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Key understandings in
mathematics learning

Paper 1: Overview

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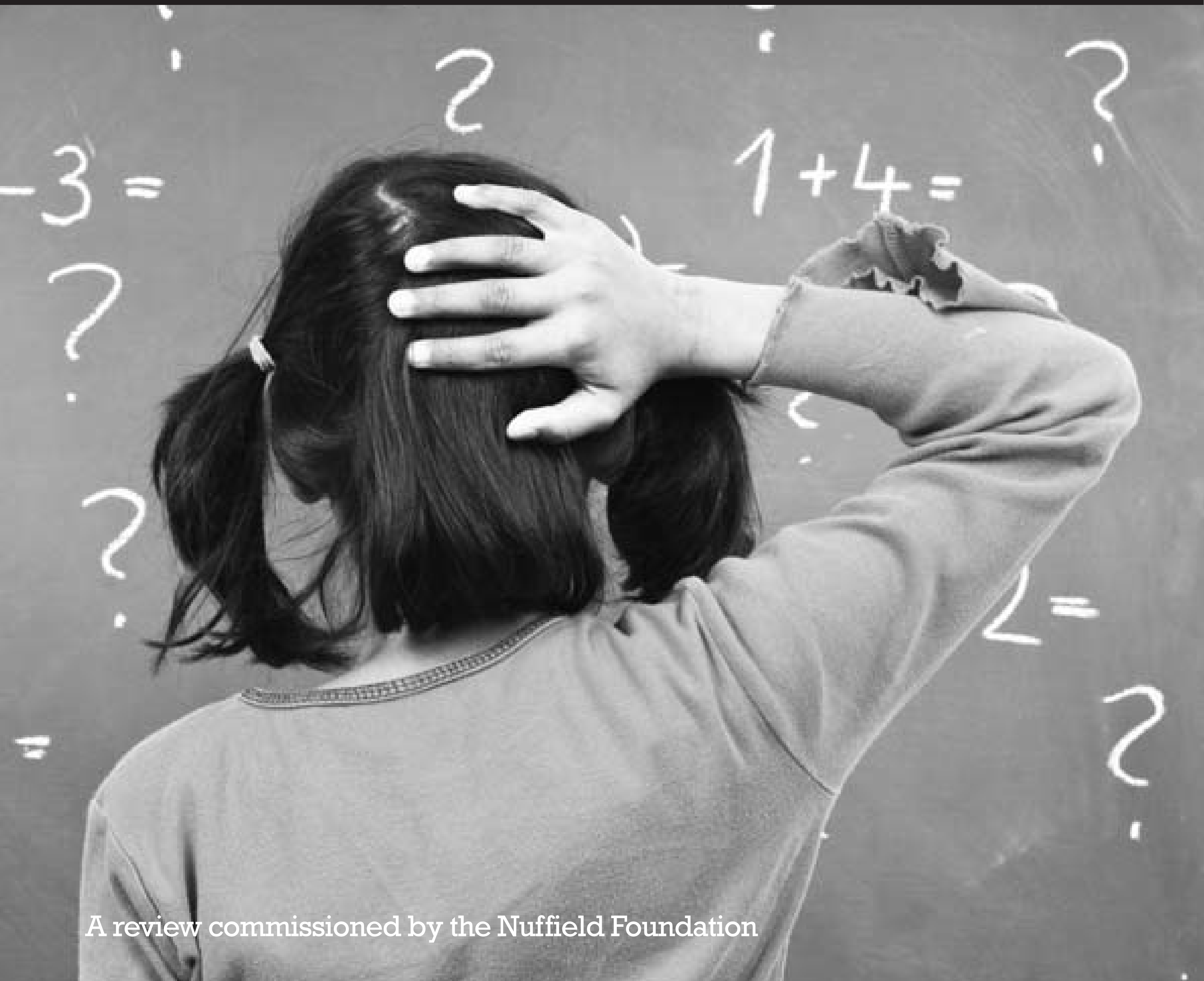

A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of findings

Aims

Our aim in the review is to present a synthesis of research on mathematics learning by children from the age of five to the age of sixteen years and to identify the issues that are fundamental to understanding children's mathematics learning. In doing so, we concentrated on three main questions regarding key understandings in mathematics.

- What insights must students have in order to understand basic mathematical concepts?
- What are the sources of these insights and how does informal mathematics knowledge relate to school learning of mathematics?
- What understandings must students have in order to build new mathematical ideas using basic concepts?

Theoretical framework

While writing the review, we concluded that there are two distinct types of theory about how children learn mathematics.

Explanatory theories set out to explain how children's mathematical thinking and knowledge change. These theories are based on empirical research on children's solutions to mathematical problems as well as on experimental and longitudinal studies. Successful theories of this sort should provide insight into the causes of children's mathematical development and worthwhile suggestions about teaching and learning mathematics.

Pragmatic theories set out to investigate what children ought to learn and understand and also identify obstacles to learning in formal educational settings.

Pragmatic theories are usually not tested for their consistency with empirical evidence, nor examined for the parsimony of their explanations vis-à-vis other existing theories; instead they are assessed in multiple contexts for their descriptive power, their credibility and their effectiveness in practice.

Our starting point in the review is that children need to learn about quantities and the relations between them and about mathematical symbols and their meanings. These meanings are based on sets of relations. Mathematics teaching should aim to ensure that students' understanding of quantities, relations and symbols go together.

Conclusions

This theoretical approach underlies the six main sections of the review. We now summarise the main conclusions of each of these sections.

Whole numbers

- Whole numbers represent both quantities and relations between quantities, such as differences and ratio. Primary school children must establish clear connections between numbers, quantities and relations.

- Children's initial understanding of quantitative relations is largely based on correspondence. One-to-one correspondence underlies their understanding of cardinality, and one-to-many correspondence gives them their first insights into multiplicative relations. Children should be encouraged to think of number in terms of these relations.

- Children start school with varying levels of ability in using different action schemes to solve arithmetic problems in the context of stories. They do not need to know arithmetic facts to solve these problems: they count in different ways depending on whether the problems they are solving involve the ideas of addition, subtraction, multiplication or division.

- Individual differences in the use of action schemes to solve problems predict children's progress in learning mathematics in school.

- Interventions that help children learn to use their action schemes to solve problems lead to better learning of mathematics in school.

- It is more difficult for children to use numbers to represent relations than to represent quantities.

Implications for the classroom

Teaching should make it possible for children to:

- connect their knowledge of counting with their knowledge of quantities
- understand additive composition and one-to-many correspondence
- understand the inverse relation between addition and subtraction
- solve problems that involve these key understandings
- develop their multiplicative understanding alongside additive reasoning.

Implications for further research

Long-term longitudinal and intervention studies with large samples are needed to support curriculum development and policy changes aimed at implementing these objectives. There is also a need for studies designed to promote children's competence in solving problems about relations.

Fractions

- Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. As with whole numbers, children need to make connections between quantities and their representations in fractions in order to be able to use fractions meaningfully.

- Two types of quantities that are taught in primary school must be represented by fractions. The first involves measurement: if you want to represent a quantity by means of a number and the quantity is smaller than the unit of measurement, you need a fraction; for example, a half cup or a quarter inch. The second involves division: if the dividend is smaller than the divisor, the result of the division is represented by a fraction; for example, three chocolates shared among four children.

- Children use different schemes of action in these two different situations. In division situations, they use correspondences between the units in the numerator and the units in the denominator. In measurement situations, they use partitioning.

- Children are more successful in understanding equivalence of fractions and in ordering fractions by magnitude in situations that involve division than in measurement situations.

- It is crucial for children's understanding of fractions that they learn about fractions in both types of situation: most do not spontaneously transfer what they learned in one situation to the other.

- When a fraction is used to represent a quantity, children need to learn to think about how the numerator and the denominator relate to the value represented by the fraction. They must think about direct and inverse relations: the larger the numerator, the larger the quantity, but the larger the denominator, the smaller the quantity.

- Like whole numbers, fractions can be used to represent quantities and relations between quantities, but they are rarely used to represent relations in primary school. Older students often find it difficult to use fractions to represent relations.

