the curriculum as this would cut across so much contextual and historical practice. Rather, the understanding of algebraic statements must be situational, and this includes learning when to use '==' to mean 'calculate'; when to use it to mean 'equal in special cases' and when to mean 'equivalent'; and when to indicate that 'these two functions are related in this way' (Saenz-Ludlow and Walgamuth, 1998). These different meanings have implications for how the letter is seen: a quantitative placeholder in a structure; a mystery number to be found to make the equality work; or a variable which co-varies with others within relationships. Saenz-Ludlow and Walgamuth showed, over a year-long study with children, that the shift towards seeing '=' to mean 'is the same as' rather than 'find the answer' could be made within arithmetic with consistent, intentional, teaching. This was a teaching experiment with eight-year-olds in which children were asked to find missing sums and addends in addition grids. The verb 'to be' was used instead of the equals sign in this and several other tasks. Another task involved finding several binary calculations whose answer was 12, this time using '='. Word problems, including some set by the children, were also used. Children also devised their own ways to represent and symbolise equality. We do not have space here to describe more of the experiment, but at the end the children had altered their initial view that '=' was an instruction to calculate. They understood '=' as giving structural information. Fujii and Stephens' (2001) research can be interpreted to show that students do get better at using 'new' meanings of the equals sign and this may be a product of repeated experience of what Boero called the 'new transformations' made possible by algebra, combined with 'new anticipations' also made possible by algebra.

Alibali and colleagues (2007) studied 81 middle school students over three years to map their understanding of equations. They found that those who had, or developed, a sophisticated understanding of the equals sign were able to deal with equivalent equations, using equivalence to transform equations and solve for unknowns. Kieran and Saldanha (2005) used a Computer Algebra System to enable five classes of upper secondary students to explore different meanings of '=' and found that given suitable tasks they were able to understand equivalence, generating for themselves two different understandings: equivalence as meaning that expressions would give them equal values for a range of input values of the variables, and equivalence as meaning that the expressions were basically transformations of the same form. Both of these understandings contribute to meaningful manipulation from one form to another. Also focusing on equivalence, Kieran and Sfard (1999) used a graphical function approach and thus enabled students to recognise that equivalent algebraic representations of functions would generate the same graphs, and hence represent the same relationships between variables.

The potential for confusion between equality and equivalence relates to confusion between finding unknowns (such as values of variables when two non-equivalent expressions are temporarily made equal) and expressing relationships between variables. Equivalence is seen when graphs coincide; equality is seen when graphs intercept.

Summary

- Learners persist in using '=' to mean 'calculate' because this is familiar and meaningful for them.
- The equals sign has different uses within mathematics; sometimes it indicates equivalence and sometimes equality; learners have to learn these differences.
- Different uses of the equals sign carry different implications for the meaning of letters: they can stand for hidden numbers, or variables, or parameters.
- Equivalence is seen when graphs coincide, and can be understood either structurally or as generation of equal outputs for every input; equality is seen when graphs intercept.

Equations and inequality

In the CMF study (Johnson, 1989), 25 classes in 21 schools in United Kingdom were tested to find out why and how students between 8 and 13 cling to guess-and-check and number-fact methods rather than new formal methods offered by teachers. The study focused on several topics, including linear equations. The findings, dependent on large scale tests and additional interviews in four schools, are summarised here and can be seen to include several tendencies already described in other, related, algebraic contexts. That the same tendencies emerge in several algebraic contexts suggest that these are natural responses to symbolic stimuli, and hence take time to overcome.

Students tended to:

- calculate each side rather than operate on them
- not use inverse operations with understanding
- use *ad hoc* number-specific methods
- interpret a box or triangle to mean 'missing number' but could not interpret a letter for this purpose
- not relate a method to the symbolic form of a method
- be unable to explain steps of their procedures
- confuse a 'changing sides' method with a 'balance' metaphor, particularly not connecting what is said to what is done, or to what is written
- test actual numbers rather than use an algebraic method
- assume different letters had different values
- think that a letter could not have the value zero.

They also found that those who used the language 'getting rid of' were more likely to engage in superficial manipulation of symbols. They singled out 'get rid of a minus' for particular comment as it has no mathematical meaning. These findings have been replicated in United Kingdom and elsewhere, and have not been refuted as evidence of common difficulties with equations.

In the same study, students were then taught using a 'function machine' approach and this led to better understanding of what an equation is and the variable nature of x. However, this approach only makes sense when an input-output model is appropriate, i.e. not for equating two functions or for higher order functions (Vergnaud 1997). Ryan and Williams (2007) found that function machines can be used by most students age 12 to 14 to solve linear equations, but only when provided. Few students chose to introduce them as a method. Most 12-year-olds could reverse operations but not their order when 'undoing' to find unknowns in this approach. Booth (1984) and Piaget and Moreau (2001) show that students who understand inversion might not understand that, when inverting a sequence of operations, the inverse operations cannot just be carried out in any order: the order in which they are carried out influences the result. Robinson, Ninowski and Gray (2006) also showed that coordinating inversion with associativity is a greater challenge than using either inversion or associativity by themselves in problem solving. Associativity is the property that $x + (y + z)$ is equal to $(x + y) + z$, so that we can add either the first two terms, and then the last, or the last two and then the first. This property applies to multiplication also. (Incidentally, note that the automatic application of BODMAS here would be unnecessary.) Students get confused about how to 'undo' such related operations, and how to undo other paired operations which are not associative. As in all such matters, teaching which is based on meaning has different outcomes (see Brown and Coles, 1999, 2001).

Once learners understand the meaning of '=' there is a range of 'intuitive' methods they use to find unknown numbers: using known facts, counting, inverse operations, and trial substitution (Kieran, 1992). These do not generalize for situations in which the unknown appears on both sides, so formal methods are taught. Formal methods each carry potential difficulties: function machines do not extend beyond 'one-sided' equations; balance methods do not work for negative signs or for non-linear equations; change-side/change-sign tends to be misapplied rather than seen as a special kind of transformation.

