Nuffield Foundation

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Key understandings in mathematics learning

Paper 2: Understanding whole numbers

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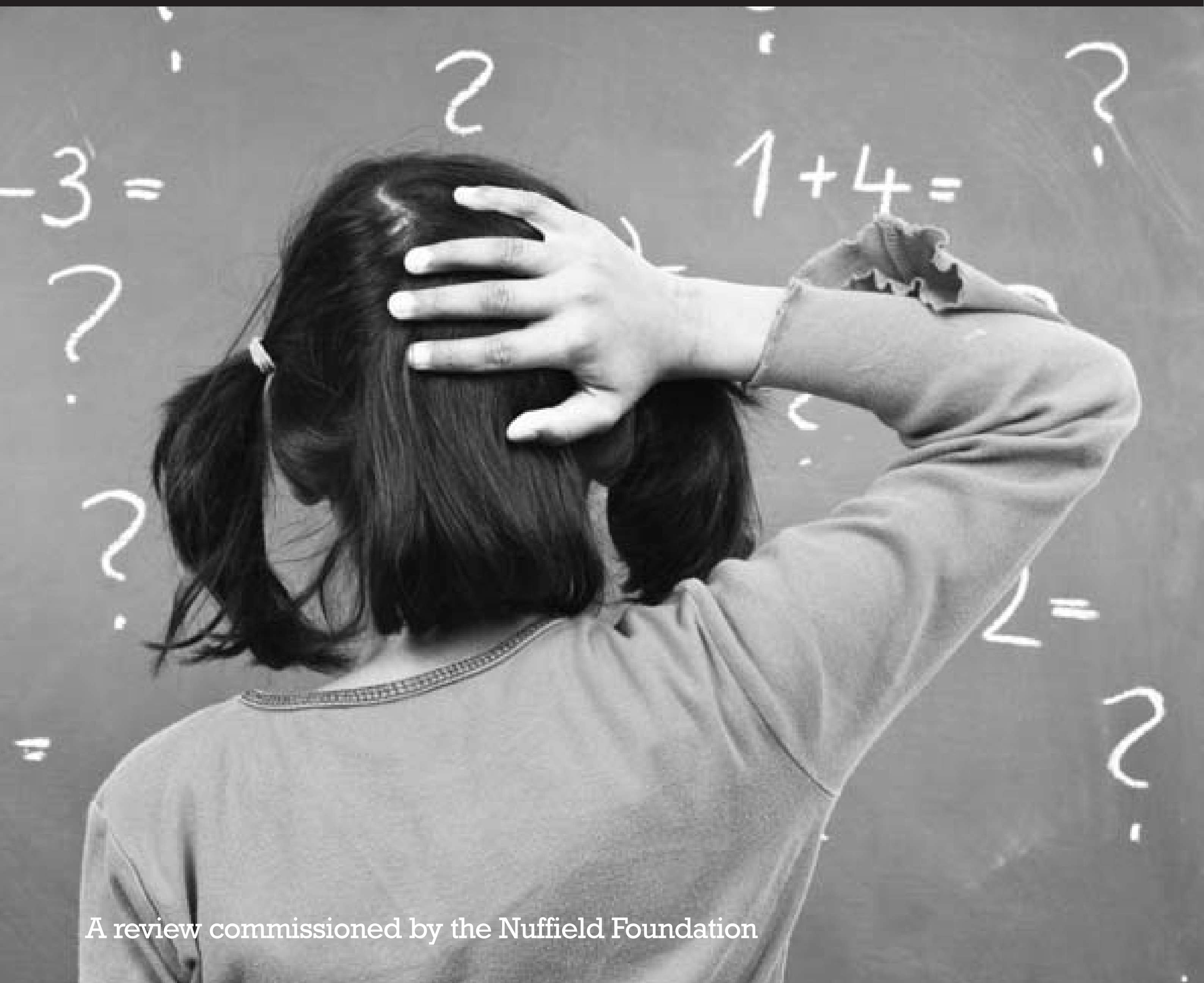

A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 2: Understanding whole numbers

Headlines

- Whole numbers are used in primary school to represent quantities and relations. It is crucial for children's success in learning mathematics in primary school to establish clear connections between numbers, quantities and relations.

- Using different schemes of action, such as setting objects in correspondence, children can judge whether two quantities are equivalent, and if they are not, make judgements about their order of magnitude. These insights are used in understanding the number system beyond simply producing a string of number words in a fixed order. It takes children some time to make links between their understanding of quantities and their knowledge of number.

- Children start school with varying levels of ability in using different action schemes to solve arithmetic problems in the context of stories. They do not need to know arithmetic facts to solve these problems: they count in different ways depending on whether the problems they are solving involve the ideas of addition, subtraction, multiplication or division.

- Individual differences in the use of action schemes to solve problems predict children's progress in learning mathematics in school.

- Interventions that help children learn to use their action schemes to solve problems lead to better learning of mathematics in school.

- It is considerably more difficult for children to use numbers to represent relations than to represent quantities. Understanding relations is crucial for their further development in mathematics in school.

In children's everyday lives and before they start school, they have experiences of manipulating and comparing quantities. For example, even at age four, many children can share sweets fairly between two recipients by using correspondences: they share giving one-for-you, one-for-me, until there are no sweets left. They do sometimes make mistakes but they know that, when the sharing is done fairly, the two people will have the same amount of sweets at the end. Even younger children know some things about quantities: they know that if you add sweets to a group of sweets, there will be more sweets there, and if you take some away, there will be fewer. However, they might not know that if you add a certain number and take away the same number, there will be just as many sweets as there were before.

At the same time that young children are developing these ideas about quantities, they are often learning to count. They learn to say the sequence of number words in the right order, they know that each object that they are counting must be counted once and only once, and that it does not matter if you count a row of sweets from left to right or from right to left, you should get to the same number.

Four-year-olds are thus amazing learners of mathematics. But they lack one thing which is crucially important: they do not at first make connections between their understanding of quantities and their knowledge of numbers. So if you ask a four-year-old, who just shared some sweets fairly between two dolls, to count the sweets that one doll has and then tell you, without counting, how many sweets the other doll has, the majority (about 60%) will tell you that they do not know. Knowing that the dolls have the same quantity is not sufficient to know that if one has 8 sweets, the other one has 8 sweets also, i.e. has the same number.

Quantities and numbers are not the same thing. We can use numbers as measures of quantities, but we can think about quantities without actually having a measure for them. Until children can understand the connections between numbers and quantities, they cannot use their knowledge of quantities to support their understanding of numbers and *vice versa*. Because the connections between quantities and numbers are many and varied, learning about these connections could take three to four years in primary school.

An important link that children must make between number and quantity is the link between the order of number words in the counting sequence and the magnitude of the quantity represented. How do children come to understand that the any number in the counting sequence is equal to the preceding number plus 1?

Different explanations have been proposed in the literature. One is that they simply see that magnitude increases as they count. But this explanation does not work well: our perception of magnitude is approximate and knowing that any number is equal to its predecessor plus 1 is a very precise piece of knowledge. A second explanation is that children use perception, language and inferences together to reach this understanding. Young children discriminate well, for example, one puppet from two puppets and two puppets from three puppets. Because they know these differences precisely, they put these two pieces of information together, and learn that two is one more than one, and three is one more than two. They then make the inference that all numbers in the counting sequence are equal to the predecessor plus one. But this sort of generalisation could not be stretched into helping children understand that any number is also equal to the last-but-one in the sequence plus 2. This process of putting together perception with language and then generalising is an explanation for only the $n + 1$ idea; it would be much better if we could have a more general explanation of how children understand the connection between quantities and the number sequence.

The third explanation for how children connect heir knowledge of quantities with the magnitude of numbers in the counting sequence is that children's schemes of action play the most important part in this development. The actions of adding and taking away help them understand part–whole relations. When they can link their understanding of part–whole relations with counting, they will understand many things about relations between numbers. A critical change in young children's behaviour when they add two sets is from 'count all' to 'count on'. If they know that they have 5 sweets, and you add 4 to the 5, they could either start from 1 and count all the sweets (count all) or they could point to the 5, and count on from there. 'Count on' is a sign that the children have linked their knowledge of part–whole relations with the counting sequence: they have understood the additive composition number. This explanation works for the relation between a number and its immediate predecessor and any of its predecessors. It is supported by much research that shows that counting on is a sign of abstraction in part–whole relations, which opens the way for children to solve many other problems: they can add a quantity to an invisible set, count coins of different denominations to form a single total, and are ready to learn to use place value to represent numbers in writing.

Adding and subtracting elements to sets also give children the opportunity to understand the inverse relation between addition and subtraction. This insight is not gained in an all-or-nothing fashion: children first apply it only to quantities and later on to number also. The majority of five-year-olds realises that if you add 3 sweets to a set of sweets and then take the *same* sweets away, the number of sweets in the set remains the same. However, many of these children will not realise that if you add 3 sweets to the set and then take 3 *other* sweets away, the number of sweets is still the same. They see that adding and taking away the *same quantity* leaves the original quantity the same but this does not immediately mean to them that adding and taking away the *same number* also leaves the original number the same. Research shows that the step from understanding the inverse relation between addition and subtraction of quantities is a useful start if one wants to teach children about the inverse relation between addition and subtraction of number.

Adding, taking away and understanding part–whole relations form one part of the story of what children know about quantities and numbers in the early years of primary school. They relate to how additive reasoning develops. The other part of the story is surprising to many people: children also know quite a lot about multiplicative reasoning when they start school.

Children use two different schemes of action to solve multiplication and division problems before they are taught about these operations in school: they use one-to-many correspondence and sharing. If five- and six-year-olds are shown, for example, four little houses in a row, told that they should imagine that in each live three dogs, and asked how many dogs live in the street, the majority can say the correct number. Many children will point three times to each house and count in this way until they complete the counting at the fourth house. They are not multiplying: they are solving the problem using one-to-many correspondence. Children can also share objects to recipients and answer problems about division. They do not know the arithmetic operations, but they can use their reasoning to count in different ways and solve the problem. So children manipulate quantities using multiplicative reasoning and solve problems before they learn about multiplication and division in school.

If children are assessed in their understanding of the inverse relation between addition and subtraction, of additive composition, and of one-to-many correspondence in their first year of school, this provides us with a good way of anticipating whether they will have difficulties in learning mathematics in school. Children who do well in these assessments go on to attain better results in mathematics assessments in school. Those who do not do well can improve their prospects through early intervention. Children who received specific instruction on these relations between quantities and how to use them to solve problems did significantly better than a similar group who did not receive such instruction.

Finally, many studies have used story problems to investigate which uses of additive reasoning are easier and which are more difficult for children of primary school age. Two sorts of difficulties have been identified.

The first relates to the need to understand that addition and subtraction are the inverse of each other. One story that requires this understanding is: Ali had some Chinese stamps in his collection and his grandfather gave him 2; now he has 8; how many stamps did he have before his grandfather gave him the 2 stamps? This problem exemplifies a situation in which a quantity increases (the grandfather gave him 2 stamps) but, because the information about the original number in his collection is missing, the problem is not solved by an addition but rather by a subtraction. The problem would also be an inverse problem if Ali had some Chinese stamps in his collection and gave 2 to his grandfather, leaving his collection with 6. In this second problem, there is a decrease in the quantity but the problem has to be solved by an increase in the number, in order to get us back to Ali's collection before he gave 2 stamps away. There is no controversy in the literature: inverse problems are more difficult than direct problems, irrespective of whether the arithmetic operation that is used to solve it is addition or subtraction.

The second difficulty depends on whether the numbers in the problem are all about quantities or whether there is a need to consider a relation between quantities. In the two problems about Ali's stamps, all the numbers refer to quantities. An example of a problem involving relations would be: In Ali's class there are 8 boys and 6 girls; how many more boys than girls in Ali's class? (Or how many fewer girls than boys in Ali's class?). The number 2 here refers neither to the number of boys nor to the number of girls: it refers to the relation (the difference) between number of boys and girls. A difference is not a quantity: it is a relation. Problems that involve relations are more difficult than those that involve quantities. It should not be surprising that relations are more difficult to deal with in numerical contexts than quantities: the majority, if not all, the experiences that children have with counting have to do with finding a number to represent a quantity, because we count things and not relations between things. We can re-phrase problems that involve relations so that all the numbers refer to quantities. For example, we could say that the boys and girls need to find a partner for a dance; how many boys won't be able to find a girl to dance with? There are no relations in this latter problem, all the numbers refer to quantities. This type of problem is significantly easier. So it is difficult for children to use numbers to represent relations. This could be one step that teachers in primary school want to help their children take, because it is a difficult move for every child.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's pre-school knowledge of quantities and counting develops separately.	**Teaching** Teachers should be aware of the importance of helping children make connections between their understanding of quantities and their knowledge of counting.
When children start school, they can solve many different problems using schemes of action in coordination with counting, including multiplication and division problems.	**Teaching** The linear view of development, according to which understanding addition precedes multiplication, is not supported by research. Teachers should be aware of children's mathematical reasoning, including their ability to solve multiplication and division problems, and use their abilities for further learning.
Three logical-mathematical reasoning principles have been identified in research, which seem to be causally related to children's later attainment in mathematics in primary school. Individual differences in knowledge of these principles predict later achievement and interventions reduce learning difficulties.	**Teaching** A greater emphasis should be given in the curriculum to promoting children's understanding of the inverse relation between addition and subtraction, additive composition, and one-to-many correspondence. This would help children who start school at risk for difficulties in learning mathematics to make good progress in the first years. **Research** Long-term longitudinal and intervention studies with large samples are needed before curriculum and policy changes can be proposed. The move from the laboratory to the classroom must be based on research that identifies potential difficulties in scaling up successful interventions.
Children's ability to solve word problems shows that two types of problem cause difficulties for children: those that involve the inverse relation between addition and subtraction and those that involve thinking about relations.	**Teaching** Systematic use of problems involving these difficulties followed by discussions in the classroom would give children more opportunities for making progress in using mathematics in contexts with which they have difficulty. **Research** There is a need for intervention studies designed to promote children's competence in solving problems about relations. Brief experimental interventions have paved the way for classroom-based research but large-scale studies are needed.