Implications for the classroom

Teaching should make it possible for children to:

- use their understanding of quantities in division situations to understand equivalence and order of fractions
- make links between different types of reasoning in division and measurement situations
- make links between understanding fractional quantities and procedures
- learn to use fractions to represent relations between quantities, as well as quantities.

Implications for further research
Evidence from experimental studies with larger samples and long-term interventions in the classroom are needed to establish how division situations relate to learning fractions. Investigations on how links between situations can be built are needed to support curriculum development and classroom teaching.

There is also a need for longitudinal studies designed to clarify whether separation between procedures and meaning in fractions has consequences for further mathematics learning.

Given the importance of understanding and representing relations numerically, studies that investigate under what circumstances primary school students can use fractions to represent relations between quantities, such as in proportional reasoning, are urgently needed.

Relations and their mathematical representation

- Children have greater difficulty in understanding relations than in understanding quantities. This is true in the context of both additive and multiplicative reasoning problems.

- Primary and secondary school students often apply additive procedures to solve multiplicative problems and multiplicative procedures to solve additive problems.

- Teaching designed to help students become aware of relations in the context of additive reasoning problems can lead to significant improvement.

- The use of diagrams, tables and graphs to represent relations in multiplicative reasoning problems facilitates children's thinking about the nature of the relations between quantities.

- Excellent curriculum development work has been carried out to design programmes that help students develop awareness of their implicit knowledge of multiplicative relations. This work has not been systematically assessed so far.

- An alternative view is that students' implicit knowledge should not be the starting point for students to learn about proportional relations; teaching should focus on formalisations rather than informal knowledge and only later seek to connect mathematical formalisations with applied situations. This alternative approach has also not been systematically assessed yet.

- There is no research that compares the results of these diametrically opposed ideas.

Implications for the classroom
Teaching should make it possible for children to:
- distinguish between quantities and relations
- become explicitly aware of the different types of relations in different situations
- use different mathematical representations to focus on the relevant relations in specific problems
- relate informal knowledge and formal learning.

Implications for further research
Evidence from experimental and long-term longitudinal studies is needed on which approaches to making students aware of relations in problem situations improve problem solving. A study comparing the alternative approaches – starting from informal knowledge versus starting from formalisations – would make a significant contribution to the literature.

Space and its mathematical representation

- Children come to school with a great deal of informal and often implicit knowledge about spatial relations. One challenge in mathematical education is how best to harness this knowledge in lessons about space.

- This pre-school knowledge of space is mainly relational. For example, children use a stable background to remember the position and orientation of objects and lines.

- Measuring length and area poses particular problems for children, even though they are able to understand the underlying logic of measurement. Their difficulties concern iteration of standard units and the need to apply multiplicative reasoning to the measurement of area.

- From an early age children are able to extrapolate imaginary straight lines, which allows them to learn how to use Cartesian co-ordinates to plot specific positions in space with little difficulty. However, they need help from teachers on how to use co-ordinates to work out the relation between different positions.

- Learning how to represent angle mathematically is a hard task for young children, even though angles are an important part of their everyday life. Initially children are more aware of angle in the context of movement (turns) than in other contexts. They need help from to teachers to be able to relate angles across different contexts.

- An important aspect of learning about geometry is to recognise the relation between transformed shapes (rotation, reflection, enlargement). This can be difficult, since children's preschool experiences lead them to recognise the same shapes as equivalent across such transformations, rather than to be aware of the nature of the transformation.

- Another aspect of the understanding of shape is the fact that one shape can be transformed into another by addition and subtraction of its subcomponents. For example, a parallelogram can be transformed into a rectangle of the same base and height by the addition and subtraction of equivalent triangles. Research demonstrates a danger that children learn these transformations as procedures without understanding their conceptual basis.

Implications for the classroom

Teaching should make it possible for children to:

- build on spatial relational knowledge from outside school
- relate their knowledge of relations and correspondence to the conceptual basis of measurement
- iterate with standard and non-standard units
- understand the difference between measurements which are/are not multiplicative
- relate co-ordinates to extrapolating imaginary straight lines
- distinguish between scale enlargements and area enlargements.

Implications for further research

There is a serious need for longitudinal research on the possible connections between children's pre-school spatial abilities and how well they learn about geometry at school.

Psychological research is needed on: children's ability to make and understand transformations and the additive relations in compound shapes; the exact cause of children's difficulties with iteration; how transitive inference, inversion and one-to-one correspondence relate to problems with geometry, such as measurement of length and area.

There is a need for intervention studies on methods of teaching children to work out the relation between different positions, using co-ordinates.

Algebra

- Algebra is the way we express generalisations about numbers, quantities, relations and functions. For this reason, good understanding of connections between numbers, quantities and relations is related to success in using algebra. In particular, understanding that addition and subtraction are inverses, and so are multiplication and division, helps students understand expressions and solve equations.

- To understand algebraic symbolisation, students have to (a) understand the underlying operations and (b) become fluent with the notational rules. These two kinds of learning, the meaning and the symbol, seem to be most successful when students know what is being expressed and have time to become fluent at using the notation.

- Students have to learn to recognise the different nature and roles of letters as: unknowns, variables, constants and parameters, and also the meanings of equality and equivalence. These meanings are not always distinct in algebra and do not relate unambiguously to arithmetical understandings.

- Students often get confused, misapply, or misremember rules for transforming expressions and solving equations. They often try to apply arithmetical meanings inappropriately to algebraic expressions. This is associated with over-emphasis on notational manipulation, or on 'generalised arithmetic', in which they may try to get concise answers.

Implications for the classroom

Teaching should make it possible for children to:

- read numerical and algebraic expressions relationally, rather than as instructions to calculate (as in substitution)
- describe generalisations based on properties (arithmetical rules, logical relations, structures) as well as inductive reasoning from sequences

- use symbolism to represent relations
- understand that letters and '=' have a range of meanings
- use hands-on ICT to relate representations
- use algebra purposefully in multiple experiences over time
- explore and use algebraic manipulation software.

Implications for further research

We need to know how explicit work on understanding relations between quantities enables students to move successfully between arithmetical to algebraic thinking.

Research on how expressing generality enables students to use algebra is mainly in small-scale teaching interventions, and the problems of large-scale implementation are not so well reported. We do not know the longer-term comparative effects of different teaching approaches to early algebra on students' later use of algebraic notation and thinking.

There is little research on higher algebra, except for teaching experiments involving functions. How learners synthesise their knowledge of elementary algebra to understand polynomial functions, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations is not known.

There is some research about the use of symbolic manipulators but more needs to be learned about the kinds of algebraic expertise that develops through their use.

Modelling, solving problems and learning new concepts in secondary mathematics

Students have to be fluent in understanding methods and confident about using them to know why and when to apply them, but such application does not automatically follow the learning of procedures. Students have to understand the situation as well as to be able to call on a familiar repertoire of facts, ideas and methods.

Students have to know some elementary concepts well enough to apply them and combine them to form new concepts in secondary mathematics. For example, knowing a range of functions and/or their representations seems to be necessary to understand the modelling process, and is certainly necessary to engage in modelling.

Understanding relations is necessary to solve equations meaningfully.

Students have to learn when and how to use informal, experiential reasoning and when to use formal, conventional, mathematical reasoning. Without special attention to meanings, many students tend to apply visual reasoning, or be triggered by verbal cues, rather than analyse situations to identify variables and relations.

In many mathematical situations in secondary mathematics, students have to look for relations between numbers, and variables, and relations between relations, and properties of objects, and know how to represent them.

Implications for the classroom

Teaching should make it possible for children to:

- learn new abstract understandings, which is neither achieved through learning procedures, nor through problem-solving activities, without further intervention
- use their obvious reactions to perceptions and build on them, or understand conflicts with them
- adapt to new meanings and develop from earlier methods and conceptualizations over time
- understand the meaning of new concepts 'know about', 'know how to', and 'know how to use'
- control switching between, and comparing, representations of functions in order to understand them
- use spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

Implications for further research

Existing research suggests that where contextual and exploratory mathematics, integrated through the curriculum, *do* lead to further conceptual learning it is related to conceptual learning being a rigorous focus for curriculum and textbook design, and in teacher preparation, or in specifically designed projects based around such aims. There is therefore an urgent need for research to identify the key conceptual understandings for success in secondary mathematics. There is no evidence to convince us that the new U.K. curricula will necessarily lead to better *conceptual* understanding of mathematics, either at the elementary level which is necessary to learn higher mathematics, or at higher levels which provide the confidence and foundation for further mathematical study.