Many errors when solving equations appear to come from misapplication of rules and processes rather than a flawed understanding of the equals sign. Filloy describes several 'cognitive tendencies' observed over several studies of students progressing from concrete to abstract understandings (e.g. Filloy and Sutherland, 1996). These tendencies are: to cling to concrete models; to use sign systems inappropriately; to make inappropriate generalizations; to get stuck when negatives appear; to misinterpret concrete actions. Problems with the balance metaphor could be a manifestation of the general tendency to cling to concrete models (Filloy and Rojano, 1989), and the negative sign cannot be related to concrete understandings or even to some syntactic rules which may have been learnt (Vlassis, 2002). Another problem is that when the 'unknown' is on both sides it can no longer be treated with simple inversion techniques as finding 'the hidden number'; $3x = 12$ entails answering the question 'what number must I multiply 3 by to get 12?'. But when balancing '$4m + 3$' with '$3m + 8$' the balance metaphor can suggest testing and calculating each side until they match, rather than solving by filling-in arithmetical facts. Vlassis devised a teaching experiment with 40 lower secondary students in two classes. The first task was a word problem which would have generated two equal expressions in one variable, and students only applied trial-and-error to this. The second task was a sequence of balance problems with diagrams provided, and all students could solve these. The final task was a sequence of similar problems expressed algebraically, two of which used negative signs. These generated a range of erroneous methods, including failure to identify when to use an inverse operation, misapplication of rules, syntactical mistakes and manipulations whose meaning was hard to identify. In subsequent exercises errors of syntax and meaning diminished, but errors with negative integers persisted. Eight months later, in a delayed interview, Vlassis' students were still using correctly the principles represented in the balance model, though not using it explicitly, but still had problems when negatives were included. In Filloy and Rojano (1989) a related tendency is described, that of students creating a personal sense of concrete action (e.g. 'I shall move this from here to here') and using them as if they are algebraic rules (also observed by Lima and Tall, 2008). More insight into how learners understand equations is given by English and Sharry (1996) who asked students to classify equations into similar types. Some classified them according to superficial syntactic aspects, and others to underlying algebraic structure. English and Sharry draw attention to the need for students to have experience of suitable structures in order to reason analogically and identify deeper similarities.

There is little research in students' understanding of inequality in algebra. In number, children may know about ranges of smaller, or larger, or 'between' numbers from their position on a numberline, and children often know that adding the same quantity to two unequal quantities maintains the inequality. There are well-known confusions about relative size of decimal numbers due to misunderstandings about the notation, but beyond the scope of this review (Hart, 1981). Research by Tsamir and others describe common problems which appear to relate to a tendency to act procedurally with unequal algebraic expressions without maintaining an understanding of the inequality (Linchevski and Sfard, 1991; Tsamir and Bazzini, 2001; Tsamir and Almog, 2001). One of these studies compares the performance of 170 Italian students to that of 148 Israeli students in higher secondary school (Tsamir and Bazzini, 2001). In both countries students had been formally taught about a range of inequalities. They were asked whether statement about the set $S =\{ x \in R: x = 3\}$ could be true or not: 'S can be the solution set of an equality and an inequality'. Only half the students understood that it could be the solution set of an inequality, and those few Italians who gave examples chose a quadratic inequality that they already knew about. Some students offered a linear inequality that could be solved to include 3 in the answer. The researchers concluded that unless an inequality question was answerable using procedural algebra it was too hard for them. Another task asked if particular solution sets satisfied $5x^4 \leq 0$. Only half were able to say that $x = 0$, the next most popular answer being $x \leq 0$. The researchers compared students' responses to both tasks. It seems that the image of 'imbalance' often used with algebraic inequalities is abandoned when manipulation is done. The 'imbalance' image does not extend to quadratic inequalities, for which a graphical image works better, but again a procedural approach is preferred by many students who then misapply it.

Summary

- Once students understand the equals sign, they are likely to use intuitive number-rules as a first resort.
- The appearance of the negative sign creates need for a major shift to abstract meanings of operations and relations, as concrete models no longer operate.
- The appearance of the unknown on both sides of an equation creates the need for a major shift towards understanding equality and variables.
- Students appear to use procedural manipulations when solving equations and inequalities without a mental image or understanding strong enough to prevent errors.
- Students appear to develop action-based rules when faced with situations which do not have obvious concrete manifestations.

- Students find it very hard to detach themselves from concrete models, images and instructions and focus on structure in equations.

Manipulatives

It is not only arithmetical habits that can cause obstacles to algebra. There are other algebraic activities in which too strong a memory for process might create obstacles for future learning. For example, a popular approach to teaching algebra is the provision of materials and diagrams which ascribe unknown numerical (dimensional) meaning to letters while facilitating their manipulation to model relationships such as commutativity and distributivity. These appear to have some success in the short term, but shifts from physical appearance to mental abstraction, and then to symbolism, are not made automatically by learners (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss and Mutch, 1997). These manipulatives provide persistent images and metaphors that may be obstructions in future work. On the other hand, the original approach to dealing with variables was to represent them as spatial dimensions, so there are strong historical precedents for such methods. There are reported instances of success in teaching this, relating to Bruner's three perspectives, enactive-iconic-symbolic (1966), where detachment from the model has been understood and scaffolded by teaching (Filloy and Sutherland, 1996; Simmt and Kieren, 1999). Detachment from the model has to be made when values are negative and can no longer be represented concretely, and also with fractional values and division operations. Spatial representations have been used with success where the image is used persistently in a range of algebraic contexts, such as expressions and equations and equivalence, and where teachers use language to scaffold shifts between concrete, numerical and relational perspectives.

Use of rod or bar diagrams as in Singapore (NMAP, 2008; Greenes and Rubenstein, 2007) to represent part/whole comparisons, reasoning, and equations, appears to scaffold thinking from actual numbers to structural relationships, so long as they only involve addition and/or repeated addition. Statements in the problem are translated into equalities between lengths. These equal lengths are constructed from rods which represent both the actual and the unknown numbers. The rod arrangements or values can then be manipulated to find the value of the unknown pieces. Equations with the variable on both sides are taught to 11 and 12 year-olds in Singapore using such an approach. The introduction of such methods into classrooms where teachers are not experienced in its use has not been researched. It has some similarities to the approach based on Cuisenaire rods championed by Gattegno in Europe. Whereas use for numbers was widespread in U.K. primary schools, use for algebra was not, possibly because the curriculum focus on substitution and simplification, rather than meaning and equivalence, provided an obstacle to sustained use.

Summary

- Manipulatives can be useful for modelling algebraic relationships and structures.
- Learners might see manipulatives as 'just something else to learn'.
- Teachers can help learners connect the use of objects, the development of imagery and the use of symbols through language.
- Students have to appreciate the limitations of concrete materials and shift to mental imagery and abstract understandings.