Understanding extensive quantities and whole numbers

Counting and reasoning

At school, children's formal learning about mathematics begins with natural numbers (1, 2, …17…103 …525…). Numbers are symbols for quantities: they make it possible for the child to specify single values precisely and also to work out the relations between different quantities. By counting, the child can tell you that there are 20 books in the pile on the teacher's desk (a single quantity), and eventually should be able to work out that there is 1 book for every child in the class if there are 20 children there, or that there are 5 more books than children (a relation between two quantities) if there are 15 children in the class.

Quantities and numbers are not the same. Thompson (1993) suggested that 'a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it. Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values' (pp. 165–166).[1]

Children experience and learn about quantities and the relations between them quite independently of learning to count. Similarly, they can learn to count quite independently from understanding quantities and relations between them. We shall argue in this section that the most important task for a child who is learning about natural numbers is to connect these numbers to a good understanding of quantities and relations. The connection should work at two levels.

First, children must realise that their knowledge of quantities and numbers should agree with one another. If Sean has 15 books and Patrick 17, Patrick has more books than Sean. Unless children understand that numbers are a precise way of expressing quantities, the number system will have no meaning for them.

Second, they must realise eventually that the number system enhances their knowledge of quantities in an increasingly powerful way. They may not be able to look at a pile of books and tell without counting that the one with 17 has more books than the one with 15; indeed, the thickness of books varies and the pile of 15 books could well be taller than the pile with 17. By counting they can know which pile has more books. When they know how to count, we can also add and subtract numbers, and work out the exact relations between them. If we understand lots of things about quantities, e.g. how to create equivalent quantities and how their equivalence is changed, but we don't have numbers to represent them, we cannot add and subtract.

In this section, therefore, we will focus on the connections that children make, and sometimes fail to make, between their growing knowledge about quantities and the number system. In many ways this is an unusual thing to do. Most existing accounts of how children learn about number are more restricted. Either they leave out the number system altogether and concentrate instead on children's ability to reason about quantities, or they are strictly confined to how well children count sets of objects.

Piaget's theory (Piaget, 1952) is an example of the first kind of theory. His view that children have to be able to reason logically about quantity in order to understand number and the number system is almost certainly right, but it left out the possibility that learning to count eventually transforms this reasoning in children by making it more powerful and more precise.

In the opposite corner, Gelman's influential theory (Gelman and Gallistel, 1978), which focuses on how children count single sets of objects and has little to say about children's quantitative reasoning, has the serious disadvantage that it by-passes children's reasoning about relations between quantities. In the end, numbers are only important because they allow us to represent quantities and make sense of quantitative relations.

The first part of this section is an account of how children connect numbers with quantity. We will start this account with a detailed list of the connections that they need to make. We argue that children need to make three types of connections between number words and quantities in order to make the most of what they learn when they begin to count: they need to understand cardinality; they need to understand ordinal numbers, and they need to understand the relation between cardinality and addition and subtraction. The second part of this section is an account of how children learn to use numbers to solve problems. We argue that numbers are used to represent quantities but that children must also learn to use them to represent transformations and relations, and that the different meanings that numbers can have affect how easily children solve problems.

Giving meaning to numbers

Young children's dissociation of quantities and numbers

Children may know that two quantities are the same and still not make the inference that the number of objects in one is the same as the number of objects in the other. Conversely, they may know how to count and yet not make use of counting when asked to create two equal sets. We review here briefly research within two different traditions, inspired by Piaget's and Gelman's theories, that shows that young children do not necessarily make a connection between what they know about quantities and what they know about counting.

Equivalence of sets in one-to-one correspondence and its connection to number words

Numbers have both cardinal and ordinal properties. Two sets have the same cardinal value when the items in one set are in one-to-one correspondence with those in the other. There are as many eggcups in a box of six egg-cups as there are eggs in a carton of six eggs, and if there are six people at the breakfast table each will have one of those eggs on its own eggcup to eat. Thus, the eggcups, eggs and people are all in one-to-one correspondence since there is one egg and eggcup for each one person, which means that each of these three sets has the same number.

We shall deal with the ordinal properties of number in a later section. At the moment, all that we need to say is that numbers are arranged in an ordered series.

To return to cardinality, Piaget argued quite reasonably that no one can understand the meaning of 'six' unless he or she also understands the number's cardinal properties, and by this he meant understanding not only that any set of six contains the same number of items as any other set of six but also that that the items in a set of six are in one-to-one correspondence with any other set of six items. So, if we are to pursue the approach of studying the links between children's quantitative reasoning and how they learn about natural numbers, we need to find out how well children understand the principle that sets which are in one-to-one correspondence with each other are equal in quantity, and also how clearly they apply what they understand about one-to-one correspondence to actual numbers like 'six'.

Piaget based his claim that young children have a very poor understanding of one-to-one correspondence on the mistakes that they make when they are shown one set of items (e.g. a row of eggs) and are asked to form another set (e.g. of eggcups) of the same number. Four- and five-year-olds often match the new set with the old one on irrelevant criteria such as two rows' lengths and make no effort to put the rows into one-to-one correspondence. Their ability to establish one-to-one correspondence between sets grows over time: it cannot be taken for granted.

However, even when children do establish a one-to-one correspondence between two sets, they do not necessarily infer that counting the elements in one set tells them how many elements there are in the other set. Piaget (1952) established this in an experiment in which he proposed to buy sweets from the children, using a one-to-one exchange between pence and sweets. For each sweet that the child gave to Piaget, he gave the child a penny. As they exchanged pence and sweets, the child was asked to count how many pence he/she had. Piaget ensured that he stopped this exchange procedure without going over the child's counting range. When he stopped the exchange, he asked the child how many pence the child had. The children were able to answer this without difficulty as they had been counting their coins. He then asked the child how many sweets he had. Piaget reports that some children were unable to make the inference that the number of sweets Piaget had was the same as the number of pence that the child himself/herself had. Unfortunately, Piaget gave no detailed description of how the ability to make this inference related to the children's age.

More recent research, which offers quantitative information, shows that many four-year-olds who do understand one-to-one correspondence well enough to share fairly do not make the inference that equivalent sets have the same number of elements. Frydman and Bryant (1988) asked four-year-old children to share a set of 'chocolates' to two recipients. At this age, children often share things between themselves, and they typically do so on a one-for-A, one-for-B, one-for-A, one-for-B basis. In this study, the children established the correspondence themselves; this contrasts with Piaget's study, where Piaget controlled the exchange of sweets and pence. When the child had done the sharing, the experimenters counted out the number of items that had been given to one recipient, which was six. Having done this, they asked the child how many chocolates had been given to the other recipient.

None of the children immediately made the inference that there were the same number of chocolates in one set as in the other, and therefore that there were also six items in the second set. Instead, every single child began to count the second set. In each case, the experimenter then interrupted the child's counting, and asked him or her if there was any other way of working out the number of items in the second recipient's share. Only 40% of the group of four-year-olds made the correct inference that the second recipient had also been given six chocolates. The failure of more than half of the children is an interesting one. The particular pre-school children who made it knew that the two recipients' shares were equal, and they also knew the number of items in one of the shares. Yet, they did not connect what they knew about the relative quantities to the number symbols. Other children, however, did make this connection, which we think is the first significant step in understanding cardinality. Whether all children will have made this connection by the time that they start learning about numbers and arithmetic at school depends on many factors: for example, the age they start school and their previous experiences with number are related to whether they have taken this important step by then (e.g. socio-economic status related to maths ability at school entry: see Ginsburg, Klein, and Starkey, 1998; Jordan, Huttenlocher, and Levine, 1992; Secada, 1992).

Counting and understanding relations between quantities

Piaget's theory of how children develop an understanding of cardinality was confronted by an alternative theory, by Gelman's nativist view of children's counting and its connection to cardinal number knowledge (Gelman and Gallistel, 1978). Gelman claimed that children are born with a genuine understanding of natural number, and that this makes it possible for them to learn and use the basic principles of counting as soon as they begin to learn the names for numbers. She outlined five basic counting principles. Anyone counting a set of objects should understand that:

- you should count every object once and only once (one-to-one correspondence principle)
- the order in which you count the actual objects (from left-to right, from right to left or from the middle outwards) makes no difference (order irrelevance principle)
- you should produce the number words in a constant order when counting: you cannot count 1-2-3 at one time and 1-3-2 at another (fixed order principle)
- whether the objects in a set are all identical to each other or all quite different has no effect on their number (the abstraction principle)
- the last number that you count is the number of items in the set (cardinal principle).

Each of these principles is justified in the sense that anyone who does not respect them will end up counting incorrectly. A child who produces count words in different orders at different times is bound to make incorrect judgements about the number of items in a set. So will anyone who does not obey the one-to-one principle.

Gelman's original observations of children counting sets of objects, and the results of some subsequent experiments in which children had to spot errors in other protagonists' counting (e.g. Gelman and Meck, 1983), all supported her idea that children obey and apparently understand all five of these principles with small sets of items long before they go to school. The young children's success in counting smaller sets allowed her to dismiss their more frequent mistakes with large sets of items as executive errors rather than failures in understanding. She agued that the children knew the principles of counting and therefore of number, but lacked some of the skills needed to carry them out. This view became known as the 'principles-before-skills hypothesis'.

These observations of Gelman's provoked a great deal of useful further research on children's counting, most of which has confirmed her original results, though with some modifications. For example, five-year-old children do generally count objects in a one-to-one fashion (one number word for each object) but not all of the time (Fuson, 1988). They tend either to miss objects or count some more than once in disorganised arrays. It is now clear that gestures play an important part in helping children keep track during counting (Albilali and DiRusso, 1999) but sometimes they point at some of the objects in a target set without counting them.

Many of the criticisms of Gelman's hypothesis were against her claims that children understand cardinality. Ironically, even critics of Gelman (e.g. Carey, 2004; LeCorre and Carey, 2007)) have in their own research accepted her all too limited definition of understanding cardinality (that it is the realisation that the last number counted represents the number of objects). However, several researchers have criticised her empirical test of cardinality. Gelman had argued that children, who count a set of objects and emphasise the last number ('one-two-three-FOUR') or repeat it ('one-two-three-four- there are four'), understand that this last number represents the quantity of the counted set. However, Fuson (Fuson, and Hall, 1983; Fuson, Richards and Briars, 1982) and Sophian (Sophian, Wood, and Vong, 1995) both made the reasonable argument that emphasising or repeating the last number could just be part of an ill-understood procedure.

Although Gelman's five principles cover some essential aspects of counting, they leave others out. The five principles, and the tools that Gelman devised to study children's understanding of these principles, only apply to what someone must know and do in order to count a single set of objects. They tell us nothing about children's understanding of numerical relations between sets. Piaget's research on number, on the other hand, was almost entirely concerned with comparisons between different quantities, and this has the confusing consequence that when Gelman and Piaget used the same terms, they gave them quite different meanings. For Piaget, understanding cardinality was about grasping that all and only equivalent sets are equal in number: for Gelman it meant understanding that the last number counted represents the number of items in a single set. When Piaget studied one-to-one correspondence, he looked at children's comparisons between two quantities (eggs and egg cups, for example): Gelman's concern with one-to-one correspondence was about children assigning one count word to each item in a set.

Since two sets are equal in quantity if they contain the same number of items and unequal if they do not, one way to compare two sets quantitatively is to count each of them and to compare the two numbers. Another, for much the same reason, is to use one-to-one correspondence: if the sets are in correspondence they are equal; if not, they are unequal. This prompts a question: how soon and how well do children realise that counting sets is a valid way, and sometimes the only feasible valid way, of comparing them quantitatively? Another way of putting the same question is to ask: how soon and how well do children realise that numbers are a measure by which they can compare the quantities of two or more different sets.