We need to understand the ways in which students learn new ideas in mathematics that depend on combinations of earlier concepts, in secondary school contexts, and the characteristics of mathematics teaching at higher secondary level which contribute both to successful conceptual learning and application of mathematics.

Common themes

We reviewed different areas of mathematical activity, and noted that many of them involve common themes, which are fundamental to learning mathematics: number, logical reasoning, reflection on knowledge and tools, understanding symbol systems and mathematical modes of enquiry.

Number

Number is not a unitary idea, which children learn in a linear fashion. Number develops in complementary strands, sometimes with discontinuities and changes of meaning. Emphasis on procedures and manipulation with numbers, rather than on understanding the underlying relations and mathematical meanings, can lead to over-reliance and misapplication of methods in arithmetic, algebra, and problem-solving. For example, if children form the idea that quantities are only equal if they are represented by the same number, a principle that they could deduce from learning to count, they will have difficulty understanding the equivalence of fractions. Learning to count and to understand quantities are separate strands of development. Teaching can play a major role in helping children co-ordinate these two forms of knowledge without making counting the only procedure that can be used to think about quantities.

Successful learning of mathematics includes understanding that number describes quantity; being able to make and use distinctions between different, but related, meanings of number; being able to use relations and meanings to inform application and calculation; being able to use number relations to move away from images of quantity and use number as a structured, abstract, concept.

Logical reasoning

The evidence demonstrates beyond doubt that children must rely on logic to learn mathematics and that many of their difficulties are due to failures to make the correct logical move that would have led them to the correct solution. Four different aspects of logic have a crucial role in learning about mathematics.

The logic of correspondence (one-to-one and one-to-many correspondence) The extension of the use of one-to-one correspondence from sharing to working out the numerical equivalence or non-equivalence of two or more spatial arrays is a vastly important step in early mathematical learning. Teaching multiplication in terms of one-to-many correspondence is more effective than teaching children about multiplication as repeated addition.

The logic of inversion Longitudinal evidence shows that understanding the inverse relation between addition and subtraction is a strong predictor of children's mathematical progress. A flexible understanding of inversion is an essential element in children's geometrical reasoning as well. The concept of inversion needs a great deal more prominence than it has now in the school curriculum.

The logic of class inclusion and additive composition Class inclusion is the basis of the understanding of ordinal number and the number system. Children's ability to use this form of inclusion in learning about number and in solving mathematical problems is at first rather weak, and needs some support.

The logic of transitivity All ordered series, including number, and also forms of measurement involve transitivity ($a > c$ if $a > b$ and $b > c$: $a = c$ if $a = b$ and $b = c$). Learning how to use transitive relations in numerical measurements (for example, of area) is difficult. One reason is that children often do not grasp the importance of iteration (repeated units of measurement).

The results of longitudinal research (although there is not an exhaustive body of such work) support the idea that children's logic plays a critical part in their mathematical learning.

Reflection on knowledge and tools

Children need to re-conceptualise their intuitive models about the world in order to access the mathematical models that have been developed in the discipline. Some of the intuitive models used by children lead them to appropriate mathematical problem solving, and yet they may not know why they succeeded. Implicit models can interfere with problem solving when students rely on assumptions that lead them astray.

The fact that students use intuitive models when learning mathematics, whether the teacher recognises the models or not, is a reason for helping them to develop an awareness of their models. Students can explore their intuitive models and extend them to concepts that are less intuitive, more abstract. This pragmatic theory has been shown to have an impact in practice.

Understanding symbol systems

Systems of symbols are human inventions and thus are cultural tools that have to be taught. Mathematical symbols are human-made tools that improve our ability to control and adapt to the environment. Each system makes specific cognitive demands on the learner, who has to understand the systems of representation and relations that are being represented; for example place-value notation is based on additive composition, functions depict covariance. Students can behave as if they understand how the symbols work while they do not understand them completely: they can learn routines for symbol manipulation that remain disconnected from meaning. This is true of rational numbers, for example.

Students acquire informal knowledge in their everyday lives, which can be used to give meaning to mathematical symbols learned in the classroom. Curriculum development work that takes this knowledge into account is not as widespread as one would expect given discoveries from past research.

Mathematical modes of enquiry

Some important mathematical modes of enquiry arise in the topics covered in this synthesis.

Comparison helps us make new distinctions and create new objects and relations Comparisons are related to making distinctions, sorting and classifying; students need to learn to make these distinctions based on mathematical relations and properties, rather than perceptual similarities.

Reasoning about properties and relations rather than perceptions Throughout mathematics, students have to learn to interpret representations before they think about how to respond. They need to think about the relations between different objects in the systems and schemes that are being represented.

Making and using representations Conventional number symbols, algebraic syntax, coordinate geometry, and graphing methods, all afford manipulations which might otherwise be impossible. Coordinating different representations to explore and extend meaning is a fundamental mathematical skill.

Action and reflection-on-action In mathematics, actions may be physical manipulation, or symbolic rearrangement, or our observations of a dynamic image, or use of a tool. In all these contexts, we observe what changes and what stays the same as a result of actions, and make inferences about the connections between action and effect.

Direct and inverse relations It is important in all aspects of mathematics to be able to construct and use inverse reasoning. As well as enabling more understanding of relations between quantities, this also establishes the importance of reverse chains of reasoning throughout mathematical problem-solving, algebraic and geometrical reasoning.

Informal and formal reasoning At first young children bring everyday understandings into school and mathematics can allow them to formalise these and make them more precise. Mathematics also provides formal tools, which do not describe everyday experience, but enable students to solve problems in mathematics and in the world which would be unnoticed without a mathematical perspective.

Epilogue

We have made recommendations about teaching and learning, and hope to have made the reasoning behind these recommendations clear to educationalists (in the extended review). We have also recognised that there are weaknesses in research and gaps in current knowledge, some of which can be easily solved by research enabled by significant contributions of past research. Other gaps may not be so easily solved, and we have described some pragmatic theories that are or can be used by teachers when they plan their teaching. Classroom research stemming from the exploration of these theories can provide new insights for further research in the future, alongside longitudinal studies which focus on learning mathematics from a psychological perspective.

Overview

Aims

Our aim in this review is to present a synthesis of research on key aspects of mathematics learning by children from the age of 5 to the age of 16 years: these are the ages that comprise compulsory education in the United Kingdom In preparing the review, we have considered the results of a large body of research carried out by psychologists and by mathematics educators over approximately the last six decades. Our aim has been to develop a theoretical analysis of these results in order to attain a big picture of how children learn, and sometimes fail to learn, mathematics and how they could learn it better. Our main target is not to provide an answer to any specific question, but to identify issues that are fundamental to understanding children's mathematics learning. In our view theories of mathematics learning should deal with three main questions regarding key understandings in mathematics:

- What insights must students have in order to understand basic mathematical concepts?
- What are the sources of these insights and how does informal mathematics knowledge relate to school learning of mathematics?
- What understandings must students have in order to build new mathematical ideas using basic concepts?

Theoretical analysis played a major role in this synthesis. Many theoretical ideas were already available in the literature and we sought to examine them critically for coherence and for consistency with the empirical evidence. Cooper (1998) suggests that there may be occasions when new theoretical schemes must be developed to provide an overarching understanding of the higher-order relations in the research domain; this was certainly true of some of our theoretical analysis of the evidence that we read for this review.

The answers to our questions should allow us to trace students' learning trajectories. Confrey (2008) defined a learning trajectory as 'a researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time.' If students' learning trajectories towards understanding specific concepts are generally understood, teachers will be much better placed to promote their advancement.

Finally, one of our aims has been to identify a set of research questions that stem from our current knowledge about children's mathematics learning and methods that can provide relevant evidence about important, outstanding issues.