Application of formulae within mathematics

Dickson's study with three classes of ten-year-olds (1989 *a*) into students' use of formulae and formal methods is based on using the formula for area of rectangle in various contexts. In order to be successful in such tasks, students have to understand what multiplication is and how it relates to area, e.g. through an array model, how to use the formula by substitution and how the measuring units for area are applied. Some students can then work out a formula for themselves without formal teaching. From this study, Dickson (1989*a*, 1989*b*) and her colleagues found several problems in how students approach formal methods in early secondary school. A third of her subjects did not use a formal method at all; a third used it in a test but could not explain it in interviews; a third used it and explained it. She found that they:

- may not have underlying knowledge on which to base formalisation (note that formalisation can happen spontaneously when they do have such

knowledge)
- base their reasoning on incorrect method
- have a sound strategy that may not match formal method
- may be taught methods leading up to formal, but not matching the formal method
- may retain other methods, which may have limited application
- may retain formalisation but lose meaning, then misapply a formal method in future
- pre-formal enactive or iconic experiences may have been forgotten
- might be able to use materials to explain formal method
- may interpret formal notation inadequately.

The research described above, taken as a whole, suggests that the problems students have with using formulae in subjects other than mathematics are due to: not being fluent with the notation; not understanding the underlying operations; experience of using such formulae in mathematics lessons being limited to abstract or confusing situations, or even to situations in which an algebraic formula is not necessary. In addition, of course, they may not understand the intended context.

Summary

- Learners are able to construct formulae for themselves, at least in words if not symbols, if they have sufficient understanding of the relationships and operations.

- Learners' problems using formulae have several possible root causes.
  1. Underlying knowledge of the situation or associated concepts may be weak.
  2. Existing working strategies may not match the formal method.
  3. Notational problems with understanding how to interpret and use the formula.

Part 2: problems arising in different approaches to developing algebraic reasoning

Since the CSMS study (Hart, 1981) there has been an expansion of teaching approaches to develop meaningful algebra as:
- expressing generalities which the child already knows, therefore is expressing something that has meaning, and comparing equivalent expressions
- describing relationships between expressions as equations, which can then be solved to find unknown values (as in word problems)
- a collection of techniques for transforming equations to either find unknown values or represent relationships between variables in different ways
- expressing functions and their inverses, in which inputs become outputs according to a sequence of operations; using multiple representations
- modelling situations by identifying variables and how they co-vary.

Each of these offers more success in some aspects than an approach based on rules for manipulating expressions, but also highlights further obstacles to reasoning. Research is patchy, and does not examine how students learn across contexts and materials (Rothwell-Hughes, 1979). Indeed, much of the research is specifically about learning in particular contexts and materials.

Expressing generalisations from patterns

One approach to address inherent difficulties in algebra is to draw on our natural propensity to observe patterns, and to impose patterns on disparate experiences (Reed 1972). In this approach, sequences of patterns are presented and students asked to deduce formulae to describe quantitative aspects of a general term in the sequence. The expectation is that this generates a need for algebraic symbolisation, which is then used to state what the student can already express in other ways, numerical, recursive, diagrammatically or enactively.

This approach is prevalent in the United Kingdom, Australia and parts of North America. The NMAP (2008) review finds no evidence that expressing generality contributes to algebraic understanding, yet others would say that this depends on the definition of algebraic understanding. Those we offered at the start of this chapter include expression of generality as an indication of understanding. In Australia, there are contradictory findings about the value of such tasks.

The following is an example of one of the items which was used in the large scale test administered to students by MacGregor and Stacey (reported in Mason and Sutherland, 2002).

Look at the numbers in this table and answer the questions:

x	*y*
1	5
2	6
3	7
4	8
5	9
6	..
7	11
8	...
....	...
...	

(i) When *x* is 2, what is *y*?
(ii) When *x* is 8, what is *y*?
(iii) When *x* is 800, what is *y*?
(iv) Describe in words how you would find *y* if you were told that *x* is ………
(v) Use algebra to write a rule connecting *x* and *y* ………..

MacGregor and Stacey found performance on these items varied from school to school. The success of 14-year-old students in writing an algebraic rule ranged from 18% in one school to 73% in another. In general students searched for a term-to-term rule (e.g. Stacey, 1989). They also tested the same students with more traditional items involving substitution to show the meaning of notation and transformation, to show equivalence and finding unknowns. From this study they concluded that students taught with a pattern-based approach to algebra did no better and no worse on traditional algebra items than students taught with a more traditional approach (MacGregor and Stacey, 1993, 1995).

Redden (1994) studied the work of 1400 10- to 13-year-olds to identify the stages through which students must pass in such tasks. First they must recognise the number pattern (which might be multiplicative), then there must be a stimulus to expression, such as being asked for the next term and then the value of uncountable term; they must then express the general rule and use symbols to express it. Some students could only process one piece of data, some could process more pieces of data, some gave only a specific example, some gave the term-to-term formula and a few gave a full functional formula. A major shift of perception has to take place to express a functional formula and this is more to do with 'seeing' the functional relationship, a shift of perception, than symbolising it. Rowland and Bills (1996) describe two kinds of generalisation: empirical and structural, the first being more prevalent than the second. Amit and Neria (2007) use a similar distinction and found that students who had followed a pattern-generalisation curriculum were able to switch representations meaningfully, distinguish between variables, constants and their relationships, and shift voluntarily from additive to multiplicative reasoning when appropriate.

Moss, Beatty and Macnab (2006) worked with nine-year-old students in a longitudinal study and found that developing expressions for pattern sequences was an effective introduction to understanding the nature of rules in 'guess the rule' problems. Nearly all of the 34 students were then able to articulate general descriptions of functions in the classic handshake problem[7] which is known to be hard for students in early secondary years. By contrast, Ryan and Williams (2007) found in large-scale testing that the most prevalent error in such tasks for 12- and 14-year-olds was giving the term-to-term formula rather than the functional formula, and giving an actual value for the *n*th term. Cooper and Warren (2007, and Warren and Cooper, 2008), worked for three years in five elementary classrooms, using patterning and expressing patterns, to teach students to express generalisations to use various representations, and to compare expressions and structures. Their students learnt to use algebraic conventions and notations, and also understood that expressions had underlying operational meanings. Clearly, students are capable of learning these aspects of algebra in certain pedagogic conditions. Among other aspects common to most such studies, Cooper and Warren's showed the value of comparing different but equivalent expressions that arise from different ways to generalise the patterns, and also introduced inverse operations in the context of function machines, and a range of mental arithmetic methods. If other research about generalising patterns applies in this study, then it must be the combination of pattern-growth with these other aspects of algebra that made the difference in the learning of their students. They point to 'the importance of understanding and communicating aspects of representational forms which allowed commonalities to be seen across or between representations'.