Most of the research on this topic suggests that it takes children some time to realise that they can, and often should, count to compare. Certainly many pre-school children seem not to have grasped the connection between counting and comparing even if they have been able to count for more than one year.

One source of evidence comes from the work by Sophian (1988), who asked children to judge whether someone else (a puppet) was counting the right way when asked to do two things. The puppet was faced with two sets of objects, and was asked in some trials to say whether the two sets were equal or not and in others to work out how many items there were on the table altogether. Sometimes the puppet did the right thing, which was to count the two sets separately when comparing them and to count all the items together when working out the grand total. At other times he got it wrong, e.g. counted all the objects as one set when asked to compare the two sets. The main result of Sophian's study was that the pre-school children found it very hard to make this judgement. Most 3-year-olds judged counting each set was the right way to count in both tasks while 4-year-olds judged counting both sets together was the right way to count in both tasks. Neither age group could identify the right way to count reliably.

A second type of study shows that even at school age many children seem not to understand fully the significance of numbers when they make quantitative comparisons. There is, for example, the striking demonstration by Pierre Gréco (1962), a colleague of Piaget's, that children will count two rows of counters, one of which is more spread out and longer than the other, and correctly say that they both have the same number (this one has six, and so does the other) but then will go on to say that there are more counters in the longer row than in the other. A child who makes this mistake understands cardinality in Gelman's sense (i.e. is able to say how many items in the set) but does not know what the word 'six' means in Piaget's sense. Barbara Sarnecka and Susan Gelman (2004) recently replicated this observation. They report that children three- and four-year-olds know that if a set had five objects and you add some to it, it no longer has five objects; however they did not know that equal sets must have the same number word.

Another source of evidence is the observation, repeated in many studies, that children, who can count quite well, nevertheless fail to count the items in two sets that they have been asked to compare numerically (Cowan, 1987; Cowan and Daniels, 1989; Michie, 1984; Saxe, Guberman and Gearhart, 1987); instead they rely on perceptual cues, like length, which of course are unreliable. Children who understand the cardinality of number should understand that they can make the comparison only by counting or using one-to-one correspondence, and yet at the age of five and six years most of them do neither, even when, as in the Cowan and Daniels study, the one-to-one cues are emphasised by lines drawn between items in the two sets that the children were asked to compare.

Finally, the criterion for the cardinality principle has itself been criticised as insufficient to show that children understand cardinality. The criticism is both theoretical and also based on empirical results. From a theoretical standpoint, Vergnaud (2008) pointed out that Gelman's cardinality criterion should actually be viewed as showing that children have some understanding of *ordinal*, not of cardinal, number: Gelman's criterion is indeed based on the *position* of the number word in the counting sequence, because the children use the *last* number word to represent the set. Vergnaud argues that ordinal numbers cannot be added whereas cardinal numbers can. He predicts that children whose knowledge of cardinal number is restricted to Gelman's cardinality principle will not be able to continue counting to answer how many objects are in a set if you add some objects to the set that they have just counted: they will need to count again from one. Research by Siegler and Robinson (1982) and Starkey and Gelman (1982) produced results in line with this prediction: 3-year-olds do not spontaneously count to solve addition problems after counting the first set. Ginsburg, Klein and Starkey (1998) also interpreted such results as indicative of an insufficient development of the concept of cardinality in young children. We return to the definition of cardinality later on, after we have discussed alternative explanations to Gelman's theory of an innate counting principle as the basis for learning about cardinality.

Three further studies will be used here to illustrate that some children who are able to use Gelman's cardinality principle do not seem to have a full grasp of when this principle should be applied; so meeting the criterion for the cardinality principle does not mean understanding cardinality.

Fuson (1988) showed that three-year-old children who seem to understand the cardinality principle continue to use the last number word in the counting sequence to say how many items are in a set even if the counting started from two, rather than from one. Counting in this unusual way should at least lead the children to reject the last word as the cardinal for the set.

Using a similar experimental manoeuvre, Freeman, Antonuccia and Lewis (2000) assessed three- and five-year-olds' rejection of the last word after counting if there had been a mistake in counting. The children participated in a few different tasks, one of which was a task where a puppet counted an array with either 3 or 5 items, but the puppet miscounted, either by counting an item twice or by skipping an item. The children were asked whether the puppet had counted right, and if they said that the puppet had not, they were asked: How many does the puppet think there are? How many are there really? All children had shown that they could count 5 items accurately (2 of 22 could count accurately to 6, another 4 could do so to 7, and the remaining 18 could count items accurately to 10). However, their competence in counting was no assurance that they realised that the puppet's answer was wrong after miscounting: only about one third of the children were able to say that the answer was not right *after* they had detected the error. The children's rejection of the puppet's wrong answer increased with age: 82% of the five-year-olds correctly rejected the puppet's answer in all three trials when a mistake had been made. However, the majority of the children could not say what the cardinal for the set was without recounting: the majority counted the set again in order to answer the question 'how many are there really?' They neither said immediately the next number when the puppet had skipped one nor used the previous number when the puppet double-counted an item. So, quite a few of the younger children passed Gelman's cardinality principle but did not necessarily see that the cardinality principle should not be applied when the counting principles are violated. Most of the older children, who rejected the use of the cardinality principle, did not use it to deduce what the correct cardinal should be; instead, all they demonstrated was that they could replace the wrong routine with the correct one, and then they could say what the number really was. Understanding that the next number is the cardinal for the set if the puppet skipped one item, without having to count again, would have demonstrated that the children have a relatively good grasp of cardinality. Freeman and his colleagues reported that only about one third of the children who detected the puppet's error were able to say what the correct number of items was without recounting. In the subsequent section we return to the importance of knowing what the next number is for the concept of cardinality.

The third study we consider here was by Bermejo, Morales and deOsuna (2004), who argued that if children really understand cardinality, and not just the Gelman's cardinality principle, they should do better than just re-implement the counting in a correct way. For example, they should be able to know how many objects are in a set even if the counting sequence is implemented backwards. If you count a set by saying 'three, two, one', and you reach the last item when you say 'one', you know that there are three objects in the set. If you count backwards from three and the label 'one' does not coincide with the last object, you know that the set does not have three objects. Just like starting to count from two, counting backwards is another way of separating out Gelman's cardinality principle from understanding cardinality: when you count backwards, the *first* number label is the cardinal for the set if there is a one-to-one correspondence between number labels and objects. Bermejo and colleagues showed that four- and six-year-old children who can say that there are three objects in a set when you count forward cannot necessarily say that if you count backwards from four and the last number label is 'two', this does not mean that there are two objects in the set. In fact, many children did not realize that there was a contradiction between the two answers: for them, the set could have three objects if you count one way and two if you count in another way. They also showed that children who were given the opportunity to discuss what the cardinal for the set was when the counting was done backwards showed marked progress in other tasks of understanding cardinality, which included starting to count from other numbers in the counting sequence than the number one, as in Fuson's task. They concluded that reflecting about the use of counting and the different actions involved in achieving a correct counting created opportunities for children to understanding cardinality better.

The evidence that we have presented so far suggests very strongly and remarkably consistently that *learning to count* and *understanding relations between quantities* are two different achievements. On the whole, children can use the procedures for counting long before they realise how counting allows them to measure and compare different quantities, and thus to work out the relations between them. We think that it is only when children establish a connection between what they know about relations between quantities and counting that they can be said to know the meaning of natural numbers.[2]

Summary

1 Natural numbers are a way of representing particular quantities and relations between quantities.

2 When children learn numbers, they must find out not just about the counting sequence and how to count, but also about how the numbers in the counting system represent quantities and relations between them.

3 One basic aspect of this representation is the cardinality of number: all sets with the same number have the same quantity of items in them.

4 Another way of expressing cardinality is to say that all sets with the number are in one-to-one correspondence with each other.

5 There is evidence that young children's first successful experiences with one-to-one correspondence come through sharing; however, even if they succeed in sharing fairly and know the number of items in one set, many do not make the inference that the number of items in the other set is the same.

6 Because of its cardinal properties, number is a measure: one can compare the quantity of items in two different sets by counting each set.

7 Several studies have shown that many children as old as six years are reluctant to count, although they know how to count, when asked to compare sets. They resort to perceptual comparisons instead.

8 This evidence suggests that learning about quantities and learning about numbers develop independently of each other in young children. But in order to understand natural numbers, children must establish connections between quantities and numbers. Thus schools must ensure that children learn not only to count but also learn to establish connections between counting and their understanding of quantities.

Current theories about the origin of children's understanding of the meaning of cardinal number

We have seen that Piaget's theory defines children's understanding of number on the basis of their understanding of relations between quantities; for him, cardinality is not just saying how many items are in sets but grasping that sets in one-to-one correspondence are equivalent in number and vice-versa. He argued that children could only be said to understand numbers if they made a connection between numbers and the relations between quantities that are implied by numbers. He also argued that this connection was established by children as they reflected about the effect of their actions on quantities: setting items in correspondence, adding and taking items away are schemes of action which form the basis for children's understanding of how to compare and to change quantities. Piaget acknowledges that learning to count can accelerate this process of reflection on actions, and so can other forms of social interaction, because they may help the children realise the contradictions that they fall into when they say, for example, that two quantities are different and yet they are labelled by the same number. However, the process that eventually leads to their understanding of the meanings of natural numbers and the implications of these representations is the child's growing understanding of relations between quantities.

Piaget's studies of children's understanding of the relations between quantities involved three different ideas that he considered central to understanding number: understanding equivalence, order, and class-inclusion (which refers to the idea that the whole is the sum of the parts, or that a set with 6 items comprises a set with 5 items plus 1). The methods used in these studies have been extensively criticised, as has the idea that children develop through a sequence of stages that can be easily traced and are closely associated with age. However, to our knowledge the core idea that children come to understand relations between quantities by reflecting upon the results of their actions is still a very important hypothesis in the study of how children learn about numbers. We do not review this vast literature here as there are several collections of papers that do so (see, for example, Steffe, Cobb and Glaserfeld, 1988; Steffe and Thompson, 2000). Later sections of this paper will revisit Piaget's theory and discuss related research.

This is not the only theory about how children come to understand the meaning of cardinal numbers. There are at least two alternative theories which are widely discussed in the literature. One is a nativist theory, which proposes that children have from birth access to an innate, inexact but powerful 'analog' system, whose magnitude increases directly with the number of objects in an array, and they attach the number words to the properties occasioning these magnitudes (Dehaene 1992; 1997; Gallistel and Gelman, 1992; Gelman and Butterworth, 2005; Xu and Spelke, 2000; Wynn, 1992; 1998). This gives all of us from birth the ability to make approximate judgements about numerical quantities and we continue through life to use this capacity. The discriminations that this system allows us to make are much like our discriminations along other continua, such as loudness, brightness and length. One feature of all these discriminations is that the greater the quantities (the louder, the brighter or the longer they are) the harder they are to discriminate (known, after the great 19th century psycho-physicist who meticulously studied perceptual sensitivity, as the 'Weber function'). To quote Carey (2004): 'Tap out as fast as you can without counting (you can prevent yourself from counting by thinking 'the' with each tap) the following numbers of taps: 4, 15, 7, and 28. If you carried this out several times, you'd find the mean number of taps to be 4, 15, 7, and 28, with the range of variation very tight around 4 (usually 4, occasionally 3 or 5) and very great around 28 (from 14 to 40 taps, for example). Discriminability is a function of the absolute numerical value, as dictated by Weber's law' (p. 63). The evidence for this analog system being an innate one comes largely from studies of infants (Xu and Spelke, 2000; McCrink and Wynn, 2004) and to a certain extent studies of animals as well, and is beyond the scope of this review. The evidence for its importance for learning about number and arithmetic comes from studies of developmental or acquired dyscalculia (e.g. Butterworth, Cipolotti and Warrington, 1996; Landerl, Bevana and Butterworth, 2004). However important this basic system may be as a neurological basis for number processing, it is not clear how the link between an analog and imprecise system and a precise system based on counting can be forged: 'ninety' does not mean 'approximately ninety' any more than 'eight' could mean 'approximately eight'. In fact, as reported in the previous section, three- and four-year-olds know that if a set has 6 items and you add one item to it, it no longer has 5 objects: they know that 'six' is not the same as 'approximately six'.

A third well-known theoretical alternative, which starts from a standpoint in agreement with Gelman's theory, is Susan Carey's (2004) hypothesis about three ways of learning about number. Carey accepts Gelman and Gallistel's (1978) limited definition of the cardinality principle but rejects their conclusions about how children first come to understand this principle. Carey argues that initially (by which she means in the first three years of life), very young children can represent number in three different ways (Le Corre and Carey, 2007). The first is the analog system, described in the previous paragraphs. However, although Carey thinks that this system plays a part in people's informal experiences of quantity throughout their lives, she does not seem to assign it a role in children's learning about the counting system, or in any other part of the mathematics that they learn about at school.