Scope of the review

As we reviewed existing research and existing theories about mathematics learning, it soon became clear to us that there are two types of theories about how children learn mathematics. The first are *explanatory* theories. These theories seek to explain how children's thinking and knowledge change. Explanatory theories are based on empirical research on the strategies that children adopt in solving mathematical problems, on the difficulties and misconceptions that affect their solutions to such problems, and on their successes and their explanations of their own solutions. They also draw on quantitative methods to describe age or school grade levels when certain forms of knowledge are attained and to make inferences about the nature of relationships observed during learning (for example, to help understand the relation between informal knowledge and school learning of mathematics).

We have called the second type of theory *pragmatic*. A pragmatic theory is rather like a road map for teachers: its aims are to set out what children must learn and understand, usually in a clear sequence, about particular topics and to identify obstacles to learning in formal educational settings and other issues which teachers should keep in mind when designing teaching. Pragmatic theories are usually not tested for their consistency with empirical evidence, nor examined for the parsimony of their explanations vis-à-vis other existing theories; instead they are assessed in multiple contexts for their descriptive power, their credibility and their effectiveness in practice.

Explanatory theories are of great importance in moving forward our understanding of phenomena and have proven helpful, for example, in the domain of literacy teaching and learning. However, with some aspects of mathematics, which tend to be those that older children have to learn about, there simply is not enough explanatory knowledge yet to guide teachers in many aspects of their mathematics teaching, but students must still be taught even when we do not know much about how they think or how their knowledge changes over time through learning. Mathematics educators have developed pragmatic theories to fill this gap and to take account of the interplay of learning theory with social and cultural aspects of educational contexts. Pragmatic theories are designed to guide teachers in domains where there are no satisfactory explanatory theories, and where explanatory theory does not provide enough information to design complex classroom teaching. We have included both types of theory in our review. We believe that both types are necessary in mathematics education but that they should not be confused with each other.

We decided to concentrate in our review on issues that are specific to mathematics learning. We recognise the significance of general pedagogical theories that stress, for example, the importance of giving learners an active role in developing their thinking and conceptual understanding, the notion of didactic transposition, the theory of situations, social theories regarding the importance of conflict and cooperation, the role of the teacher, the role and use of language, peer collaboration and argumentation in the classroom. These are important ideas but they apply to other domains of learning as well, and we decided not to provide an analysis of such theories but to mention them only in the context of specific issues about mathematical learning.

Another decision that we made about the scope of the synthesis was about how to deal with cultural differences in teaching and learning mathematics. The focus of the review is on mathematics learning by U.K. students during compulsory education. We recognise that there are many differences between learners in different parts of the world; so, we decided to include mostly research about learners who can be considered as reasonably similar to U.K. students, i.e. those living in Western cultures with a relatively high standard of living and plenty of opportunities to attend school. Thus the description of students who participated in the studies is not presented in detail and will often be indicated only in terms of the country where the research was carried out. In order to offer readers a notion of the time in students' lives when they might succeed or show difficulties with specific problems, we used age levels or school grade levels as references. These ages and years of schooling are not to be generalised to very different circumstances where, for example, children might be growing up in cultures with different number systems or largely without school participation. Occasional reference to research with other groups is used but this was purposefully limited, and it was included only when it was felt that the studies could shed light on a specific issue.

We also decided to concentrate on key understandings that offer the foundation for mathematics learning rather than on the different technologies used in mathematics. Wartofsky (1979) conceives technology as any human made tool that improves our ability to control and adapt to the environment. Mathematics uses many such tools. Some representational tools, such as counting and written numbers, are part of traditional mathematics learning in primary school. They improve our abilities in amazing ways: for example, counting allows us to represent precisely quantities which we could not discriminate perceptually and written numbers in the Hindu-Arabic system create the possibility of column arithmetic, which is not easily implemented with oral number when quantities are large or even with written Roman numerals. We have argued elsewhere (Nunes, 2002) that systems of signs enhance, structure and empower their users but learners must still construct meanings that allow them to use these systems. Our choice in this review was to consider how learners construct meanings rather than explore in depth the enabling role of mathematical representations. We discuss in much greater detail how they learn to use whole and rational numbers meaningfully than how they calculate with these numbers. Similarly, we discuss how they might learn the meaning and power of algebraic representation rather than how they might become fluent with algebraic manipulation. Psychological theories (Luria, 1973; Vygotsky, 1981) emphasise the empowering role of culturally developed systems of signs in human reasoning but stress that learners' construction of meanings for these signs undergoes a long development process in order for the signs to be truly empowering. Similarly, mathematics educators stress that technology is aimed not to replace, but to enhance mathematical reasoning (Noss and Hoyles, 1992).

Our reason for not focusing on technologies in this synthesis is that there are so many technological resources used today for doing mathematics that it is not possible to consider even those used or potentially useful in primary school in the required detail in this synthesis. We recognise this gap and strongly suggest that at least some of these issues be taken up for a synthesis at a later point, as some important comparative work already exists in the domain of column arithmetic (e.g. Anghileri, Beishuizen and Putten, 2002; Treffers, 1987) and the use of calculators (e.g. Ruthven, 2008).

We wish to emphasise, therefore, that this review is not an exhaustive one. It considers a part of today's knowledge in mathematics education. There are other, more specific aspects of the subject which, usually for reasons of space, we decided to by-pass. We shall explain the reasons for these choices as we go along.

Methods of the review

We obtained the material for the synthesis through a systematic search of peer reviewed journals, edited volumes and refereed conference proceedings.[1] We selected the papers that we read by first screening the abstracts: our main criteria for selecting articles to read were that they should be on a relevant topic and that they should report either the results of empirical research or theoretical schemes for understanding mathematics learning or both. We also consulted several books in order to read researchers' syntheses of their own empirical work and to access earlier well-established reviews of relevant research; we chose books that provide useful frameworks for research and theories in mathematics learning.

We hope that this review will become the object of discussion within the community of researchers, teachers and policy makers. We recognise that it is only one step towards making sense of the vast research on how students' thinking and knowledge of mathematics develops, and that other steps must follow, including a thorough evaluation of this contribution.

Teaching and learning mathematics: What is the nature of this task?

Learning mathematics is in some ways similar (but of course not identical) to language learning: in mathematics as well as in language it is necessary to learn symbols and their meaning, and to know how to combine them meaningfully.

Learning meanings for symbols is often more difficult than one might think. Think of learning the meaning of the word 'brother'. If Megan said to her four-year-old friend Sally 'That's my brother' and pointed to her brother, Sally might learn to say correctly and appropriately 'That's Megan's brother' but she would not necessarily know the meaning of 'brother'. 'Brother of' is a phrase that is based on a set of relationships, and in order to understand its meaning we need to understand this set of relationships, which includes 'mother of' and 'father of'. It is in this way that learning mathematics is very like learning a language: we need to learn mathematical symbols and their meanings, and the meaning of these symbols is based on sets of relations.

In the same way that Megan might point to her brother, Megan could count a set of pens and say: 'There are 15 pens here'. Sally could learn to count and say '15 pens' (or dogs, or stars). But '15' in mathematics does not just refer to the result of counting a set: it also means that this set is equivalent to all other sets with 15 objects, has

fewer objects than any set with 16 or more, and has more objects than any set with 14 or fewer. Learning about numbers involves more than understanding the operations that are carried out to determine the word that represents the quantity. In the context of learning mathematics, we would like students to know, without having to count, that some operations do and others do not change a quantity. For example, we would like them to understand that there would only be more pens if we added some to the set, and fewer if we subtracted some from the set, and that there would still be 15 pens if we added and then subtracted
(or vice versa) the same number of pens to and from the original set.

The basic numerical concepts that we want students to learn in primary school have these two sides to them: on the one hand, there are quantities, operations on quantities and relations between quantities, and on the other hand there are symbols, operations on symbols and relations between symbols. Mathematics teaching should aim to ensure that students' understanding of quantities, relations and symbols go together. Anything we do with the symbols has to be consistent with their underlying mathematical meaning as well as logically consistent and we are not free to play with meaning in mathematics in quite the same ways we might play with words.