As Carraher, Martinez & Schliemann (2007) show, it is important to nurture the transition from empirical (term-to-term) generalizations (called naïve induction by Radford, 2007), to generalisations that follow from explicit statements about mathematical relations between independent and dependent variables, and which might not be 'seen' in the data. Steele (2007) indicates some of the ways in which a few successful 12 to 13 year old students go about this transition when using various forms of data, pictorial, diagrammatic and numerical, but bigger studies show that this shift is not automatic and benefits from deliberate tuition. Radford further points out that once a functional relation is observed, expressing it is a further process involving integration of signs and meaning. Stephens' work (see Mason, Stephens and Watson, in press) shows that the opportunity and ability to exemplify relationships between variables as number pairs, and to express the relationship within the pairs, are necessary predictors of the ability to focus on and express a functional relationship. This research also illustrates that such abilities are developmental, and hence indicates the kind of learning experiences required to make this difficult shift.

Rivera and Becker (2007), looking longitudinally at middle school students' understanding of sequences of growing diagrammatic patterns in a teaching experiment, specify three forms of generalization that students engage with: constructive standard, constructive nonstandard, and deconstructive. It is the deconstruction of diagrams and situations that leads most easily to the functional formula, they found, rather than reasoning inductively from numbers. However, their students generally reverted to arithmetical strategies, as reported in many other studies of this and other shifts towards algebra.

Reed (1972) hypothesised that classifying is a natural act that enables us to make distinctions, clump ideas, and hence deal with large amounts of new information. It is therefore useful to think of what sort of information learners are trying to classify in these kinds of task. Reed found that people extract prototypes from the available data and then see how far other cases are from this prototype. Applying this to pattern-growth and sequence tasks makes it obvious that term-to-term descriptions are far easier and likely to be dominant when the data is expressed sequentially, such as in a table. We could legitimately ask the question: is it worth doing these kinds of activity if the shifts to seeing and then expressing functional relationships are so hard to make? Does this just add more difficulties to an already difficult subject? To answer this, we looked at some studies in which claims are made of improvements in seeing and expressing algebraic relationships, and identifying features of pedagogy or innovation which may have influenced these improvements. Yeap and Kaur (2007) in Singapore found a wider range of factors influencing success in unfamiliar generalisation tasks than has been reported in studies which focus on rehearsed procedures. In a class of 38 ten-year-old students they set tasks, then observed and interviewed students about the way they had worked on them. Their aim was to learn more about the strategies students had used and how these contributed towards success. The task was to find the sums of consecutive odd numbers: $1 + 3 + 5 + \ldots + (2n - 1)$. Students were familiar with adding integers from 1 to 100, and also with summing multiples. They were given a sequence of subtasks: a table of values to complete, to find the sum or $1 + 3 + \ldots + 99$ and to find the sum of $51 + 53 + \ldots + 99$. The researchers helped students by offering simpler versions of the same kinds of summation if necessary. Nearly all students were able to recognize and continue the pattern of sums (they turn out to be the square numbers); two-thirds were able to transfer their sense of structure to the 'sum to 99' task, but only one-third completed the 'sum from 51' task – the one most dissimilar to the table-filling tasks, requiring adaptation of methods and use of previous knowledge to make an argument. The researcher had a series of designed prompts to help them, such as to find the sum from 1 to 49, and then see what else they needed to get the sum to 99. Having found an answer, students then had to find it again using a different method. They found that success depended on:

- the ability to see structures and relationships
- prior knowledge
- metacognitive strategies

- critical-thinking strategies
- the use of organizing heuristics such as a table
- the use of simplifying heuristics such as trying out simpler cases
- task familiarity
- use of technology to do the arithmetic so that large numbers can be handled efficiently.

As with all mathematics teaching, limited experience is unhelpful. Some students only know one way to construct cases, one way to accomplish generalisation (table of values and pattern spotting), and have only ever seen simple cases used to start sequence generation, rather than deliberate choices to aid observations. Students in this situation may be unaware of the necessity for critical, reflective thinking and the value of simplifying and organising data. Furthermore, this collection of studies on expressing generality shows that construction, design, choice and comparison of various representational means does not happen spontaneously for students who are capable of using them. Choosing when and why to switch representations has long been known to be a mark of successful mathematics students (Krutetskii, 1976) and therefore this is a strategy which needs to be deliberately taught. Evidence from Blanton and Kaput's intervention study with 20 teachers (2005) is that many primary children were able to invent and solve 'missing number' sentences using letters as placeholders, symbolize quantities in patterns, devise and use graphical representations for single variables, and some could write simple relations using letters, codes, 'secret messages' or symbols. The intervention was supportive professional development which helped teachers understand what algebraic reasoning entails, and gave them resources, feedback, and other support over five years. Ainley (1996) showed that supportive technology can display the purpose of formal representations, and also remove the technical difficulties of producing new representations. Ten-year-old students in her study had worked for a few years in a computer-rich environment and used spreadsheets to collect data from purposeful experiments. They then generated graphs from the data and studied these, in relation to the data, to make conjectures and test them. One task was designed to lead to a problematic situation so that students would have to look for a shortcut, and she observed that the need to 'teach the computer' how to perform a calculation led to spontaneous formal representation of a variable.

So, if it is possible for students to learn to make these generalizations only with a great deal of pedagogic skill and technical know-how, why should it be pursued? The reason is that skill in the meaning and use of algebra enables further generalizations to be made, and transformations of mathematical relationships to be used and studied. The work required to understand the functional relationship is necessary to operate at a higher level than merely using algebra to symbolize what you do, as with term-to-term formulae. It is algebra that provides the means to building concepts upon concepts, a key aspect of secondary mathematics, by providing expression of abstract relationships in ways that can be manipulated. In algebra, the products are not answers, but structures, relationships, and information about relationships and special instances of them. These tasks provide contexts for that kind of shift, but do not guarantee that it will take place.