In her theory, the second of Carey's three systems, which she calls the 'parallel individuation' system, plays the crucial part in making it possible for children to learn how to connect number with the counting system. This system makes it possible for infants to recognise and represent very small numbers exactly (not approximately like the analog system). The system only operates for sets of 1, 2 and 3 objects and even within this restricted scope there is marked development over children's first three years.

Initially the system allows very young children to recognise sets of 1 as having a distinct quantity. The child understands 1 as a quantity, though he or she does not at first know that the word 'one' applies to this quantity. Later on the child is able to discriminate and recognise – in Carey's words 'to individuate' – sets of 1 and 2 objects, and still later, around the age of three- to four-years, sets of 1, 2 and 3 objects as distinct quantities. In Carey's terms young children progress from being 'one-knowers' to becoming 'two-knowers' and then 'three-knowers'.

During the same period, these children also learn number words and, though their recognition of 1, 2 and 3 as distinct quantities does not in any way depend on this verbal learning, they do manage to associate the right count words ('one', 'two' and 'three') with the right quantities. This association between parallel individuation and the count list eventually leads to what Carey (2004) calls 'bootstrapping': the children lift themselves up by their own intellectual bootstraps. They do so, some time in their fourth or fifth year, and therefore well before they go to school.

This bootstrapping takes two forms. First, with the help of the constant order of number words in the count list, the children begin to learn about the ordinal properties of numbers: 2 always comes after 1 in the count list and is always more numerous than 1, and 3 is more numerous than 2 and always follows 2. Second, since the fact that the count list that the children learn goes well beyond 3, they eventually infer that the number words represent a continuum of distinct quantities which also stretches beyond 'three'. They also begin to understand that the numbers above three are harder to discriminate from each other at a glance than sets of 1, 2 and 3 are, but that they can identify by counting. In Carey's words 'The child ascertains the meaning of 'two' from the resources that underlie natural language quantifiers, and from the system of parallel individuation, whereas she comes to know the meaning of 'five' through the bootstrapping process – i.e., that 'five' means one more than four, which is one more than three – by integrating representations of natural language quantifiers with the external serial ordered count list'. Carey called this new understanding 'enriched parallel individuation' (Carey, 2004; p. 65).

Carey's main evidence for parallel individuation and enriched parallel individuation came from studies in which she used a task, originally developed by Wynn, called 'Give – a number'. In this, an experimenter asks the child to give her a certain number of objects from a set of objects in front of them: 'Could you take two elephants out of the bowl and place them on the table?' Children sometimes put out the number asked for and sometimes just grab objects apparently randomly. Using this task Carey showed that different three-, four- and five-year-old children can be classified quite convincingly as 'one-' 'two-' or 'three-knowers' or as 'counting-principle-knowers'. The one-knowers do well when asked to provide one object but not when asked the other numbers while the two- and three-knowers can respectively provide up to two and three objects successfully. The 'counting-principle-knowers' in contrast count quantities above three or four.

The evidence for the existence of these three groups certainly supports Carey's interesting idea of a radical developmental change from 'knowing' some small quantities to understanding that the number system can be extended to other numbers in the count list. The value of her work is that it shows developmental changes in children's learning about the counting system. These had been by-passed both by Piaget and his colleagues because their theory was about the underlying logic needed for this learning and not about counting itself, and also by Gelman, because her theory about counting principles was about innate or rapidly acquired structures and not about development. However, Carey's explanation of children's counting in terms of enriched parallel individuation suffers the limitation that we have mentioned already: it has no proper measure of children's understanding of cardinality in its full sense. Just knowing that the last number that you counted is the number of the set is not enough.

The third way in which children learn number, according to Carey's theory, is through a system which she called 'set-based quantification': this is heavily dependent on language and particularly on words like 'a' and 'some' that are called 'quantifiers'. Thus far the implications of this third hypothesised system for education are not fully worked out, and we shall not discuss it further.

Carey's theory has been subjected to much criticism for the role that it attributes to induction or analogy in the use of the 'next' principle and to language. Gelman and Butterworth (2005), for example, argue that groups that have very restricted number language still show understanding of larger quantities; their number knowledge is not restricted to small numerosities as suggested in Carey's theory. Rips, Asmuth and Bloomfield (2006; 2008) address it more from a theoretical standpoint and argue that the bootstrapping hypothesis presupposes the very knowledge of number that it attempts to explain. They suggest that, in order to apply the 'next number' principle, children would have to know already that 1 is a set included in 2, 2 in 3, and 3 in 4. If they already know this, then they do not need to use the 'next number' principle to learn about what number words mean.

Which of these approaches is right? We do not think that there is a simple answer. If you hold, as we do, that understanding number is about understanding an ordered set of symbols that represent quantitative relations, Piaget's approach definitely has the edge. Both Gelman's and Carey's theory only address the question of how children give meaning to number words: neither entertains the idea that numbers represent quantities and relations between quantities, and that it is necessary for children to understand this system of relations as well as the fact that the word 'five' represents a set with 5 items in order to learn mathematics. Their research did nothing to dent Piaget's view that children of five years and six years are still learning about very basic relations between quantities, sometimes quite slowly.

Summary

1 Piaget's studies of learning about number concentrated on children's ability to reason logically about quantitative relations, and bypassed their acquisition of the counting system.

2 In contrast many current theories concentrate on children learning to count, and omit children's reasoning about quantitative relations. The most notable omission in these theories is the question of children's understanding of cardinality.

3 Gelman's studies of children's counting, nevertheless, did establish that even very young children systematically obey some basic counting principles when they do count.

Ordinal number

Numbers, as we have noted, come in a fixed order, and this order represents a quantitative series. Numbers are arranged in an ascending scale: 2 is more than 1 and 3 more than 2 and so on. Also the next number in the scale is always 1 more than the number that precedes it. Ordinal numbers indicate the position of a quantity in a series.

Piaget developed much the same argument about ordinal number as about cardinal number. He claimed that children learn to count, and therefore to produce numbers in the right fixed order, long before they understand that this order represents an ordinal series. This claim about children's difficulties with ordinality was based on his experiments on 'seriation' and also on 'transitivity'.

In his 'seriation' experiments, Piaget and his colleagues (Piaget, 1952) showed children a set of sticks all different in length and arranged in order from smallest to largest, and then jumbled them up and asked the child to re-order them in the same way. However, the children were asked to do so not by constructing the visual display all at once, which they would be able to do perceptually and by trial-and-error, but by giving the sticks to the experimenter one by one, in the order that they think they should be placed.

This is a surprisingly difficult task for young children and, at the age range that we are considering here (five- to six-years), children tend to form groups of ordered sticks instead of creating a single ordered series. Even when they do manage to put the sticks into a proper series, they tend then to fail an additional test, which Piaget considered to be the acid test of ordinal understanding: this was to insert another stick which he then gave them into its correct place in the already created series, which was now visible. These difficulties, which are highly reliable and have never been refuted or explained away, are certainly important, but they may not be true of number. Children's reactions to number series may well be different precisely because of the extensive practice that they have with producing numbers in a fixed order.

Recently, however, Brannon (2002) made the striking claim that even one-year-old-children understand ordinal number relations. The most direct evidence that Brannon offered for this claim was a study in which she showed the infant sequences of three cards, each of which depicted a different number of squares. Each three-card sequence constituted either an increasing or a decreasing series. In some sequences the number of squares increased from card to card e.g. 2, 4, 8 and 3, 6, 12: in others the numbers decreased e.g. 16, 8, 4.

Brannon's results suggested that 11-month-old infants could discriminate the two kinds of sequence (after seeing several increasing sequences they were more interested in looking at a decreasing than at yet another increasing sequence, and vice versa), and she concluded that even at this young age children have some understanding of seriation.

However, her task was a very weak test of the understanding of ordinality. It probably shows that children of this age are to some extent aware of the relations 'more' and 'less', but it does not establish that the children were acting on the relation between all three numbers in each sequence.

The point here is that in order to understand ordinality the child must be able to co-ordinate a set of 'more' and 'less' relations. This means understanding that *b* is smaller than *a* and at the same time larger than *c* in an $a > b > c$ series. Piaget was happy to accept that even very young children can see quite clearly that *b* is smaller than a and at another time that it is larger than *c*, but he claimed that in order to form a series children have to understand that intermediary quantities like *b* are *simultaneously* larger than some values and smaller than others. Of course, Brannon did not show whether the young children in her study could or could not grasp these two-way relations.

Piaget's (1921) most direct evidence for children's difficulties with two-way relations came from another kind of task – the transitivity task. The relations between quantities in any ordinal series are transitive. If A > B and B > C, then it follows that A > C, and one can draw this logical conclusion without ever directly comparing A with C. This applies to number as well: since 8 is more than 4 and 4 more than 2, 8 is more than 2.

Piaget claimed that children below the age of roughly eight years are unable to make these inferences because they find it difficult to understand that B can be simultaneously smaller than one quantity (A) and larger than another (C). In his experiments on transitivity Piaget did find that children very rarely made the indirect inference between A and C on the basis of being shown that A > C and B > C, but this was not very strong evidence for his hypothesis because he failed to check the possibility that the children failed to make the inference because they had forgotten the premises – a reason which has nothing directly to do with logic or with reasoning.

Subsequent studies, in which care was taken to check how well the children remembered the premises at the time that they were required to make the A > C inference (Bryant and Trabasso, 1971; Bryant and Kopytynska, 1976 ; Trabasso, 1977) consistently showed that children of five years or older do make the inference successfully, provided that they remember the relevant premises correctly. Young children's success in these tasks throws some doubt on Piaget's claim that they do not understand ordinal quantitative relations, but by and large there is still a host of unanswered questions about children's grasp of ordinality. We shall return to the issue of transitivity in the section on Space and Geometry.

Above all we need a comprehensive set of seriation and transitivity experiments in which the quantities are numbers (discontinuous quantities), and not continuous quantities like the rods of different lengths that have been the staple diet of previous work on these subjects.

Summary

1 The count list is arranged in order of the magnitude of the quantities represented by the numbers. The relations between numbers in this series are transitive: if A > B and B > C, then A > C.

2 Piaget argued that young children find ordinal relations, as well as cardinal relations, difficult to understand. He attributed these difficulties to an inability, on the part of young children, to understand that, in an A > B > C series, B is simultaneously smaller than A and larger than B.

3 Piaget's evidence for this claim came from studies of seriation and transitivity. The difficulties that children have in the seriation experiment, in which they have to construct an ordered series of sticks, are surprising and very striking.

4 However, the criterion for constructing the series in the seriation experiment (different lengths of some sticks) cannot be applied by counting. Therefore, seriation studies do not deal directly with children's understanding of natural number. The question of the seriation of number is still an open one.

Cardinality, additive reasoning and extensive quantities

So far we have discussed how children give meaning to number and how easy or difficult it is for them to make connections between what they understand about quantity and the numbers that they learn when they begin to count. Now we turn to another aspect of cardinal number, its connection with addition and subtraction – or, more generally, with additive reasoning. There are undeniable connections between the concept of cardinality and additive reasoning and we shall explore them in this section, which is about the additive composition of number, and in the subsequent section, which is about the inverse relations between addition and subtraction.

Piaget (1952) included in his definition of children's understanding of number their realisation that a quantity (and its numerical representation) is only changed by addition or subtraction, not by other operations such as spreading the elements or bunching them together. This definition, he indicated, is valid for the domain of extensive quantities, which are measured by the addition of units because the whole is the sum of the parts. If the quantity is made of discrete elements (e.g. a set of coins), the task of measuring it and assigning a number to it is easy: all the children have to do is to count. If the quantity is continuous (e.g. a ribbon), the task of measuring it is more difficult: normally a conventional unit would be applied to the quantity and the number that represents the quantity is the number of iterations of these units. Extensive quantities differ from intensive quantities, which are measured by the ratio between two other quantities. For example, the concentration of a juice is measured by the ratio between amount of concentrate and amount of water used to make the juice. These quantities are considered in Paper 4.

His studies of children's understanding of the conservation of quantities have been criticised on methodological grounds (e.g. Donaldson, 1978; Light, Buckingham and Robbins, 1979; Samuel and Bryant, 1984) but, so far as we know, his idea that children must realise that extensive quantities change either by addition or by subtraction has not been challenged.

Piaget (1952) also made the reasonable suggestion that you cannot understand what 'five' is unless you also know that it is composed of numbers smaller than it. Any set of five items contains a sub-set of 4 items and another sub-set of 1, or one sub-set of 3 and another of 2. A combination of or, in other words, an addition of each of these pairs of sets produces a set of five. This is called the additive composition of number, which is an important aspect of the understanding of relations between numbers.

Piaget used the idea of class-inclusion to describe this aspect of number; others (e.g. Resnick and Ford, 1981) have called it part-whole relations. Piaget's studies consisted in asking children about the quantitative relations between classes, one of which was included in the other. For example, in some tasks children were asked to compare the number of dogs with the number of animals in sets which included other animals, such as cats. For an adult, there is no need to know the actual number of dogs, cats, and animals in such a task: there will be always more animals than dogs because the class of dogs is included in the class of animals. However, some children aged four- to six-years do not necessarily think like adults: if the number of dogs is quite a bit larger than the number of cats, the children might answer that there are more dogs than animals. According to Piaget, this answer which to an adult seems entirely illogical, was the result of the children's difficulties with thinking of the class of dogs as simultaneously included in the class of animals and excluded from it for comparison purposes. Once they mentally excluded the dogs from the set of animals, they could no longer think of the dogs as part of the set of animals: they would then be unable to focus on the fact that the whole (the overall class, animals) is always larger than one part (the included class, dogs).