This necessary connection is often neglected in theories about mathematics learning and in teaching practices. Theories that appear to be contradictory have often focused either on students' understanding of quantities or on their understanding of symbols and their manipulations. Similarly, teaching is often designed with one or the other of these two kinds of understanding in sight, and the result is that there are different ways of teaching that have different strengths and weaknesses.

Language learners eventually reach a time when they can learn the meaning of new words simply by definitions and connections with other words. Think of words like 'gene' and 'theory': we learn their meanings from descriptions provided by means of other words and from the way they are used in the language. Mathematics beyond primary school often works similarly: new mathematical meanings are learned by using previously learned mathematical meanings and ways of combining these. There are also other ways in which mathematics and language learning are similar; perhaps the most important of these other similarities is that we can use language to represent a large variety of meanings, and mathematics has a similar power. But, of course, mathematics learning differs from language learning: mathematics contains its own distinct concepts and modes of enquiry which determine the way that mathematics is used. This specificity of mathematical concepts is reflected in the themes that we chose to analyse in our synthesis.

The framework for this review

As we start our review, there is a general point to be made about the theoretical position that we have reached from our review of research on children's mathematics. On the whole, the teaching of the various aspects of mathematics proceeds in a clear sequence, and with a certain amount of separation in the teaching of different aspects. Children are taught first about the number sequence and then about written numbers and arithmetical operations using written numbers. The teaching of the four arithmetical operations is done separately. At school children learn about addition and subtraction separately and before they learn about multiplication and division, which also tend to be taught quite separately from each other. Lessons about arithmetic start years before lessons about proportions and the use of mathematical models.

This order of events in teaching has had a clear effect on research and theories about mathematical learning. For example, it is a commonplace that research on multiplication and division is most often (though there are exceptions; see Paper 4) carried out with children who are older than those who participate in research on addition and subtraction. Consequently, in most theories additive reasoning is hypothesised (or assumed) to precede multiplicative reasoning. Until recently there have been very few studies of children's understanding of the connection between the different arithmetical operations because they are assumed to be learned relatively independently of each other.

Our review of the relevant research has led us to us to a different position. The evidence quite clearly suggests that there is no such sequence, at any rate in the onset of children's understanding of some of these different aspects of mathematics. Much of this learning begins, as our review will show, in informal circumstances and before children go to school. Even after they begin to learn about mathematics formally, there are clear signs that they can embark on genuinely multiplicative reasoning, for example, at a time when the instruction they receive is all about addition and subtraction. Similar observations can be made about learning algebra; there are studies that show that quite young children are capable of expressing mathematical generalities in algebraic terms, but these are rare: the majority of studies focus on the ways in which learners fail to do so at the usual age at which this is taught.

Sequences do exist in children's learning, but these tend not to be about different arithmetic operations (e.g. not about addition before multiplication). Instead, they take the form of children's understanding of new quantitative relations as a result of working with and manipulating relations that have been familiar for some time. An example, which we describe in detail in Paper 2, is about the inverse relation between addition and subtraction. Young children easily understand that if you add some new items to a set of items and then subtract exactly the same items, the number of items in the set is the same as it was initially (inversion of identity), but it takes some time for them to extend their knowledge of this relation enough to understand that the number of items in the set will also remain the same if you add some new items and then subtract an equal number items from the set, which are not the same ones you had added (inversion of quantity: $a + b - b = a$). Causal sequences of this kind play an important part in the conclusions that we reach in this review.

Through our review, we identified some key understandings which we think children must achieve to be successful learners of mathematics and which became the main topics for the review. In the paragraphs that follow, we present the arguments that led us to choose the six main topics. Subsequently, each topic is summarised under a separate heading. The research on which these summaries are based is analysed in Papers 2 to 7.

The main points that are discussed here, before we turn to the summaries, guided the choice of papers in the review.

Quantity and number

The first point is that there is a distinction to be made between quantity and number and that children must make connections as well as distinctions between quantity and number in order to succeed in learning mathematics.

Thompson (1993) suggested that 'a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it. Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values' (pp. 165–166). Children experience and learn about quantities and the relations between them quite independently of learning to count. Similarly, they can learn to count quite independently from understanding quantities and relations between them. It is crucial for children to learn to make both connections and distinctions between number and quantity. There are different theories in psychology regarding how children connect quantity and number; these are discussed in Paper 2.

The review also showed that there are two different types of quantities that primary school children have to understand and that these are connected to different types of numbers. In everyday life, as well as in primary school, children learn about quantities that can be counted. Some are discrete and each item can be counted as a natural unit; other quantities are continuous and we use measurement systems, count the conventional units that are part of the system, and attribute numbers to these quantities. These quantities which are measured by the successive addition of items are termed extensive quantities. They are represented by whole numbers and give children their first insights into number.

In everyday life children also learn about quantities that cannot be counted like this. One reason why the quantity might not be countable in this way is that it may be smaller than the unit; for example, if you share three chocolate bars among four people, you cannot count how many chocolate bars each one receives. Before being taught about fractions, some primary school students are aware that you cannot say that each person would be given one chocolate bar, because they realise that each person's portion would be smaller than one: these children conclude that they do not know a number to say how much chocolate each person will receive (Nunes and Bryant, 2008). Quantities that are smaller than the unit are represented by fractions, or more generally by rational numbers.

Rational numbers are also used to represent quantities which we do not measure directly but only through a relation between two other measures. For example, if we want to say something about the concentration of orange squash in a glass, we have to say something about the ratio of concentrate used to water. This type of quantity, measurable by ratio, is termed intensive quantity and is often represented by rational numbers.

In Papers 2 and 3 we discuss how children make connections between whole and rational numbers and the different types of quantities that they represent.

Relations

Our second general point is about relations. Numbers are used to represent quantities as well as relations; this is why children must establish a connection between quantity and number but also distinguish between them. Measures are numbers that are connected to a quantity. Expressions such as 20 books, 3 centimetres, 4 kilos, and ½ a chocolate are measures. Relations, like quantities, do not have to be quantified. For example, we can simply say that two quantities are equivalent or different. This is a qualitative statement about the relation between two quantities. But we can quantify relations and we use numbers to do so: for example, when we compare two measures, we are quantifying a relation. If there are 20 children in a class and 17 books, we can say that there are 3 more children than books. The number 3 quantifies the relation. We can say 3 more children than books or 3 books fewer than children; the meaning does not change when the wording changes because the number 3 does not refer to children or to books, but to the relation between the two measures.

A major use of mathematics is to quantify relations and manipulate these representations to expand our understanding of a situation. We came to the conclusion from our review that understanding relations between quantities is at the root of understanding mathematical models. Thompson (1993) suggested that 'Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships… A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities' (p. 165). Elsewhere, Thompson (1994) emphasised that 'a quantitative operation is non-numerical; it has to do with the *comprehension* (italics in the original) of a situation.' (p. 187). So relations, like quantities, are different from numbers but we use numbers to quantify them.

Paper 4 of this synthesis discusses the quantification of relations in mathematics, with a focus on the sorts of relations that are part of learning mathematics in primary school.

The coordination of basic concepts and the development of higher order concepts

Students in secondary school have the dual task of refining what they have learned in primary school and understanding new concepts, which are based on reflections about and combinations of previous concepts. The challenge for students in secondary school is to learn to take a different perspective with respect to their mathematics knowledge and, at the same time, to learn about the power of this new perspective. Students can understand much about using mathematical representations (numbers, diagrams, graphs) for quantities and relations and how this helps them solve problems. Students who have gone this far understand the role of mathematics in representing and helping us understand phenomena, and even generalising beyond what we know. But they may not have understood a distinct and crucial aspect of the importance of mathematics: that, above and beyond helping represent and explore what you know, it can be used to discover what you do not know. In this review, we consider two related themes of this second side of mathematics: algebraic reasoning and modelling. Papers 6 and 7 summarise the research on these topics.

In the rest of this opening paper we shall summarise our main conclusions from our review. In other words, Papers 2 to 7 contain our detailed reviews of research on mathematics learning; each of the six subsequent sections about a central topic in mathematics learning is a summary of Papers 2 to 7.