Assumptions, such as that which appears to be made in Redden's study, that understanding term-to-term relationships is a route to understanding functional relations contradict the experience of mathematicians that algebra expresses the structure of relations, and this can be adduced from single cases which are generic enough to illustrate the relationship through diagrams or other spatial representations. Numerical data has to be backed up with further information about relationships. For example, consider this data set:

x	y
1	1
2	4
3	9
...	...

While it is possible for these values to be examples of the function $y = x^2$ it is also possible that they exemplify $y = x^2 + (x - 1)(x - 2)(x - 3)$. Without further information, such as x being the side of a square and y being its area, we cannot deduce a functional formula, and inductive reasoning is misapplied. There is much that is mathematically interesting in the connection between term-to-term and functional formulae, such as application of the method of differences, and students have to learn how to conjecture about algebraic relationships, but to only approach generalisation from a sequence perspective is misleading and, as we have seen from these studies, very hard without the support of specially-designed tasks comparing and transforming equivalent structural generalisations.

Summary

- Learners naturally make generalisations based on what is most obviously related; this depends on the visual impact of symbols and diagrams.
- Seeing functional, abstract, relationships is hard and has to be supported by teaching.
- Deconstruction of diagrams, relationships, situations is more helpful in identifying functional relationships than pattern-generation.
- Development of heuristics to support seeing structural relationships is helpful.
- There is a further shift from seeing to expressing functional relationships.
- Learners who can express relationships correctly and algebraically can also exemplify relationships with number pairs, and express the relationships within the pairs; but not all those who can express relationships within number pairs can express the relationship algebraically.
- Learners who have combined pattern-generalisation with function machines and other ways to see relationships can become more fluent in expressing generalities in unfamiliar situations.
- Conflicting research results suggest that the nature of tasks and pedagogy make a difference to success.
- Functional relationships cannot be deduced from sequences without further information about structure.

Using an equation-centred approach to teaching algebra

There are new kinds of problem that arise in an equation-centred approach to teaching algebra in addition to those described earlier: the solution of equations to find unknown values, and the construction of equations from situations. The second of these new problems is considered in Paper 7. Here we look at difficulties that students had in teaching studies designed to focus on typical problems in finding unknowns.

Students in one class of Booth's (1984) intervention study (which took place with four classes in lower secondary school) had a teacher who emphasised throughout that letters had numerical value. These students were less likely than others to treat a letter as merely an object. In her study, discussion about the meaning of statements before formal activity seemed to be beneficial, and those students who were taught a formal method seemed to understand it better some time after the lesson, maybe after repeated experiences. However, some students did not understand it at all. As with all intervention studies, the teaching makes a difference. Linchevski and Herscovics (1996) taught six students to collect like terms and then decompose additive terms in order to focus on 'sides' or equations as expressions which needed to be equated. While this led to them being better able to deal with equations, there were lingering problems with retaining the sign preceding the letter rather than attaching the succeeding sign.

Several other intervention studies (e.g. van Ameron, 2003; Falle, 2005) confirm that the type of equation and the nature of its coefficients often make non-formal methods available to learners, even if they have had significant recent teaching in formal methods. These studies further demonstrate that students will use *ad hoc* methods if they seem more appropriate, given that they understand the meaning of an equation; where they did not understand they often misapplied formal methods. Falle's study included more evidence that the structure $a/x = b$ caused particular problems as learners interpreted 'division' as if it were commutative. As with other approaches to teaching algebra, using equations as the central focus is not trouble-free.

Summary

- As with all algebraic expressions, learners may react to the visual appearance without thinking about the meaning.
- Learners need to know what the equation is telling them.
- Learners need to know why an algebraic method is necessary; this is usually demonstrated when the unknown is negative, or fractional, and/or when the unknown is on both sides. They are likely to choose *ad hoc* arithmetic methods such as guess- and-check, use of known number facts, compensation or trial-and-adjustment if these are more convenient.

- Learners' informal methods of making the sides equal in value may not match formal methods.
- 'Undoing' methods depend on using inverse operations with understanding.
- Fluent technique may be unconnected to explaining the steps of their procedures.
- Learners can confuse the metaphors offered to 'model' solving equations, e.g. 'changing sides' with 'balance'.
- Metaphors in common use do not extend to negative coefficients or 'unknowns' or non-linear equations.
- Non-commutative and associative structures are not easily used with inverse reasoning.
- As in many other contexts, division and rational structures are problematic.

Spreadsheets

Learners have to know how to recognise structures (based on understanding arithmetical operations and what they do), express structures in symbols, and calculate particular cases (to stimulate inductive understanding of concepts) in order to use algebra effectively in other subjects and in higher mathematics. Several researchers have used spreadsheets as a medium in which to explore what students might be able to learn (e.g. Schwartz and Yerushalmy, 1992; Sutherland and Rojano, 1993; Friedlander and Tabach, 2001). The advantages of using spreadsheets are as follows.

- In order to use spreadsheets you have to know the difference between parameters (letters and numbers that structure the relationship) and variables, and the spreadsheet environment is low-risk since mistakes are private and can easily be corrected.
- The physical act of pointing the cursor provides an enactive aspect to building abstract structures.
- Graphical, tabular and symbolic representations are just a click away from each other and are updated together.
- Correspondences that are not easy to see in other media can be aligned and compared on a spreadsheet, e.g. sequences can be laid side by side, input and output values for different functions can be compared, and graphs can be related directly to numerical data.
- Large data sets can be used so that questions about patterns and generalities become more meaningful.

In Sutherland and Rojano's work, two small groups of students 10- to 11-years-old with no formal algebraic background were given some algebraic spreadsheet tasks based on area. It was found that they were less likely to use arithmetical approaches when stuck than students reported in non-spreadsheet research, possibly because these arithmetical approaches are not easily available in a spreadsheet environment. Sutherland and Rojano used three foci known to be difficult for students: the relation between functions and inverse functions, the development of equivalent expressions and word problems. The arithmetic methods used included whole/part approaches and trying to work from known to unknown. Most of the problems, however, required working from the unknown to the known to build up relationships. In a similar follow-up study 15-year-old students progressively modified the values of the unknowns until the given totals were reached (Sutherland and Rojano, 1993). There was some improvement in post-tests over pre-tests for the younger students, but most still found the tasks difficult. One of the four intervention sessions involved students constructing equivalent spreadsheet expressions. Some students started by constructing expressions that generated equality in specific cases, rather than overall equivalence. Students who had started out by using particular arithmetical approaches spontaneously derived algebraic expressions in the pencil-and-paper tasks of the post-test. This appears to confound evidence from other studies that an arithmetical approach leads to obstacles to algebraic generalization. The generation of numbers, which can be compared to the desired outputs, and adjusted through adapting the spreadsheet formula, may have made the need for a formula more obvious. The researchers concluded that comparing expressions which referred only to numbers, to those which referred to variables, appeared to have enabled students to make this critical shift.