Piaget and his colleagues (Piaget, 1952; Inhelder, Sinclair and Bovet, 1974) did use a number of conditions to try to eliminate alternative hypotheses for children's difficulties. For example, they asked the children whether in the whole world there would be more dogs or more animals. This question used the same linguistic format but could be answered without an understanding of the necessary relation between a part and a whole: the children could think that there are many types of animals in the world and therefore there say that there are more animals than dogs. Children are indeed more successful in answering this question than the class-inclusion one. Another manipulation Piaget and his colleagues used was to ask the children to circle with a string the dogs and the animals: this had no effect on the children's performance, and they continued to exclude the dogs mentally from the class of animals. The only manipulation that helped the children was to ask the children to first think of the set of animals without separating out the dogs, then replace the dogs with visual representations that marked their inclusion in the class of animals, while the dogs were set in a separate class: the children were then able to create a simultaneous representation of the dogs included in the whole and as a separate part and answer the question correctly. After having answered the question in this situation, some children went on to answer it correctly when other class-inclusion problems were presented (for example, about flowers and roses) without the support of the extra visual signs.

The Piagetian experiments on class-inclusion have been criticised on many grounds: for example, it has been argued that the question the children are asked is an anomalous question because it uses disjunction (dogs *or* animals) when something can be simultaneously a dog and *an* animal (Donaldson, 1978; Markman, 1979). However, Piaget's hypothesis that part-whole relations are an important aspect of number understanding has not been challenged. As discussed in the previous sections, it has been argued (e.g. by Rips, Asmuth and Bloomfield, 2008) that it is most unlikely that a child will understand the ordinality of number until she has grasped the connection between the next number and the plus-one compositions: i.e. that a set of 5 items contains a set of 4 items plus a set of 1, and a set of 4 items is composed of a set of 3 plus 1, and so on.

For exactly the same reasons, the understanding of additive composition of number is essential in any comparison between two numbers. To judge the difference, for example, between 7 and 4, something which as we shall see is not always easy for young children, you need to know that 7 is composed of 4 and 3, which means that 7 is 3 greater than 4.

Of course, even very young children have a great deal of informal experience of quantities increasing or decreasing as a result of additions and subtractions. There is good evidence that pre-schoolers do understand that additions increase and subtractions decrease quantities (Brush, 1978; Cooper, 1984; Klein, 1984) but this does not mean that they realise that the only changes that affect quantity are addition and subtraction. It is possible that their understanding of these changes is qualitative in the sense that it lacks precision. We can take as an example what happens when young children are shown two sets that are unequal and are arranged in one-to-one correspondence, as in Figure 2.1, so that it is possible for the children to see the size of the difference (say one set has 10 objects and the other 7). The experimenter proposes to add to the smaller set fewer items than the difference (i.e. she proposes to add 2 to the set with 7). Some preschoolers judge that the set to which elements are added will become larger than the other. Others think that it is now the same as the other set. It is only at about six- or seven-years that children actually take into account the precise difference between the sets in order to know whether they will be the same or not after the addition of items to the smaller set (Klein, 1984; Blevins-Knabe, Cooper, Mace and Starkey, 1987).

The basic importance of the additive composition of number means that learning to count and learning to add and subtract are not necessarily two successive and separate intellectual steps, as common sense might suggest. At first glance, it seems quite a plausible suggestion that children must understand number and know about the counting system in order to do any arithmetic, like adding and subtracting. It seems simply impossible that they could add 6 and 4 or subtract 4 from 6 without knowing what 6 and 4 mean. However, we have now seen that this link between counting and arithmetic must work in the opposite direction as well, because it is also impossible that children could know what 6 or 4 or any other number mean, or anything about the relations between these numbers, without also understanding something about the additive composition of number.

Given its obvious importance, there is remarkably little research on children's grasp of additive composition of number. The most relevant information, though it is somewhat indirect, comes from the well-known developmental change from 'counting-all' to 'counting-on'. As we have seen, five-year-old children generally know how to count the number of items in a set within the constraints of one-to-one counting.

However, their counting is not always economic. If, for example, they are given a set of 7 items which they duly count and then 6 further items are added to this set and the children are asked about the total number in the newly increased set, they tend to count all the 13 items in front of them including the subset that they counted before. Such observations have been replicated many times (e.g. Fuson, 1983; Nunes and Bryant, 1996; Wright, 1994) and have given origin to a widely used analysis of children's progress in understanding cardinality (e.g. Steffe, Cobb and von Glaserfeld, 1988; Steffe, Thompson and Richards, 1982; Steffe, von Glasersfeld, Richards and Cobb, 1983). This counting of all the items is not wrong, of course, but the repeated counting of the initial items is unnecessary. The children could just as well and much more efficiently have counted on from the initial set (not '1, 2, 3,…..13' but '8, 9, 10…..13').

According to Vergnaud (2008), the explanation for children's uneconomical behaviour is conceptual: as referred in the previous section, he argues that their understanding of number may be simply *ordinal* (i.e. what they know is that the last number word represents the set) and so they cannot add because ordinal numbers cannot be added. They can, however, count a new, larger set, and give to it the label of the last number word used in counting.

Studies of young children's reactions to the kind of situation that we have just described have consistently produced two clear results. The first is that young children of around the age of five years consistently count all the items. The second is that between the ages of five and seven years, there is a definite developmental shift from counting-all to counting-on: as children grow older they begin to adopt the more economic strategy of counting-on from the previously counted subset. This new strategy is a definite sign of children's eventual recognition of the additive composition of the new set: they appear to understand that the total number of the new set will contain the original 7 items plus the newly added 6 items. The fact that younger children stick to

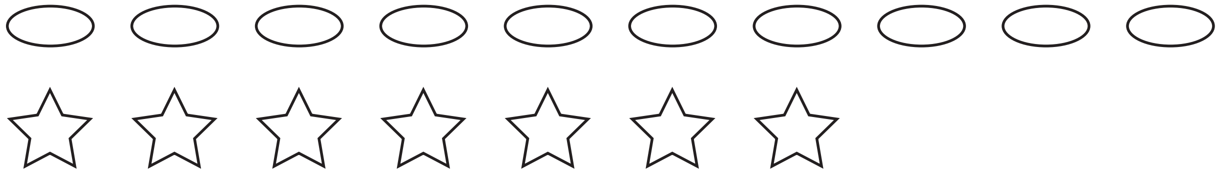

Figure 2.1: Two sets in correspondence; the difference between sets is easily seen.

counting-all does not establish that they cannot understand the additive composition of the new set (as is often the case, it is a great deal easier to establish that children do understand some principle than that they do not). However, the developmental change that we have just described does suggest an improvement in children's understanding of additive relations between numbers during their first two years at school.

The study of the connections that children make, or fail to make, between understanding number, additive composition and additive reasoning plainly supports the Piagetian thesis that children give meaning to numbers by establishing relations between quantities though their schemes of action. They do need to understand that addition increases and subtraction decreases the number of items in a set. This forms a foundation for their understanding of the precise way in which the number changes: adding 1 to set a creates a number that is equal to $a + 1$. This number can be seen as a whole that includes the parts a and 1. Instead of relying on the 'next number' induction or analogy, children use addition and the logic of part-whole to understand numbers.

Summary

1 In order to understand number as an ordinal series, children have to realise that numbers are composed of combinations of smaller numbers.

2 This realisation stems from their progressive understanding of how addition affects number: at first they understand that addition increases the number of items in a set without being precise about the extent of this increase but, as they co-ordinate their knowledge of addition with their understanding of part-whole relations, they can also become more precise about additive composition.

3 Young children's tendency to count-all rather than to count-on suggests either that they do not understand the additive composition of number or that their grasp of additive composition is too weak for them to take advantage of it.

4 Their difficulties suggest that children should be taught about additive composition, and therefore about addition, as they learn about the counting system.

The decade structure and additive composition

Additive composition and the understanding of number and counting are linked for another important reason. The power and the effectiveness of counting rest largely on the invention of base systems, and these systems depend on additive composition. The base-10 system, which is now widespread, frees us from having to remember long strings of numbers, as indeed any base system does. In English, once we know the simple rules for the decimal system and remember the number words for 1 to 20, for the decades, and then for a hundred, a thousand and a million, we can generate most of the natural numbers that we will ever need to produce with very little effort or difficulty. The link between understanding additive composition and adopting a base system is quite obvious. Base systems rest on the additive composition of number and the decade structure is in effect a clear reminder that 'fourteen' is a combination of 10 and 4, and 'thirty-five' of three 10s and 5.

Additive composition is the basic concept that underlies any counting system with a base, oral or written. This includes of course the Hindu-Arabic place-value system that we use to write numbers. For example, the decimal system explicitly represents the fact that all the numbers between 10 and a 100 must be a combination of one or more decades and a number less than 10: 17 is a combination of 10 and 7 and 23 a combination of two 10s and 3. The digits express the additive composition of any number from 10 on: e.g. in 23, 2 represents two 10s which are added to 3, which represents three units.

Additive composition is also at the root of our ability to count money using coins and notes of different denominations. When we have, for example, one 10p and five 1p coins, we can only count the 10p and the 1p coins together if we understand about additive composition.

The data from the 'shop task', a test devised by Nunes and Schliemann (1990), suggest that initially children find it hard to combine denominations in this way. In the shop task children are shown a set of toys in a 'shop', are given some (real or artificial) money and are asked to choose a toy that they would like to buy. Then the experimenter asks them to pay a certain sum for their choice. Sometimes the child can pay for this with coins of one denomination only: for example, the experimenter charges a child 15p for a toy car and the child has that number of pence to make the purchase or the charge is 30p and the child can pay with three 10p coins. In other trials, the child can pay only by combining denominations: the car costs 15p and the child, having fewer 1p coins than that, must pay with the combination of a 10p and five 1p coins. Although the values that the children are asked to pay when they use only 10p coins are larger than those they pay when using combinations of different values, children can count in tens (ten, twenty, thirty etc.) using simple correspondences between the counting labels and the coins. This task does not require understanding additive composition. So Nunes and Schliemann predicted that the mixed denomination trials would be significantly more difficult than the other trials in the task. They found that the mixed denominations trials were indeed much harder for the children than the single denomination trials and that there was a marked improvement between the ages of five and seven years in children's performance in the combined denomination trials. This work was originally carried out in Brazil and the results have been confirmed in other research in the United Kingdom (Krebs, Squire and Bryant, 2003; Nunes *et al*., 2007).

A fascinating observation in this task is that children don't change from being unable to carry out the additive composition to counting on from ten as they add 1p coins to the money they are counting. The show the same count-all behaviour that they show when they have a set of objects and more object are added to the set. However, as there are no visible 1p coins within the 10p, they point to the 10p ten times as they count, or they lift up 10 fingers and say 'ten', and only then go on to count 'eleven, twelve, thirteen etc.' This repeated pointing to count invisible objects has been documented also by Steffe and his colleagues (e.g. Steffe, von Glasersfeld, Richards and Cobb, 1983), who interpreted it, as we do, as a significant step in coordinating counting with a more mature understanding of cardinality.

In a recent training study (Nunes *et al*., 2007), we encouraged children who did not succeed in the shop-task to use the transition behaviour we had observed, and asked them to show us ten with their fingers; we then pointed to their fingers and the 10p coin and asked the children to say how much there was in each display; finally, we encouraged them to go on and count the money. Our study showed that some children seemed to be able to grasp the idea of additive composition quite quickly after this demonstration and others took some time to do so, but all children benefited significantly from brief training sessions using this procedure.

Since children appear to be finding out about the additive nature of the base-10 system and at the same time (their first two years at school) about the additive composition of number in general, one can reasonably ask what the connection between these two is. One possibility is that children must gain a full understanding of the additive composition of number before they can understand the decade structure. Another is that instruction about the decade structure is children's first entrée to additive composition. First they learn that 12 is a combination of 10 and 2 and then they extend this knowledge to other combinations (e.g. 12 is also a combination of 8 and 4). The results of a recent study by Krebs, Squire and Bryant (2003), in which the same children were given the shop task and counting all/counting on tasks, favour the second hypothesis. All the children who consistently counted on (the more economic strategy) also did well on the shop task, but there were some children who scored well in the shop task but nevertheless tended not to count on. However, no child scored well in the counting all/on task but poorly in the shop task. This pattern suggests that the cues present in the language help children learn about the decimal system first and then extend their new understanding of additive composition to combinations that do not involve decades.

Summary

1 The decimal system is a good example of an invented and culturally transmitted mathematical tool. It enhances our power to calculate and frees us from having to remember extended sequences of number.