Key understandings in mathematics: A summary of the topics reviewed

Understanding extensive quantities and whole numbers

Natural numbers are a way of representing quantities that can be counted. When children learn numbers, they must find out not just about the counting sequence and how to count, but also about how the numbers in the counting system represent quantities and relations between them. We found a great deal of evidence that children are aware of quantities such as the size of objects or the amount of items in groups of objects long before they learn to count or understand anything about the number system. This is quite clear in their ability to discriminate objects by size and sets by number when these discriminations can be made perceptually.

Our review also showed that children learn to count with surprisingly little difficulty. Counting is an activity organised by principles such as the order invariance of number labels, one-to-one correspondence between items and counting labels, and the use of the last label to say how many items are in the set. There is no evidence of children being taught these principles systematically before they go to school and yet most children starting school at the age of five years are already able to respect these principles when counting and identify other people's errors when they violate counting principles.

However, research on children's numerical understanding has consistently shown that at first they make very little connection between the number words that they learn and their existing knowledge about quantities such as size and the amounts of items. Our review showed that Thompson's (1993) theoretical distinction between quantities and number is hugely relevant to understanding children's mathematics. For example, many four-year-old children understand how to share objects equally between two or more people, on a one-for-A, one-for-B basis, but have some difficulty in understanding that the number of items in two equally shared sets must be the same, i.e. that if there are six sweets in one set, there must be six in the other set as well. To make the connection between number words and quantities, children have to grasp two aspects of number, which are *cardinal number* and *ordinal number*. By *cardinal number*, we mean that two sets with the same number of items in them are equal in amount. The term *ordinal number* refers to the fact that numbers are arranged in an ordered series of increasing magnitude: successive numbers in the counting sequence are greater than the preceding number by 1. Thus, 2 is a greater quantity than 1 and 3 than 2 and it follows that 3 must also greater than 1.

There are three different theories about how children come to co-ordinate their knowledge of quantities with their knowledge of counting.

The first is Piaget's theory, which maintains that this development is based on children's schemas of action and the coordination of the schemas with each other. Three schemas of action are relevant to natural number: adding, taking away, and setting objects in correspondence. Children must also understand how these schemas relate to each other. They must, for example, understand that a quantity increases by addition, decreases by subtraction, and that if you add and take away the same amount to an original quantity, that quantity stays the same. They must also understand the additive composition of number, which involves the coordination of one-to-one correspondence with addition and subtraction: if the elements of two sets are placed in correspondence but one has more elements than the other, the larger set is the sum of the smaller set plus the number of elements for which there is no corresponding item in the smaller set. Research has shown that this insight is not attained by young children, who think that adding elements to the smaller set will make it larger than the larger set without considering the number added.

A second view, in the form of a nativist theory, has been suggested by Gelman and Butterworth (Gelman & Butterworth, 2005). They propose that from birth children have access to an innate, inexact but powerful 'analog' system, whose magnitude increases directly with the number of objects in an array, and they attach the number words to the properties occasioning these magnitudes. According to this view both the system for knowing about quantities and the principles of counting are innate and are naturally coordinated.

A third theoretical alternative, proposed by Carey (2004), starts from a standpoint in agreement with Gelman's theory with respect to the innate analog system and counting principles. However, Carey does not think that these systems are coordinated naturally: they become so through a 'parallel individuation' system, which allows very young children to make precise discriminations between sets of one and two objects, and a little later, between two and three objects. During the same period, these children also learn number words and, through their recognition of 1, 2 and 3 as distinct quantities, they manage to associate the right count words ('one', 'two' and 'three') with the right quantities. This association between parallel individuation and the count list eventually leads to what Carey (2004) calls 'bootstrapping': the children lift themselves up by their own intellectual bootstraps by inducing a rule that the next count word in the counting system is exactly one more than the previous one. They do so, some time between the age of three – and five years and, therefore, before they go to school.

One important point to note about these three theories is that they use different definitions of cardinal number, and therefore different criteria for assessing whether children understand cardinality or not. Piaget's criterion is the one that we have mentioned already and which we ourselves think to be right: it is the understanding that two or more sets are equal in quantity when the number of items in them is the same (and vice-versa). Gelman's and Carey's less demanding criterion for understanding cardinality is the knowledge that the last count word for the set represents the set's quantity: if I count 'one, two, three' items and realise that means that that there are three in the set, I understand cardinal number. In our view, this second view of cardinality is inadequate for two reasons: first, it is actually based on the position of the count word and is thus more related to ordinal than cardinal number; second, it does not include any consideration of the fact that cardinal number involves inferences regarding the equivalence of sets. Piaget's definition of cardinal and ordinal number is much more stringent and it has not been disputed by mathematics educators. He was sceptical of the idea that children would understand cardinal and ordinal number concepts simply from learning how to count and the evidence we reviewed definitely shows that learning about quantities and numbers develop independently of each other in young children.

This conclusion has important educational implications. Schools must not be satisfied with teaching children how to count: they must ensure that children learn not only to count but also to establish connections between counting and their understanding of quantities.

Piaget's studies concentrated on children's ability to reason logically about quantitative relations. He argued that children must understand the inverse relation between addition and subtraction and also additive composition (which he termed class-inclusion and was later investigated under the label of part-whole relations) in order to truly understand number. The best way to test this sort of causal hypothesis is through a combination of longitudinal and intervention studies. Longitudinal studies with the appropriate controls can suggest that A is causally related to B if it is a specific predictor of B at a later time. Intervention studies can test these causal ideas: if children are successfully taught A and, as a consequence, their learning of B improves, it is safe to conclude that the natural, longitudinal connection between A and B is also a causal one .

It had been difficult in the past to use this combination of methods in the analysis of children's mathematics learning for a variety of reasons. First, researchers were not clear on what sorts of logical reasoning were vital to learning mathematics. There are now clearer hypothesis about this: the inverse relation between addition and subtraction and additive composition of number appear as key concepts in the work of different researchers. Second, outcome measures of mathematics learning were difficult to find. The current availability of standardised assessments, either developed for research or by policy makers for monitoring the performance of educational systems, makes both longitudinal and intervention studies possible, as these can be seen as valid outcome measures. Our own research has shown that researcher designed and government designed standardised assessments are highly correlated and, when used as outcome measures in longitudinal and intervention studies, lead to convergent conclusions. Finally, in order to carry out intervention studies, it is necessary to develop ways of teaching children the key concepts on which mathematics learning is grounded. Fortunately, there are currently successful interventions that can be used for further research to test the effect that learning about these key concepts has on children's mathematics learning.

Our review identified two longitudinal studies that show that children's understanding of logical aspects of number is vital for their mathematics learning. One was carried out in the United Kingdom and showed that children's understanding of the inverse relation between addition and subtraction and of additive composition at the beginning of school are specific predictors of their results in National Curriculum maths tests (a government designed and administered measure of children's mathematics learning) about 14 months later, after controlling for their general cognitive ability, their knowledge of number at school entry, and their working memory.

The second study, carried out in Germany, showed that a measure of children's understanding of the inverse relation between addition and subtraction when they were eight years old was a predictor of their performance in an algebra test when they were in university; controlling for the children's performance in an intelligence test given at age eight had no effect on the strength of the connection between their understanding of the inverse relation and their performance in the algebra measure.

Our review also showed that it is possible to improve children's understanding of these logical aspects of number knowledge. Children who were weak in this understanding at the beginning of school and improved this understanding through a short intervention performed significantly better than a control group that did not receive this teaching. Together, these studies allow us to conclude that it is crucial for children to coordinate their understanding of these logical aspects of quantities with their learning of numbers in order to make good progress in mathematics learning.

Our final step in this summary of research on whole numbers considered how children use additive reasoning to solve word problems. Additive reasoning is the logical analysis of problems that involve addition and subtraction, and of course the key concepts of additive composition and the inverse relation between addition and subtraction play an essential role in this reasoning. The chief tool used to investigate additive reasoning is the word problem. In word problems a scene is set, usually in one or two sentences, and then a question is posed. We will give three examples.