A recent area of research is in the use of computer algebra systems (CAS) to develop algebraic reasoning. Kieran and Saldanha (2005) have had some success with getting students to deal with equations as whole meaningful objects within CAS.

Summary

Use of spreadsheets to build formulae:
- allows large data sets to be used
- provides physical enactment of formula construction
- allows learners to distinguish between variables and parameters
- gives instant feedback
- does not always lock learners into arithmetical and empirical viewpoints.

Functional approach

Authors vary in their use of the word 'function'. Technically, a function is a relationship of dependency between variables, the independent variables (input) which vary by some external means, and the dependent variables (output) which vary in accordance with the relationship. It is the relationship that is the function, not a particular representation of it, however in practice authors and teachers refer to 'the function' when indicating a graph or equation. An equivalence such as temperature conversion is not a function, because these are just different ways to express the same thing, e.g. $t = 9/5\ C + 32$ where t is temperature in degrees Fahrenheit and C temperature in degrees Celsius (Janvier 1996). Thus a teaching approach which focuses on comparing different expressions of the same generality is concerned with structure and would afford manipulation, while an approach which focuses on functions, such as using function machines or multiple representations, is concerned with relationships and change and would afford thinking about pairs of values, critical inputs and outputs, and rates of change.

Function machines

Some researchers report that students find it hard to use inverses in the right order when solving equations. However, in Booth's work (1984) with function machines she found that lower secondary students were capable of instructing the 'machine' by writing operations in order, using proper algebraic syntax where necessary, and could make the shift to understanding the whole expression. They could then reverse the flow diagram, maintaining order, to 'undo' the function.

We have discussed the use of function machines to solve equations above.

Multiple representations

A widespread attempt to overcome the obstacles of learning algebra has been to offer learners multiple representations of functions because:
- different representations express different aspects more clearly
- different representations constrain interpretations – these have to be checked out against each other
- relating representations involves identifying and understanding isomorphic structures (Goldin 2002).

By and large these methods offer graphs, equations, and tabular data and maybe a physical situation or diagram from which the data has been generated. The fundamental idea is that when the main focus is on meaningful functions, rather than mechanical manipulations, learners make sensible use of representations (Booth, 1984; Yerushalmy, 1997; Ainley, Nardi and Pratt, 1999; Hollar and Norwood, 1999).

A central issue is that in most contexts for a letter to represent anything, the student must understand what is being represented, yet it is often only by the use of a letter that what is being represented can be understood. This is an essential shift of abstraction. It may be that seeing the use of letters alongside other representations can help develop meaning, especially through isomorphisms.

This line of thought leads to a substantial body of work using multiple representations to develop understandings of functions, equations, graphs and tabular data. All these studies are teaching experiments with a range of students from upper primary to first year undergraduates. What we learn from them is a range of possibilities for learning and new problems to be overcome. Powell and Maher (2003) have suggested that students can themselves discover isomorphisms. Others have found that learners can recognise similar structures (English and Sharry, 1996) but need experience or prompts in order to go beyond surface features. This is because surface features contribute to the first impact of any situation, whether they are visual, aural, the way the situation is first 'read', or the first recognition of similarity.

Hitt (1998) claims that 'A central goal of mathematics teaching is taken to be that the students be able to pass from one representation type to another without falling into contradictions.' (p. 134). In experiments with teachers on a course he asked them to match pictures of vessels with graphs to represent the relationship between the volume and height of liquid being poured into them. The most common errors in the choice of functions were due to misinterpretation of the graphical representation, and misidentification of the independent variable in the situation. Understanding the representation, in addition to understanding the situation, was essential. The choice of representation, in addition to understanding, is also influential in success. Arzarello, Bazzini and Chiappini, (1994) gave 137 advanced mathematics students this problem: 'Show that if you add a 4-digit number to the 4-digit number you get if you reverse the digits, the answer is a multiple of eleven'. There were three strategies used by successful students, and the most-used was to devise a way to express a 4-digit number as the sum of multiples of powers of ten. This strategy leads immediately to seeing that the terms in the sum combine to show multiples of eleven. The relationship between the representation and its meaning in terms of 'eleven' was very close. 'Talk' can structure a choice of representations that most closely resemble the mathematical meaning (see also Siegler and Stern, 1998).

Even (1998) points to the ability to select, use, move between and compare representations as a crucial mathematical skill. She studied 162 early students in 8 universities (the findings are informative for secondary teaching) and found a difference between those who could only use individual data points and those who could adopt a global, functional approach. Nemirovsky (1996) demonstrates that the Cartesian relationship between graphs and values is much easier to understand pointwise, from points to line perhaps via a table of values, than holistically, every point on a line representing a particular relationship.

Some studies such as Computer-Intensive Algebra (e.g. Heid, 1996) and CARAPACE (Kieran, Boileau and Garancon,1996) go some way towards understanding how learners might see the duality of graphs and values. In a study of 14 students aged about 13, the CARAPACE environment (of graphs, data, situations and functions) seemed to support the understanding of equality and equivalence of two functions. This led to findings of a significant improvement in dealing with 'unknown on both sides' equations over groups taught more conventionally. The multiple-representation ICT environment led to better performance in word problems and applications of functions, but students needed additional teaching to become as fluent in algebra as 'conventional' students. But teaching to fluency took only six weeks compared to one year for others. This result seems to confirm that if algebra is seen to have purpose and meaning then the technical aspects are easier to learn, either because there is motivation, or because the learner has already developed meanings for algebraic expressions, or because they have begun to develop appropriate schema for symbol use. When students first had to express functions, and only then had to answer questions about particular values, they had fewer problems using symbols.