2 Once we know the rules for the decade system and the names of the different classes and orders (tens, hundreds, thousands etc.), we can use the system to count by generating numbers ourselves.

3 However, the system also makes some quite difficult intellectual demands. Children find it hard at first to combine different denominations, such as tens and ones.

4 Teachers should be aware that the ability to combine denominations rests on a thorough grasp of additive composition.

5 There is some evidence that experience with the structure of the decimal system may enhances children's understanding of additive composition. There is also evidence that it is possible to use money to provoke children's progress in understanding additive composition.

The inverse relation between addition and subtraction

The research we have considered so far suggests that by the age of six or seven children understand quite a lot about number: they understand equivalence well enough to know that if two sets are equivalent they can infer the number that describes one by counting the other; they understand that addition and subtraction are the operations that change the number in a set; they understand additive composition and what must be added to one set to make it equivalent to the other; they understand that they can count on if you add more elements to a set; and they understand about ordinal number and can make transitive inferences. However, there is an insight about how addition and subtraction affect the number of elements in a set that we still need to consider. This is the insight that addition is the inverse of subtraction and vice versa, and thus that equal additions and subtractions cancel each other out: 27 + 19 – 19 = 27 and 27 – 19 + 19 = 27.

It is easy to see that one cannot understand either addition or subtraction or even number fully without also knowing about the inverse relation of each of these operations to the other. It is absolutely essential when adding and subtracting to understand that these are reversible actions. Otherwise one will not understand that one can move along the number scale in two opposite directions – up and down.

The understanding of any inverse relation should, according to Piaget (1952), be particularly hard for young children, since in his theory young children are not able to carry out 'reversible' thought processes. Children in the five- to eight-year range do not see that if 4 + 8 = 12, therefore 12 – 8 = 4 because they do not realise that the original addition (+8) is cancelled out by the inverse subtraction (–8). This claim is a central part of Piaget's theory about children's arithmetical learning, but he never tested it directly, even though it would have been quite easy to do so.

In one of his last publications, Piaget and Moreau (2001) did report an ingenious, but rather too complicated, study of the inverse relations between addition and subtraction and also between multiplication and division. They asked children, aged from six- to ten-years, to choose a number but not to tell them what this was. Then they asked the child first to add 3 to this number, next to double the sum and then to add 5 to the result of the multiplication. Next, they asked the child what the result was, and went on to tell him or her what was the number that s/he chose to start with. Finally the experimenters asked each child to explain how they had managed to work out what this initial number was.

Piaget and Moreau reported that this was a difficult task. The youngest children in the sample did not understand that the experimenters had performed the inverse operations, subtracting where the child had added and dividing where s/he had multiplied. The older children did show some understanding that this was how the experimenters reached the right number, but did not understand that the order of the inverse operations was important. The experimenters accounted for the younger children's difficulties by arguing that these children had failed to understand the adult's use of inversion (equal additions and subtractions and equal multiplications and divisions) because they did not understand the principle of inversion.

This was a highly original study but Piaget and Moreau's conclusions from it can be questioned. One alternative explanation for the children's difficulties is that they may perfectly have understood the inverse relations between the different operations, but they may still not have been able to work out how the adult used them to solve the problem. The children, also, had to deal with two kinds of inversion (addition-subtraction and multiplication-division) in order to explain the adult's correct solution, and so their frequent failures to produce a coherent explanation may have been due to their not knowing about one of the inverse relations, e.g. between multiplication and division, even though they were completely at home with the other, e.g. between addition and subtraction.

Nevertheless, some following studies seemed to confirm that young school-children are often unaware aware that inverse transformations cancel each other out in $a + b - b$ sums. In two studies (Bisanz and Lefevre, 1990; Stern, 1992), the vast majority of the younger children did no better with inverse $a + b - b$ sums in which they could take advantage of the inversion principle than with control $a + b - c$ sums where this was not possible. For example, Stern reported that only 13% of the seven-year-old children and 48% of the nine-year-olds in her study used the inversion principle consistently, when some of the problems that they were given were $a + b - b$ sums and others $a + b - c$ sums.

This overall difficulty was confirmed in a further study by Siegler and Stern (1998), who gave eight-year-old German children inversion problems in eight successive sessions. Their aim was to see whether the children improved in their use of the inversion principle to solve problems. The children were also exposed to other traditional scholastic problems (e.g. $a + b - c$), which could not be solved by using the inverse principle. In the last of the eight sessions, Siegler and Stern also gave the children control problems, which involved sequences such as $a + b + b$, so that the inversion principle was not appropriate for solution. The experimenters recorded how well children distinguished the problems that could be solved through the inversion principle from those that had to be solved in some other way.

The study showed that the children who were given lots of inversion problems in the first seven sessions tended to get better at solving these problems over these sessions, but in the final session in which the children were given control as well as inversion problems they often, quite inappropriately, overgeneralised the inversion strategy to the control sums: they would give a as the answer in $a + b + b$ control problems as well as in inverse $a + b - b$ problems. Their relatively good performance with the inversion problems in the previous sessions, therefore, was probably not the result of an increasing understanding of inversion. They seem to have learned some lower-level and totally inadequate strategy, such as 'if the first number (a) is followed by another number (b) which is then repeated, the answer must be a'.

The pervasive failures of the younger school children in these studies to take advantage of the inversion principle certainly suggest that it is extremely difficult for them to understand and to learn how to use this principle, as Piaget first suggested. However, in all these tasks the problems were presented either verbally or in written form. Other studies, which employed sets of physical objects, paint a different picture. (Bryant, Christie and Rendu, 1999; Rasmussen, Ho and Bisanz, 2003). For example, Bryant *et al*. used sets of bricks to present five- and six-year-old children with $a + b - b$ inversion problems and $a + a - b$ control problems. In this particular task young children did a great deal better with the inversion problems than with the control problems, which is good evidence that they were using the inversion principle when they could. In the same study the children were also given equivalent inversion and control problems as verbal sums (27 + 14 – 14): they used the inversion principle much less often in this task than in the task with bricks, a result which resonates well with Hughes' (1981) discovery that pre-school children are much more successful at working out the results of additions and subtractions in problems that involve concrete objects than in abstract, verbal sums.

The fact that young children are readier to use the inversion principle in concrete than in abstract problems suggests that they may learn about inversion initially through their actions with concrete material. Bryant *et al*. raised this possibility, and they also made a distinction between two levels in the understanding of the inverse relation between addition and subtraction. One is the level of *identity*: when identical stuff is added to and then subtracted from an object, the final state of this object is the same as the initial state. Young children have many informal experiences of inverse transformations at this level. A child gets his shirt dirty (mud is added to it) and then it is cleaned (mud is subtracted) and the shirt is as it was before. At meal-times various objects (knives, forks etc.) are put on the dining room table and then subtracted when the meal is over; the table top is as empty after the meal, as it was before.

Note that understanding the inversion of identity may not involve quantity. The child can understand that, if the same (or identical) stuff is added and then removed, the *status quo* is restored without having to know anything about the quantity of the stuff.

The other possible level is the understanding of the inversion of *quantity*. If I have 10 sweets and someone gives me 3 more and then I eat 3, I have the same number left as at the start, and it doesn't matter whether the 3 sweets that I ate are the same 3 sweets as were given to me or different ones. Provided that I eat the same number as I was given, the quantitative *status quo* is now restored.

In a second study, again using toy bricks, Bryant *et al*. established that five- and six-year-old children found problems, called identity problems, in which exactly the same bricks were added to and then subtracted from the initial set (or vice versa), easier than other problems, called quantity problems, in which the same number of bricks was added and then subtracted (or vice versa), but the actual bricks subtracted were quite different from the bricks that had been added before. Bryant *et al*. also found a greater improvement with age in children's performance in quantity inversion problems than in identity inversion problems. These results point to a developmental hypothesis: children's understanding of the inversion of identity precedes, and may provide the basis for, their understanding of the inversion of quantity. First they understand that adding and subtracting the same stuff restores the physical *status quo*. Then they extend this knowledge to quantity, realising now that adding and subtracting the same quantity restores the quantitative *status quo*, whether the addend and subtrahend are the same stuff or not.

However, the causal determinants of learning about inversion might vary between children. Certainly there are many reports of substantial individual differences within the same age groups in the understanding of the inversion principle. Many of the seven- and nine-year-olds in Bisanz and LeFevre's study (1990) used the inversion principle to solve appropriate problems but over half of them did not. Over half of the ten-year-olds tested in Stern's (1992) original study did take advantage of the principle, but around 40% seemed unable to do so.

Recent work by Gilmore (Gilmore and Bryant, 2006; Gilmore and Papadatou-Pastou, 2008) suggests that the underlying pattern of these individual differences might take a more complex and also a more interesting form than just a dichotomy between those who do and those who do not understand the inversion principle. She used cluster analysis with samples of six- to eight-year-olds who had been given inversion and control problems (again the control problems had to be solved through calculation), and consistently found three groups of children. One group appeared to have a clear understanding of inversion and good calculation skills as well; these children did better in the inversion than in the control problems, but their scores in the control problems were also relatively high. Another group consisted of children who seemed to have little understanding of inversion and whose calculation skills were weak as well. The remaining group of children had a good understanding of inversion, but their calculation skills were weak: in other words, these children did better in the inversion than in the control problems, but their scores in the control problems were particularly low. Thus, the discrepancy between knowing about inversion and knowing how to calculate went one way but not the other. Gilmore identified a group of children who could use the inversion principle and yet did not calculate well, but she found no evidence at all for the existence a group of children who could calculate well but were unable to use the inversion principle. Children, therefore, do not have to be good at adding and subtracting in order to understand the relation between these two operations. On the contrary, they may need to understand the inverse relation before they can learn to add and subtract efficiently.

How can knowledge of inversion facilitate children's ability to calculate? Our answer to this is only hypothetical at this stage, but it is worth examining here. If children understand well the principle of inversion, they may use their knowledge of number facts more flexibly, and thus succeed in more problems where calculation is required than children who cannot use their knowledge flexibly. For example, if they know that $9 + 7 = 16$ and understand inversion, they can use this knowledge to answer two more questions: $16 – 9 = ?$ and $16 – 7 = ?$ Similarly, the use of 'indirect addition' to solve difficult subtraction problems depends on knowing and using the inverse relation between addition and subtraction. One must understand inversion to be able to see, for example, that an easy way to solve the subtraction $42 – 39$ is to convert it into an addition: the child can count up from 39 to 42, find that this is 3, and will then know that $42 – 39$ must equal 3. In our view, no one could reason this way without also understanding the inverse relation between addition and subtraction.

If this hypothesis is correct, it has fascinating educational implications. Children spend much time at home and in school practising number facts, perhaps trying to memorise them as if they were independent of each other. However, a mixture of learning about number facts and about mathematical principles that help them relate one number fact to others, such as inversion, could provide them with more flexible knowledge as well as more interesting learning experiences. So far as we know, there is no direct evidence of how instruction that focuses both on number facts and principles works in comparison with instruction that focuses only on number facts. However, there is some preliminary evidence on the role of inversion in facilitating children's understanding of the relation between the sum $a + b = c$ and $c – b$ (or $– a$) $= ?$

Some researchers have called this 'the complement' question and analysed its difficulty in a quite direct way by telling children first that $a + b = c$ and then immediately asking them the $c - a = ?$ question (Baroody, Ginsburg and Waxman, 1983; Baroody, 1999; Baroody and Tiilikainen, 2003; Resnick, 1983; Putnam, de Bettencourt and Leinhardt, 1990). These studies established that the step from the first to the second sum is extremely difficult for children in their first two or three years at school, and most of them fail to take it. Only by the age of about eight years do a majority of children use the information from the addition to solve the subtraction, and even at this age many children continue to make mistakes. Would they be able to do better if their understanding of the inverse relation improved?

The study by Siegler and Stern (1998) described earlier on, with eight-year-olds, seems to suggest that it is not that easy to improve children's understanding of the inverse relation between addition and subtraction: after solving over 100 inversion problems, distributed over 7 days, the children did very poorly in using it selectively; i.e. using it when it was appropriate, and not using it when it was not appropriate. However, the method that they used had several characteristics, which may not have facilitated learning. First, the problems were all presented simply as numbers written on cards, with no support of concrete materials or stories. Second, the children were encouraged to answer correctly and also quickly, if possible, but they did not receive any feedback on whether they were correct. Finally, they were asked to explain how they had solved the problem, but if they indicated that they had used the inverse relation to solve it, they were neither told that this was a good idea nor asked to think more about it if they had used it inappropriately. In brief, it was not a teaching study.

Recently we completed two studies on teaching children about the inverse relation between addition and subtraction (Nunes, Bryant, Hallett, Bell and Evans, 2008). Our aims were to test whether it is possible to improve children's understanding of the inverse relation and to see whether they would improve in solving the complement problem after receiving instruction on inversion.