A Bob has three marbles and Bill has four: how many marbles do they have altogether? *Combine problem.*

B Wendy had four pictures on her wall and her parents gave her three more: how many does she have now? *Change problem.*

C Tom has seven books: Jane has five: how many more books does Tom have than Jane? *Compare problem*.

The main interest of these problems is that, although they all involve very simple and similar additions and subtractions, there are vast differences in the level of their difficulty. When the three kinds of problem are given in the form that we have just illustrated, the Compare problems are very much harder than the Combine and Change problems. This is not because it is too difficult for the children to subtract 5 from 7, which is how to solve this particular Compare problem, but because they find it hard to work out what to do so solve the problem. Compare problems require reasoning about relations between quantities, which children find a lot more difficult than reasoning about quantities.

Thus the difficulty of these problems rests on how well children manage to work out the arithmetical relations that they involve. This conclusion is supported by the fact that the relatively easy problems become a great deal more difficult if the mathematical relations are less transparent. For example, the usually easy Change problem is a lot harder if the result is given and the children have to work out the starting point. For example, Wendy had some pictures on her wall but then took 3 of them down: now she has 4 pictures left on the wall: how many were there in the first place? The reason that children find this problem a relatively hard one is that the story is about subtraction, but the solution is an addition. Pupils therefore have to call on their understanding of the inverse relation between adding and subtracting to solve this problem.

One way of analysing children's reactions to word problems is with the framework devised by Vergnaud, who argued that these problems involve quantities, transformations and relations. A Change problem, for example, involves the initial quantity and a transformation (the addition or subtraction) which leads to a new quantity, while Compare problems involve two quantities and the relation between them. On the whole, problems that involve relations are harder than those involving transformations, but other factors, such as the story being about addition and the solution being a subtraction or vice versa also have an effect.

The main impact of research on word problems has been to reinforce the idea with which we began this section. This idea is that in teaching children arithmetic we must make a clear distinction between numerical analysis and the children's understanding of quantitative relations. We must remember that there is a great deal more to arithmetical learning than knowing how to carry out numerical procedures. The children have to understand the quantitative relations in the problems that they are asked to solve and how to analyse these relations with numbers.

Understanding rational numbers and intensive quantities

Rational numbers, like whole numbers, can be used to represent quantities. There are some quantities that cannot be represented by a whole number, and to represent these quantities, we must use rational numbers. Quantities that are represented by whole numbers are formed by addition and subtraction: as argued in the previous section, as we add elements to a set and count them (or conventional units, in the case of continuous quantities), we find out what number will be used to represent these quantities. Quantities that cannot be represented by whole numbers are measured not by addition but by division: if we cut one chocolate, for example, in equal parts, and want to have a number to represent the parts, we cannot use a whole number.

We cannot use whole numbers when the quantity that we want to represent numerically:

- is smaller than the unit used for counting, irrespective of whether this is a natural unit (e.g. we have less than one banana) or a conventional unit (e.g. a fish weighs less than a kilo)
- involves a ratio between two other quantities (e.g. the concentration of orange juice in a jar can be described by the ratio of orange concentrate to water; the probability of an event can be described by the ratio between the number of favourable cases to the total number of cases). These quantities are called intensive quantities.

We have concluded from our review that there are serious problems in teaching children about fractions and that intensive quantities are not explicitly considered in the curriculum.

Children learn about quantities that are smaller than the unit through division. Two types of action schemes are used by children in division situations: partitioning, which involves dividing a whole into equal parts, and correspondence situations, where two quantities are involved, a quantity to be shared and a number of recipients of the shares.

Partitioning is the scheme of action most often used in primary schools in the United Kingdom to introduce the concept of fractions. Research shows that children have quite a few problems to solve when they partition continuous quantities: for example, they need to anticipate the connection between number of cuts and number of parts, and some children find themselves with an even number of parts (e.g. 6) when they wanted to have an odd number (e.g. 5) because they start out by partitioning the whole in half. Children also find it very difficult to understand the equivalence between fractions when the parts they are asked to compare do not look the same. For example, if they are shown two identical rectangles, each cut in half but in different ways (e.g. horizontally and diagonally), many 9- and 10-year-olds might say that the fractions are not equivalent; in some studies, almost half of the children in these age levels did not recognize the equivalence of two halves that looked rather different due to being the result of different cuts. Also, if students are asked to paint 2/3 of a figure divided into 9 parts, many 11- to 12-year-olds may be unable to do so, even though they can paint 2/3 of a figure divided into 3 parts; in a study in the United Kingdom, about 40% of the students did not successfully paint 2/3 of figures that had been divided into 6 or 9 sections.

Different studies that we reviewed showed that students who learn about fractions through the engagement of the partitioning schema in division tend to reply on perception rather than on the logic of division when solving problems: they are much more successful with items that can be solved perceptually than with those that cannot. There is a clear lesson here for education: number understanding should be based on logic, not on perception alone, and teaching should be designed to guide children to think about the logic of rational numbers.

The research that we reviewed shows that the partitioning scheme develops over a long period of time. This has led some researchers to develop ways to avoid asking the children to partition quantities by providing them with pre-divided shapes or with computer tools that do the partitioning for the children. The use of these resources has positive effects, but these positive effects seem to be obtained only after large amounts of instruction.

In some studies, the students had difficulties with the idea of improper fractions even after prolonged instruction. For example, one student argued with the researcher during instruction that you cannot have eight sevenths if you divided a whole into seven parts.

In contrast to the difficulties that children have with partitioning, children as young as five or six years in age are quite good at using correspondences in division, and do so without having to carry out the actual partitioning. Some children seem to understand even before receiving any instruction on fractions that, for example, two chocolates shared among four children and four chocolates shared among eight children will give the children in the two groups equivalent shares of chocolate; they demonstrate this equivalence in action by showing that in both cases there is one chocolate to be shared by two children.

Children's understanding of quantities smaller than one is often ahead of their knowledge of fractional representations when they solve problems using the correspondence scheme. This is true of understanding equivalence and even more so of understanding order. Most children at the age of eight or so realise that dividing 1 chocolate among three children will give bigger pieces than dividing one chocolate among four children. This insight that they have about quantities is not necessarily connected with their understanding of ordering fractions by magnitude: the same children might say that 1/3 is less than 1/4 because three is less than four. So we find in the domain of rational numbers the same distinction found in the domain of whole numbers between what children know about quantities and what they know about the numbers used to represent quantities.

Research shows that it is possible to help children connect their understanding of quantities with their understanding of fractions and thus make progress in rational number knowledge. Schools could make use of children's informal knowledge of fractional quantities and work with problems about situations, without requiring them to use formal representations, to help them consolidate this reasoning and prepare them for formalization.

Reflecting about these two schemes of action and drawing insights from them places children in different paths for understanding rational number. When children use the correspondence scheme, they can achieve some insight into the equivalence of fractions by thinking that, if there are twice as many things to be shared and twice as many recipients, then each one's share is the same. This involves thinking about a direct relation between the quantities. The partitioning scheme leads to understanding equivalence in a different way: if a whole is cut into twice as many parts, the size of each part will be halved. This involves thinking about an inverse relation between the quantities in the problem. Research consistently shows that children understand direct relations better than inverse relations and this may also be true of rational number knowledge.

The arguments children use when stating that fractional quantities resulting from sharing are or are not equivalent have been described in one study in the United Kingdom. These arguments include the use of correspondences (e.g. sharing four chocolates among eight children can be shown by a diagram to be equivalent to sharing two chocolates among four children because each chocolate is shared among two children), scalar arguments (twice the number of children and twice the number of chocolates means that they all get the same), and an understanding of the inverse relation between the number of parts and the size of the parts (i.e. twice the number of pieces means that each piece is halved in size). It would be important to investigate whether increasing teachers' awareness of children's own arguments would help teachers guide children's learning in this domain of numbers more effectively.