There were further benefits in the CARAPACE study: they found that their students could switch from variable to unknown correctly more easily than has been found in other studies; the students saw a single-value as special case of a function, but their justifications tended to relate to tabular data and were often numerical, not relating to the overall function or the context. The students had to consciously reach for algebraic methods, even to use their own algorithms, when the situations became harder. Even in a multi-representational environment, using functions algebraically has to be taught; this is not spontaneous as long as numerical or graphical data is available. Students preferred to move between numerical and graphical data, not symbolic representations (Brenner, Mayer, Moseley, Brar, Duran, Reed and Webb,1997). This finding must depend on task and pedagogy, because by contrast Lehrer, Strom and Confrey,(2002) give examples where coordinating quantitative and spatial representations appears to develop algebraic reasoning through representational competence. Even (1998) argues that the flexibility and ease with which we hope students will move from representation to representation depends on what general strategy students bring to mathematical situations, contextual factors and previous experience and knowledge. We will look further at this in the next paper.

Further doubts about a multiple representation approach are raised by Amit and Fried (2005) in lessons on linear equations with 13 – 14 year-olds: 'students in this class did not seem to get the idea that representations are to be selected, applied, and translated'. The detail of this is elaborated through the failed attempts of one student, who did make this link, to persuade her peers about it. Hirschhorn (1993) reports on a longitudinal comparative study at three sites in which those taught using multiple representations and meaningful contexts did significantly better in tests than others taught more conventionally, but that there was no difference in attitude to mathematics. All we really learn from this is that the confluence of opportunity, task and explanation are not sufficient for learning. Overall the research suggests that there are some gains in understanding functions as meaningful expressions of variation, but that symbolic representation is still hard and the least preferred choice.

The effects of multiple representational environments on students' problem-solving and modeling capabilities are described in the next paper.

Summary

- Learners can compare representations of a relationship in graphical, numerical, symbolic and data form.
- Conflicting research results suggest that the nature of tasks and pedagogy make a difference to success.
- The hardest of these representations for learners is the symbolic form.
- Previous experience of comparing multiple representations, and the situation being modelled, helps students understand symbolic forms.
- Learners who see 'unknowns' as special cases of equality of two expressions are able to distinguish between unknowns and variables.
- Teachers can scaffold the shifts between representations, and perceptions beyond surface features, through language.
- Some researchers claim that learners have to understand the nature of the representations in order to use them to understand functions, while others claim that if learners understand the situations, then they will understand the representations and how to use them.

What students could do if taught, but are not usually taught

Most research on algebra in secondary school is of an innovative kind, in which particular tasks or teaching approaches reveal that learners of a particular age are, in these circumstances, able to display algebraic behaviour of particular kinds. Usually these experiments contradict curriculum expectations of age, or order, or nature, of learning. For example, in a teaching experiment over several weeks with 8-year-old students, Carraher, Brizuela and Schliemann (2000) report that young learners are able to engage with problems of an algebraic nature, such as expressing and finding the unknown heights in problems such as: Tom is 4 inches taller than Maria, Maria is 6 inches shorter than Leslie; draw their heights. They found that young learners could learn to express unknown heights with letters in expressions, but were sometimes puzzled by the need to use a letter for 'any number' when they had been given a particular instance. This is a real source for confusion, since Maria can only have one height. Students can naturally generalise about operations and methods using words, diagrams and actions when given suitable support (Bastable and Schifter, 2008). They can also see operators as objects (Resnick, Lesgold and Bill, 1990). These and other studies appear to indicate that algebraic thinking can develop in primary school.

In secondary school, students can work with a wider range of examples and a greater degree of complexity using ICT and graphical approaches than when confined to paper and pencil. For example, Kieran and Sfard (1999) used graphs successfully to help 12- and 13-yearold students to appreciate the equivalence of expressions. In another example, Noss, Healy and Hoyles (1997) constructed a matchsticks microworld which requires students to build up LOGO procedures for drawing matchstick sequences. They report on how the software supported some 12- to 13-year-old students in finding alternative ways to express patterns and structures of Kieran's second and third kinds. Microworlds provide support for students' shifts from particular cases to what *has* to be true, and hence support moves towards using algebra as a reasoning tool.

In a teaching study with 11 year-old students, Noble, Nemirovsky, Wright and Tierney, (2001) suggest that students can recognise core mathematical structures by connecting all representations to personally-constructed environments of their own, relevant for the task at hand. They asked pairs of students to proceed along a linear measure, using steps of different sizes, but the same number of steps each, and record where they got to after each step. They used this data to predict where one would be after the other had taken so many steps. The aim was to compare rates of change. Two further tasks, one a number table and the other a software-supported race, were given and it was noticed from the ways in which the students talked that they were bringing to each new task the language, metaphors and competitive sense which had been generated in the previous tasks. This enabled them to progress from the measuring task to comparing rates in multiple contexts and representations. This still supports the fact that students recognize similarities and look for analogical prototypes within a task, but questions whether this is related to what the teacher expects in any obvious way. In a three-year study with 16 lower secondary students, Lamon (1998) found that a year's teaching which focused on modelling sequential situations was so effective in helping students understand how to express relationships that they could distinguish between unknowns, variables and parameters and could also choose to use algebra when appropriate – normally these aspects were not expected at this stage, but two years further on.

Lee (1996) describes a long series of teaching experiments: 50 out of 200 first year university students committed themselves to an extra study group to develop their algebraic awareness. This study has implications for secondary students, as their algebraic knowledge was until then rule-based and procedural. She forced them, from the start, to treat letters as variables, rather than as hidden numbers. By many measures this group succeeded in comparative tests, and there was also evidence of success beyond testing, improvements in attitude and enjoyment. However, the impact of commitment to extra study and 'belonging' to a special group might also have played a part. Whatever the causal factors, this study shows that the notion of variable can be taught to those who have previously failed to understand, and can form a basis for meaningful algebra.

Summary

With teaching:

- Young children can engage with missing number problems, use of letters to represent unknown numbers, and use of letters to represent generalities that they have already understood.

- Young children can appreciate operations as objects, and their inverses.

- Students can shift towards looking at relationships if encouraged and scaffolded to make the shift, through language or microworlds, for example.

- Students can shift from seeing letters as unknowns to using them as variables.

- Students will develop similarities and prototypes to make sense of their experience and support future action.

- Students can shift from seeing cases as particular to seeing algebraic representations as statement about what has to be true.

- Comparison of cases and representations can support learning about functions and learning how to use algebra to support reasoning.

Part 3: Conclusions and recommendations

Conclusions

Error research about elementary algebra and pre-algebra is uncontentious and the findings are summarised above. However, it is possible for young learners to do more than is normally expected in the curriculum, e.g. they will accept the use of letters to express generalities and relationships which they already understand. Research about secondary algebra is less coherent and more patchy, but broadly can be summarised as follows.