One of the studies was with eight-year-olds, i.e. children of the same age as those who participated in the Siegler and Stern study. Our study was considerably briefer, as it involved a pre-test, two teaching sessions, and a post-test. In the pre-test the children answered inverse problems ($a + b - b$), control problems ($a + b - c$) and complement problems ($a + b = c$; $c - a/b = ?$). During the training, they only worked on inversion problems. So if the taught groups improved significantly on the complement problems, this would have to be a consequence of realising the relevance of the inverse principle to this type of problem.

For the teaching phase, the children were randomly assigned to one of three groups: a Control group, who only received practice in calculation; a Visual Demonstration group and a Verbal Calculator group, both receiving instruction on the inverse relation. The form of the instruction varied between the two groups.

The Visual Demonstration group was taught with the support of concrete materials, and started with a series of trials that took advantage of the identity inversion. First the children counted the number of bricks in a row of Unifix bricks, which was subsequently hidden under a cloth so that no counting was possible after that. Next, the experimenter added some bricks to the row and subtracted others. The child was then asked how many bricks were left under the cloth. The number of bricks added and subtracted was either the same or differed by one; this required the children to attend during all trials, as the answer was not in all examples the same number as before the additions and subtractions, but they could still use the inverse principle easily because the difference of one did not make the task too different from an exact inversion trial. When they had given their answer, they received feedback and explained how they had found the answer. If they had not used the inversion principle, they were encouraged to think about it (e.g. How many were added? How many were taken away? Would the number be the same or different?).

The Verbal Calculator group received the same number of trials but no visual demonstration. After they had provided their answer, they were encouraged to repeat the trial verbally as they entered the operations into a calculator and checked the answer. Thus they would be saying, for example, 'fourteen plus eight minus eight is' and looking at the answer.

As explained, we had three types of problems in the pre- and post-test: inversion, control and complement problems, which were transfer problems for our intervention group, as they had not learned about these directly during the training. We did not expect the groups to differ in the control trials, as the amount of experience they had between pre- and post-test was limited, but we expected the experimental teaching groups to perform significantly better than the Control group in the inversion and transfer problems.

The results were clear:

- Both taught groups made more progress than the Control group from the pre-test to the post-test in the inversion problems.
- The Visual Demonstration group made more progress than the Control group in the transfer problems; the Verbal Calculator group's improvement did not differ from the improvement in the Control group in the transfer problems.
- The children's performance did not improve significantly in the control problems in any of the groups.

Thus with eight-year-olds both Visual and Verbal methods can be used to promote children's reflection about the inverse relation between addition and subtraction. Although the two methods did not differ when directly compared to each other, they differed when compared with the fixed-standard provided by the control group: the Visual Demonstration method was effective in promoting transfer from the types of items used in the training to new types of items, of a format not presented during the training, and the Verbal method did not.

In our second teaching study, we worked with much younger children, whose mean age was just five years. We carried out the study using the same methods, with a pre-test, two teaching sessions, and a post-test, but this time all the children were taught using the Visual Demonstration method. Because the children were so young, we did not use complement problems to assess transfer, but we included a delayed post-test, given to the children about three weeks after they had completed the training in order to see whether the effects of the intervention, if any, would remain significant at a later date without further instruction.

The intervention showed significant effects for the children in one school but not for the children in the other school; the effects persisted until the delayed post-test was given. Although we cannot be certain, we think that the difference between the schools was due to the fact that in the school where the intervention did not have a significant effect we were unable to find a quite room to work with the children without interruptions and the children had difficulty in concentrating.

The main lesson from this second study was that it is possible for this intervention to work with such young children and for the effects to last without further instruction, but it is not certain that it will do so.

Finally, we need to consider whether knowing about inversion is really as important as we have claimed here. Two studies support this claim. The first was by Stern (2005). She established in a longitudinal study that German children's performance in inversion tasks, which they solved in their second year at school, significantly predicted their performance in an algebra assessment given about 15 years later, when they were 23 years old and studying in university. In fact, the brief inversion task that she gave to the children had a higher correlation with their performance in the algebra test than the IQ test given at about the same time as the inversion task. Partialling out the effect of IQ from the correlation between the inversion and the algebra tests did not affect this predictive relation between the inversion task and the algebra test.

The second study was by our own team (Nunes *et al*., 2007). It combined longitudinal and intervention methods to test whether the relation between reasoning principles and mathematics learning is a causal one. The participants in the longitudinal study were tested in their first year in school. In the second year, they completed the mathematics achievement tests administered by the teachers and designed centrally in the United Kingdom. The gap between our assessment and the mathematics achievement test was about 14 months. One of the components of our reasoning test was an assessment of children's understanding of the inverse relation between addition and subtraction; the others were additive composition (assessed by the shop task) and correspondence (in particular, one-to-many correspondence). We found that children's performance in the reasoning test significantly predicted their mathematics achievement even after controlling for age, working memory, knowledge of mathematics at school entry, and general cognitive ability. We did not report the specific connection between the inversion problems and the children's mathematics achievement in the original paper, so we report it here. We used a fixed-order regression analysis so that the connection between the inversion task and mathematics achievement could be considered after controlling for the children's age, general cognitive ability and working memory. The inversion task remained a significant predictor of the children's mathematics achievement, and explained 12% extra variance. This is a really remarkable result: 6 inversion problems given about 14 months before the mathematics achievement test made a significant contribution to predicting children's achievement after such stringent controls.

Our study also included an intervention component. We identified children who were underperforming in the logical assessment for their age at the beginning of their first year in school and created a control and an intervention group. The control children received no intervention and the intervention group received instruction on the reasoning principles for one hour a week for 12 weeks during the time their peers were participating in mathematics lessons. So they had no extra time on maths but specialised instruction on reasoning principles. We then compared their performance in the state-designed mathematics achievement tests with that of the control group. The intervention group significantly outperformed the control group. The mean for the control group in the mathematics assessment was at the 28th percentile using English norms; the intervention group's mean was just above the 50th percentile, i.e. above the mean. So a group of children who seemed at risk for difficulties with mathematics caught up through this intervention. In the intervention study it is not possible to separate out the effect of inversion; the children received instruction on three reasoning principles that we considered of great importance as a basis for their learning. It would be possible to carry out separate studies of how each of the three reasoning principles that we taught the children affects their mathematics performance but we did not consider this a desirable approach, as our view is that each one of them is central to children's mathematics learning.

The combination of longitudinal and intervention methods in the analysis of the causes of success and difficulties in learning to read is an approach that was extremely successful (Bradley and Bryant, 1983). The study by Nunes *et al*., (2007) shows that this combination of methods can also be used successfully in the analysis of how children learn mathematics. However, three caveats are called for here. First, the study involved relatively small samples: a replication with a larger sample is highly recommended. Second, it is our view that it is also necessary to attempt to replicate the results of the intervention in studies carried out in the classroom. Experimental studies, such as ours, provide a proof of existence: they show that it is possible to accomplish something under controlled conditions. But they do not show that it is possible to accomplish the same results in the classroom. The step from the laboratory to the classroom must be carefully considered (see Nunes and Bryant, 2006, for a discussion of this issue). Finally, it is clear to us that developmental processes that describe children's development when they do not have any special educational needs (they do not have brain deficiencies, for example, and have hearing and sight within levels that grant them access to information normally accessed by children) may need further analysis when we want to understand the development of children who do have special educational needs. We exemplify here briefly the situation of children with severe or profound hearing loss. The vast majority of deaf children are born to hearing parents (about 90%), who may not know how to communicate with their children without much additional learning. Mathematics learning involves logical reasoning, as we have argued, and also involves learning conventional representations for numbers. Knowledge of numbers can be used to accelerate and promote children's reflections about their schemes of action, and this takes place through social interaction. Parents teach children a lot about counting before they go to school (Schaeffer, Eggleston and Scott, 1974; Young-Loveridge, 1989) but the opportunities for these informal learning experiences may be restricted for deaf children. They would enter school with less knowledge of counting and less understanding of the relations between addition, subtraction, and number. This does not mean that they have to develop their understanding of numbers in a different way from hearing children, but it does mean that they may need to learn in a much more carefully planned environment so that their learning opportunities are increased and appropriate for their visual and language skills. In brief, there may be special children whose mathematical development requires special attention. Understanding their development may or may not shed light on a more general theory of mathematics learning.

Summary

1 The inversion principle is an essential part of additive reasoning: one cannot understand either addition or subtraction unless one also understands their relation to each other.

2 Children probably first recognise the inverse relation between adding and subtracting the identical stuff. We call this the inversion of identity.

3 The understanding of the inversion of quantity is a step-up. It means understanding that a quantity stays the same if the same number of items is added to it and subtracted from it, even though the added and subtracted items are different from each other.

4 The inversion of quantity is more difficult for young children to understand, but in tasks that involve concrete objects, many children in the five- to seven-year age range do grasp this form of inversion to some extent.

5 There are however strong individual differences among children in this form of understanding. Children in the five- to eight-year range fall into three main groups. Those who are good a calculating and also good at using the inversion principle, those who are weak in both things, and those who are good at using the inversion principle, but weak in calculating.

6 The evidence suggests that children's understanding of the inversion principle plays an important causal role in their progress in learning about mathematics. Children's understanding of inversion is a good predictor of their mathematical success, and improving this understanding has the result of improving children's mathematical knowledge in general.

Additive reasoning and problem solving

In this section we continue to analyse children's ability to solve additive reasoning problems. Additive reasoning refers to reasoning used to solve problems where addition or subtraction are the operations used to find a solutions. We prefer to use this expression, rather than addition and subtraction problems, because it is often possible to solve the same problem either by addition or by subtraction. For example, if you buy something that costs £35, you may pay with two £20-notes. You can calculate your change by subtraction (40 – 35) or by addition (35 + 5). So, problems are not addition or subtraction problems in themselves, but they can be defined by the type of reasoning that they require, additive reasoning.

Although preschoolers' knowledge of addition and subtraction is limited, as we argued in the previous section, it is clear that their initial thinking about these two arithmetical operations is rooted in their everyday experiences of seeing quantities being combined with, or taken away from, other quantities. They find purely numerical problems like 'what is 2 and 1 more?' a great deal more difficult than problems that involve concrete situations, even when these situations are described in words and left entirely to the imagination (Ginsburg, 1977; Hughes, 1981; 1986; Levine, Jordan and Huttenlocher, 1992).

The type of knowledge that children develop initially seems to be related to two types of action: putting more elements in a set (or joining two sets) and taking out elements from one set (or separating two sets). These schemes of action are used by children to solve arithmetic problems when they are presented in the context of stories.

By and large, three main kinds of story problem have been used to investigate children's additive reasoning:

- the *Change problem* ('Bill had eight apples and then he gave three of them away. How many did he have left?').
- the *Combine problem* ('Jane has three dolls and Mary has four. How many do they have altogether?').
- the *Compare problem* ('Sam has five books and Sarah has eight. How many more books does Sarah have than Sam?').

A great deal of research (e.g. Brown, 1981; Carpenter, Hiebert and Moser, 1981; Carpenter and Moser, 1982; De Corte and Verschaffel, 1987; Kintsch and Greeno, 1985; Fayol, 1992; Ginsburg, 1977; Riley, Greeno and Heller, 1983; Vergnaud, 1982) has shown that in general, the Change and Combine problems are much easier than the Compare problems. The most interesting aspect of this consistent pattern of results is that problems that are solved by the same arithmetic operation – or in other words, by the same sum – can differ radically in how difficult they are.

Usually pre-school children do make the appropriate moves in the easiest Change and Combine problems: they put together and count up (counting on or counting all) and separate and count the relevant set to find the answer. Very few pre-school children seem to know addition and subtraction facts, and so they succeed considerably more if they have physical objects (or use their fingers) in order to count. Research by Carpenter and Moser (1982) gives an indication of how pre-school children perform in the simpler problems. These researchers interviewed children (aged about four to five years) twice before they had had been given any instruction about arithmetic in school; we give here the results of each of these interviews, as there is always some improvement worth noting between the testing occasions.

For Combine problems (given two parts, find the whole), 75% and 82% of the answers were correct when the numbers were small and 50% and 71% when the numbers were larger; only 13% of the responses with small numbers were obtained through the recall of number facts and this was the largest percentage of recall of number facts observed in their study. For Change problems (Tim had 11 candies; he gave 7 to Martha; how many did he have left?), the pre-schoolers were correct 42% and 61% with larger numbers (Carpenter and Moser do not report the figures for smaller numbers) at each of the two interviews; only 1% of recall of number facts is reported. So, pre-school children can do relatively well on simple Change and Combine problems before they know arithmetic facts; they do so by putting sets together or by separating them and counting.

This classification of problems into three types – Combine, Change and Compare – is not sufficient to describe story problem-solving. In a Change problem, for example, the story might provide the information about the initial state and the change (Tim had 11 candies; he gave 7 to Martha); the child is asked to say what the final state is. But it is also possible to provide information, for example, about the transformation and the end state (Tim had some candies; he gave Martha 7 and he has 4 left) and ask the child to say what the initial state was (how many did he have before he gave candies to Martha?). This sort of analysis has resulted in more complex classifications, which consider which information is given and which information must be supplied by the children in the answer. Stories that describe a situation where the *quantity decreases*, as in the example above, but have a missing initial state can most easily be solved by an *addition*. The conflict between the decrease in quantity and the operation of addition can be solved if the children understand the inverse relation between subtraction and addition: by adding the number that Tim still has and the number he gave away, one can find out how many candies he had before.

Different analyses of word problems have been proposed (e.g. Briars and Larkin, 1984; Carpenter and Moser, 1982; Fuson, 1992; Nesher, 1982; Riley, Greeno and Heller, 1983; Vergnaud, 1982). We focus here on some aspects of the analysis provided by Gérard Vergnaud, which allows for the comparison of many different types of problems and can also be used to help understand the level of difficulty of further types of additive reasoning problems, involving directed numbers (i.e. positive and negative numbers).

First, Vergnaud distinguishes between *numerical* and *relational* calculation. *Numerical calculation* refers to the arithmetic operations that the children carry out to find the answer to a problem: in the case of additive reasoning, addition and subtraction are the relevant operations. *Relational calculation* refers to the operations of thought that the child must carry out in order to handle the relations involved in the problem. For example, in the problem 'Bertrand played a game of marbles and lost 7 marbles. After the game he had 3 marbles left. How many marbles did he have before the game?', the relational calculation is the realisation that the solution requires using the inverse of subtraction to go from the final state to the initial state and the numerical calculation would be 7 + 3.

Vergnaud proposes that children perform these relational calculations in an implicit manner: to use his expression, they rely on 'theorems in action'. The children may say that they 'just know' that they have to add when they solve the problem, and may be unable to say that the reason for this is that addition is the inverse of subtraction. Vergnaud reports approximately twice as many correct responses by French pre-school children (aged about five years) to a problem that involves no relational calculation (about 50% correct in the problem: Pierre had 6 marbles. He played a game and lost 4; how many did he have after the game?) than to the problem above (about 26% correct responses), where we are told how many marbles Bertrand lost and asked how many he had before the game.

Vergnaud also distinguished three types of meanings that can be represented by natural numbers: quantities (which he calls measures), transformations and relations. This distinction has an effect on the types of problems that can be created starting from the simple classification in three types (change, combine and compare) and their level of difficulty.

First, consider the two problems below, the first about combining a quantity and a transformation and the second about combining two transformations.

- Pierre had 6 marbles. He played one game and lost 4 marbles. How many marbles did he have after the game?

- Paul played two games of marbles. He won 6 in the first game and lost 4 in the second game. What happened, counting the two games together?

French children, who were between pre-school and their fourth year in school, consistently performed better on the first than on the second type of problem, even though the same arithmetic calculation (6-4) is required in both problems. By the second year in school, when the children are about seven years old, they achieve about 80% correct responses in the first problem, and they only achieve a comparable level of success two years later in the second problem. So, combining transformations is more difficult than combining a quantity and a transformation.

Brown (1981) confirmed these results with English students in the age range 11 to 16. In her task, students are shown a sign-post that indicates that Grange is 29 miles to the west and Barton is 58 miles to the east; they are asked how do they work out how far they need to drive to go from Grange to Barton. There were eight choices of operations connecting these two numbers for the students to indicate the correct one. The rate of correct responses to this problem was 73%, which contrasts with 95% correct responses when the problem referred to a union of sets (a combine problem).

The children found problems even more difficult when they needed to de-combine transformations than when they had to combine them. Here is an example of a problem with which they needed to de-combine two transformations, because the story provides the result of combining operations and the question that must be answered is about the state of affairs before the combination took place.

- Bruno played two games of marbles. He played the first and the second game. In the second game he lost 7 marbles. His final result, with the two games together, was that he had won 3 marbles. What happened in the first game?

This de-combination of transformations was still very difficult for French children in the fourth year in school (age about nine years): they attained less than 50% correct responses.

Vergnaud's hypothesis is that when children combine transformations, rather than quantities, they have to go beyond natural numbers: they are now operating in the domain of whole numbers. Natural numbers are counting numbers. You can certainly count the number of marbles that Pierre had before he started the game, count and take away the marbles that he lost in the second game, and say how many he had left at the end. In the case of Paul's problem, if you count the marbles that he won in the first game, you need to count them as 'one more, two more, three more etc.': you are actually not counting marbles but the relation between the number that he now has to the number he had to begin with. So if the starting point in a problem that involves transformations is not known, the transformations are now relations. Of course, children who do solve the problem about Paul's marbles may not be fully aware of the difference between a transformation and a relation, and may succeed exactly because they overlook this difference. This point is discussed in Paper 4, when we consider in detail how children think about relations.

Finally, problems where children are asked to quantify *relations* are usually difficult as well:

- Peter has 8 marbles. John has 3 marbles. How many more marbles does Peter have than John?

The question in this problem is neither about a quantity (i.e. John's or Peter's marbles) nor about a transformation (no-one lost or got more marbles): it is about the relation between the two quantities. Although most pre-school children can say that Peter has more marbles, the majority cannot quantify the relation (or the difference) between the two. The best known experiments that demonstrate this difficulty were carried out by Hudson (1983) in the United States. In a series of three experiments, he showed the children some pictures and asked them two types of question:

- Here are some birds and some worms. How many more birds than worms?

- Here are some birds and some worms. The birds are racing to get a worm. How many birds won't get worms?

The first question asks the children to quantify the relation between the two sets, of worms and birds; the second question asks the children to imagine that the sets were matched and quantify the set that has no matching elements. The children in the first year of school (mean age seven years) attained 64% correct responses to the first question and 100% to the second question; in nursery school (mean age four years nine months) and kindergarten (mean age 6 years 3 months), the rate of correct responses was, respectively, 17% and 25% to the first question and 83% and 96% to the second question.

It is, of course, difficult to be completely certain that the second question is easier because the children are asked a question about quantity whereas the first question is about a relation. The reason for this ambiguity is that two things have to change at the same time for the story to be different: in order to change the target of the question, so that it is either a quantity or a relation, the language used in the problem also varies: in the first problem, the word 'more' is used, and in the second it is not.

Hudson included in one of his experiments a pre-test of children's understanding of the word 'more' (e.g. Are there more red chips or more white chips?') and found that they could answer this question appropriately. He concluded that it was the linguistic difficulty of the 'How many more…?' question that made the problem difficult, not simply the difficulty of the word 'more'. We are not convinced by his conclusion and think that more research about children's understanding of how to quantify relations is required. Stern (2005), on the other hand, suggests that both explanations are relevant: the linguistic form is more difficult and quantifying relations is also more difficult than using numbers to describe quantities.

In the domain of directed numbers (i.e. positive and negative numbers), it is relatively easier to study the difference between attributing numbers to quantities and to relations without asking the 'how many more' question. Unfortunately, studies with larger sample, which would allow for a quantitative comparison in the level of difficulty of these problems, are scarce. However, some indication that quantifying relations is more difficult for students is available in the literature.

Vergnaud (1982) pointed out that relationships between people could be used to create problems that do not contain the question 'how many more'. Among others, he suggested the following example.

- Peter owes 8 marbles to Henry but Henry owes 6 marbles to Peter. What do they have to do to get even?

According to his analysis, this problem involves a composition of relations.

Marthe (1979) compared the performance of French students in the age range 11 to 15 years in two problems involving such composition of relations with their performance in two problems involving a change situation (i.e. quantity, transformation, quantity). In order to control for problem format, all four problems had the structure $a + x = b$, in which x shows the place of the unknown. The problems used large numbers so that students had to go through the relational calculation in order to determine the numerical calculation (with small numbers, it is possible to work in an intuitive manner, sometimes starting from a hypothetical amount and adjusting the starting point later to make it fit). An example of a problem type using a composition of relations is shown below.

- Mr Dupont owes 684 francs to Mr. Henry. But Mr Henry also owes money to Mr Dupont. Taking everything into account, Mr Dupont must give back 327 francs to Mr Henry. What amount did Mr Henry owe to Mr Dupont?

Marthe did find that problems about relations were quite a bit more difficult than those about quantities and transformations; there was a difference of 20% between the rates of correct responses for the younger children and 10% for the older children. However, the most important effect in these problems seemed to be whether the students had to deal with numbers that had the same or different signs: problems with same signs were consistently easier than those with different signs.

In summary, different researchers have argued that it is one thing to learn to use numbers to represent quantities and a quite different one to use numbers to quantify relations. Relations are more abstract and more challenging for students. Thompson (1993) hypothesises that learning to quantify and think about numbers as measures of relations is a crucial step that students must take in order to understand algebra. We are completely sympathetic to this hypothesis, but we think that the available evidence is a bit thin.

More than two decades ago, Dickson, Brown and Gibson (1984) reviewed research on additive reasoning and problem solving, and pointed out how difficult it is to come to firm conclusions when no single study has covered the variety of problems that any theoretical model would aim to compare. We have to piece the evidence together from diverse studies, and of course samples vary across different locations and cohorts. In the last decade research on additive reasoning has received less attention in research on mathematics education than before. Unfortunately, this has left some questions with answers that are, at best, based on single studies with limited numbers of students. It is time to use a new synthesis to re-visit these questions and seek for unambiguous answers within a single research programme.

Summary

1 In word problems children are told a brief story which ends in an arithmetical question. These problems are widely used in school textbooks and also as a research tool.

2 There are three main kinds of word problem: Combine, Change and Compare.

3 Vergnaud argued that the crucial elements in these problems were Quantities (measures), Transformations and Relations. On the whole, problems that involve Relations are harder than those involving Transformations.

4 However, other factors also affect the level of difficulty in word problems. Any change in sign is often hard for children to handle: when the story is about an addition but the solution is to subtract, as in missing addend problems, children often fail to use the inverse operation.

5 Overall the extreme variability in the level of difficulty of different problems, even when these demand exactly the same mathematical solution (the same simple additions or subtractions) confirms the view that there is a great deal more to arithmetical learning than knowing how to carry out particular operations.

6 Research on word problems supports a different approach, which is that arithmetical learning depends on children making a coherent connection between quantitative relations and the appropriate numerical analysis.

Overall conclusions and educational implications

- Learning about quantities and numbers are two different matters: children can understand relations between quantities and not know how to make inferences about the numbers that are used to represent the quantities; they might also learn to count without making a connection between counting and what it implies for the relations between quantities.

- Some ideas about quantities are essential for understanding number: equivalence between quantities, their order of magnitude, and the part-whole relations implicit in determining the number of elements in a set.

- These core ideas, in turn, require that children come to understand yet other logical principles: transitive relations in equivalence and order, which operations change quantities and which do not, and the inverse relation between addition and subtraction. These notions are central to understanding numbers and how they represent quantities; children who have a good grasp of them learn mathematics better in school. Children who have difficulties with these ideas and do not receive support to come to grips with them are at risk for difficulties in learning mathematics, but these difficulties can be prevented to a large extent if they receive appropriate instruction.

- There is no question that word problems give us a valuable insight into children's reasoning about addition and subtraction. They demonstrate that there is a great deal more to understanding these operations than just learning how to add and subtract. Children's solutions do depend on their ability to reason about the relations between quantities in a logical manner. There is no doubt about these conclusions, even if there is need for further research to pin down some of the details.

- Learning to count and to use numbers to represent quantities is an important element in this developmental process. Children can more easily reason about the relation between addition, subtraction, and number when they know how to represent quantities by counting. But this is not a one-way relation: it is by adding, subtracting, and understanding the inverse relation between these operations that children understand additive composition and learn to solve additive reasoning problems.

- The major implication from this review is that schools should take very seriously the need to include in the curriculum instruction that promotes reflection about relations between quantities, operations, and the quantification of relations.

- These reflections should not be seen as appropriate only for very young children: when natural numbers start to be used to represent relations, directed numbers become a new domain of activity for children to re-construct their understanding of additive relations. The construction of a solid understanding of additive relations is not completed in the first years of primary school: some problems are still difficult for students at the age of 15.

Endnotes

1 Gelman and Butterworth (2005) make a similar distinction between numerosity and the representation of number: 'we need to distinguish possession of the concept of numerosity itself (knowing that any set has a numerosity that can be determined by enumeration) from the possession of rerepresentations (in language) of particular numerosities' (pp. 6). However, we adopt here the term 'quantities' because it has an established definition and use in the context of children's learning of mathematics.

2 It is noted here that evidence from cases studies of acquired dyscalculia (a cognitive disorder affecting the ability to solve mathematics problems observed in patients after neurological damage) is consistent with the idea that understanding quantities and number knowledge can be dissociated: calculation may be impaired and conservation of quantities may be intact in some patients whereas in others calculation is intact and conceptual knowledge impaired (Mittmair-Delazer, Sailer and Benke, 1995). Dissociations between arithmetic skills and the meaning of numbers were extensively described by McCloskey (1992) in a detailed review of cases of acquired dyscalculia.

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