Teaching algebra by offering situations in which symbolic expressions make mathematical sense, and what learners have to find is mathematically meaningful (e.g. through multiple representations, expressing generality, and equating functions) is more effective in leading to algebraic thinking and skill use than the teaching of technical manipulation and solution methods as isolated skills. However, these methods need to be combined through complex pedagogy and do not in themselves bring about all the necessary learning. Technology can play a big part in this. There is a difference between using ICT in the learner's control and using ICT in the teacher's control. In the learner's control the physical actions of moving around the screen and choosing between representations can be easily connected to the effects of such moves, and feedback is personalised and instant.

There is a tenuous relationship between what it means to understand and use the affordances of algebra, as described in the previous paragraph, and understanding and using the symbolic forms of algebra. Fluency in understanding symbolic expressions seems to develop through use, and also contributes to effective use – this is a two-way process. However, this statement ignores the messages from research which is purely about procedural fluency, and which supports repetitive practice of procedures in carefully constructed varying forms. Procedural research focuses on obstacles such as dealing with negative signs and fractions, multiple operations, task complexity and cognitive load but not on meaning, use, relationships, and dealing with unfamiliar situations.

Recommendations

For teaching

These recommendations require a change from a fragmented, test-driven, system that encourages an emphasis on fluent procedure followed by application.

- Algebra is the mathematical tool for working with generalities, and hence should permeate lessons so that it is used wherever mathematical meaning is expressed. Its use should be commonplace in lessons.

- Teachers and writers must know about the research about learning algebra and take it into account, particularly research about common errors in understanding algebraic symbolisation and how they arise.

- Teachers should avoid using published and web-based materials which exacerbate the difficulties by over-simplifying the transition from arithmetic to algebraic expression, mechanising algebraic transformation, and focusing on algebra as 'arithmetic with letters'.

- The curriculum, advisory schemes of work, and teaching need to take into account how shifts from arithmetical to algebraic understanding take time, multiple experiences, and clarity of purpose.

- Students at key stage 3 need support in shifting to representations of generality, understanding relationships, and expressing these in conventional forms.

- Students have to change focus from calculation, quantities, and answers to structures of operations and relations between quantities as variables. This shift takes time and multiple experiences.

- Students should have multiple experiences of constructing algebraic expressions for structural relations, so that algebra has the purpose of expressing generality.

- The role of 'guess-the-sequence-rule' tasks in the algebra curriculum should be reviewed: it is mathematically incorrect to state that a finite number of numerical terms indicates a unique underlying generator.

- Students need multiple experiences over time to understand: the role of negative numbers and the negative sign; the role of division as inverse of multiplication and as the fundamental operation associated with rational numbers; and the meaning of equating algebraic expressions.

- Teachers of key stage 3 need to understand how hard it is for students to give up their arithmetical approaches and adopt algebraic conventions.

- Substitution should be used purposefully for exemplifying the meaning of expressions and equations, not as an exercise in itself. Matching terms to structures, rather than using them to practice substitution, might be more useful.

- The affordances of ICT should be exploited fully, in the learner's control, in the teacher's control, and in shared control, to support the shifts of understanding that have to be made including constructing objects in order to understand structure.

- Teachers should encourage the use of symbolic manipulation, using ICT, as a set of tools to support transforming expressions for mathematical understanding.

For policy

- The requirements listed above signal a training need on a national scale, focusing solely on algebra as a key component in the drive to increase mathematical competence and power.

- There are resource implications about the use of ICT. The focus on providing interactive whiteboards may have drawn attention away from the need for students to be in control of switching between representations and comparisons of symbolic expression in order to understand the syntax and the concept of functions. The United Kingdom may be lagging behind the developed world in exploring the use of CAS, spreadsheets and other software to support new kinds of algebraic thinking.

- In several other countries, researchers have been able to develop differently-sequenced curricula in which students have been able to use algebra as a way of expressing general and abstract notions as these arise. Manipulation, solution of equations, and other technical matters to do with symbols develop as well as with formal teaching, but are better understood and applied. Similar development in the United Kingdom has not been possible due to an over-prescriptive curriculum and frequent testing which forces a focus on technical manipulation.

- Textbooks which promulgate an 'arithmetic with letters' approach should be avoided; this approach leads inevitably to the standard, obvious errors and hence turns students off algebra and mathematics in favour of short-term gains.

- Symbolic manipulators, graph plotters and other algebraic software are widely available and used to allow people to focus on meaning, application and implications. Students should know how to use these and how to incorporate them into mathematical explorations and extended tasks.

- We need to be free to draw on research and explore its implications in the United Kingdom, and this may include radical re-thinking of the algebra curriculum and how it is tested. This may happen as part of the 'functional maths' agenda but its foundations need to be established when students are introduced to algebra.

For research

- Little is known about school learning of algebra in the following areas.

- The experiences that an average learner needs, in educational environments conducive to change, to shift from arithmetical to algebraic thinking.

- The relationship between understanding the nature of the representations in order to use them to understand functions, and understanding the situations as an aid to understanding the representations and how to use them.

- Whether teaching experiments using functional, multi-representational, equation or generalisation approaches have an impact on students' typical notation-related difficulties. In other words, we do not know if and how semantic-focused approaches to algebra have any impact on persistent and well-known syntactic problems.

- How learners' synthesise their knowledge to understand quadratic and other polynomials, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations.

- Whether and how the use of symbolic manipulators to transform syntax supports algebraic understanding in school algebra.

- Using algebra to justify and prove generalities, rather than generate and express them.

- How students make sense of different metaphors for solving equations (balance, doing-undoing, graphical, formal manipulation).

Endnotes

1 The importance of inverses was discussed in the paper on natural numbers

2 In the paper on rational numbers we talk more about the relationship between fractions and rational numbers, and we often use these words interchangeably.

3 The advantage of this is that spotting like terms might be easier, but this can also mask some other characteristics such as physical meaning (e.g. E = mc2) and symmetry e.g. x2y + y2x.

4 This should be contrasted with the problems young learners have with expressing relations using number, described in our paper on functional relations. Knowing that relations are themselves number-like objects does not necessarily mean we have to calculate them.

5 This is discussed in detail in our papers on whole numbers and rational numbers and outlined here.

6 A very common mnemonic to remind people to do: brackets, 'of', division, multiplication, addition and subtraction in that order. It does not always work.

7 If n people all shake hands with each other, how many handshakes will there be?

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