Some researchers have argued that a better starting point for teaching children about fractions is the use of situations where children can use correspondence reasoning than the use of situations where the scheme of partitioning is the relevant one. Our review of children's understanding of the equivalence and order of fractions supports this claim. However, there are no intervention studies comparing the outcomes of these two ways of introducing children to the use of fractions, and intervention studies would be crucial to solve this issue: one thing is children's informal knowledge but the outcomes of its formalization through instruction might be quite another. There is now considerably more information regarding children's informal strategies to allow for new teaching programmes to be designed and assessed. There is also considerable work on curriculum development in the domain of teaching fractions in primary school. Research that compares the different forms of teaching (based on partitioning or based on correspondences) and the introduction of different representations (decimal or ordinary) is now much more feasible than in the past. Intervention research, which could be carried out in the classroom, is urgently needed. The available evidence suggests that testing this hypothesis appropriately could result in more successful teaching and learning of rational numbers.

In the United Kingdom ordinary fractions continue to play an important role in primary school instruction whereas in some countries greater attention is given to decimal representation than to ordinary fractions in primary school. Two reasons are proposed to justify the teaching of decimals before ordinary fractions . First, decimals are common in metric measurement systems and thus their understanding is critical for learning other topics, such as measurement, in mathematics and science. Second, decimals should be easier than ordinary fractions to understand because decimals can be taught as an extension of place value representation; operations with decimals should also be easier and taught as extensions of place value representation.

It is certainly true that decimals are used in measurement and thus learning decimals is necessary but ordinary fractions often appear in algebraic expressions; so it is not clear *a priori* whether one form of representation is more useful than the other for learning other aspects of mathematics. However, the second argument, that decimals are easier than ordinary fractions, is not supported in surveys of students' performance: students find it difficult to make judgements of equivalence and order as much with decimals as with ordinary fractions. Students aged 9 to 11 years have limited success when comparing decimals written with different numbers of digits after the decimal point (e.g. 0.5 and 0.36): the rate of correct responses varied between 36% and 52% in the three different countries that participated in the study, even though all the children have been taught about decimals.

Some researchers (e.g. Nunes, 1997; Tall, 1992; Vergnaud, 1997) argue that different representations shed light on the same concepts from different perspectives. This would suggest that a way to strengthen students' learning of rational numbers is to help them connect both representations. Case studies of students who received instruction that aimed at helping students connect the two forms of representation show encouraging results. However, the investigation did not include the appropriate controls and so it does not allow for establishing firmer conclusions.

Students can learn procedures for comparing, adding and subtracting fractions without connecting these procedures with their understanding of equivalence and order of fractional quantities, independently of whether they are taught with ordinary or decimal fractions representation. This is not a desired outcome of instruction, but seems to be a quite common one. Research that focuses on the use of children's informal knowledge suggests that it is possible to help students make connections between their informal knowledge and their learning of procedures but the evidence is limited and the consequences of this teaching have not been investigated systematically.

Research has also shown that students do not spontaneously connect their knowledge of fractions developed with extensive quantities smaller than the unit with their understanding of intensive quantities. Students who succeed in understanding that two chocolates divided among four children and four chocolates divided among 8 children yield the same size share do not necessarily understand that a paint mixture made with two litres of white and two of blue paint will be the same shade as one made with four litres of white and four of blue paint.

Researchers have for some time distinguished between different situations where fractions are used and argued that connections that seem obvious to an adult are not necessarily obvious to children. There is now evidence that this is so. There is a clear educational implication of this result: if teaching children about fractions in the domain of extensive quantities smaller than the unit does not spontaneously transfer to their understanding of intensive quantities, a complete fractions curriculum should include intensive quantities in the programme.

Finally, this review opens the way for a fresh research agenda in the teaching and learning of fractions. The source for the new research questions is the finding that children achieve insights into relations between fractional quantities before knowing how to represent them. It is possible to envisage a research agenda that would not focus on children's misconceptions about fractions, but on children's possibilities of success when teaching starts from thinking about quantities rather than from learning fractional representations.

Understanding relations and their graphical representation

Children form concepts about quantities from their everyday experiences and can use their schemas of action with diverse representations of the quantities (iconic, numerical) to solve problems. They often develop sufficient awareness of quantities to discuss their equivalence and order as well as how quantities are changed by operations. It is significantly more difficult for them to become aware of the relations between quantities and operate on relations.

The difficulty of understanding relations is clear both with additive and multiplicative relations between quantities. Children aged about eight to ten years can easily say, for example, how many marbles a boy will have in the end if he started a game with six marbles, won five in the first game, lost three in the second game, and won two in the third game. However, if they are not told how many marbles the boy had at the start and are asked how many more or fewer marbles this boy will have after playing the three games, they find this second problem considerable harder, particularly if the first game involves a loss.

Even if the children are taught how to represent relations and recognise that winning five in the first game does not mean having five marbles, they often interpret the results of operations on relations as if they were quantities. Children find both additive and multiplicative relations significantly more difficult than understanding quantities.

There is little evidence that the design of mathematics curricula has so far taken into account the importance of helping students become aware of the difference between quantities and relations. Some researchers have carried out experimental teaching studies which suggest that it is possible to promote students' awareness of additive relations as different from quantities; this was not an easy task but the instruction seemed to have positive results (but note that there were no control groups). Further research must be carried out to analyse how this knowledge affects mathematics learning: longitudinal and intervention studies would be crucial to clarify this. If positive results are found, there will be imperative policy implications.

The first teaching that children receive in school about multiplicative relations is about proportions. Initial studies on students' understanding of proportions previously led to the conclusion that students' problems with proportional reasoning stemmed from their difficulties with multiplicative reasoning. However, there is presently much evidence to show that, from a relatively early age (about five to six years in the United Kingdom), many children (our estimate is about two-thirds) already have informal knowledge that allows them to solve multiplicative reasoning problems.

Multiplicative reasoning problems are defined by the fact that they involve two (or more) measures linked by a fixed ratio. Students' informal knowledge of multiplicative reasoning stems from the schema of one-to-many correspondence, which they use both in multiplication and division problems. When the product is unknown, children set the elements in the two measures in correspondence (e.g. one sweet costs 4p) and figure out the product (how much five sweets will cost) by counting or adding. When the correspondence is unknown (e.g. if you pay 20p for five sweets, how much does each sweet cost), the children share out the elements (20p shared in five groups) to find what the correspondence is.

This informal knowledge is currently ignored in U.K. schools, probably due to the theory that multiplication is essentially repeated addition and division is repeated subtraction. However, the connections between addition and multiplication on the one hand, and subtraction and division on the other hand, are procedural and not conceptual. So students' informal knowledge of multiplicative reasoning could be developed in school from an earlier age.

Even after being taught other methods to solve proportions problems in school, students continue to use one-to-many correspondences reasoning to solve proportions problems; these solutions have been called building up methods. For example, if a recipe for four people is to be adapted to serve six people, students figure out that six people is the same as four people plus two people; so they figure out what half the ingredients will be and add this to the quantity required for four people. Building up methods have been documented in many different countries and also among people with low levels of schooling. A careful analysis of the reasoning in building-up methods suggests that the students focus on the quantities as they solve these problems, and find it difficult to focus on the relations between the quantities.

Research carried out independently in different countries has shown that students sometimes use additive reasoning about relations when the appropriate model is a multiplicative one. Some recent research has shown that students also use multiplicative reasoning in situations where the appropriate model is additive. These results suggest that children use additive and multiplicative models implicitly and do not make conscious decisions regarding which model is appropriate in a specific situation. We concluded from our review that students' problems with proportional reasoning stems from their difficulties in becoming explicitly aware of relations between quantities. Greater awareness of the models implicit in their solutions would help them distinguish between situations that involve different types of relations: additive, proportional or quadratic, for example.

The educational implication from these findings is that schools should take up the task of helping students become more aware of the models that they use implicitly and of ways of testing their appropriateness to particular situations. The differences between additive and multiplicative situations rests on the relations between quantities; so it is likely that the critical move here is to help students become aware of the relations between quantities implicit in the procedure they use to solve problems.

Two radically different approaches to teaching proportions and linear functions in schools can be identified in the literature. These constitute pragmatic theories, which can guide teachers, but have as yet not been tested systematically. The first, described as functional and human in focus, is based on the notion that students' schemas of action should be the starting point for this teaching. Through instruction, they should become progressively more aware of the relations between quantities that can be identified in such problems. Diagrams, tables and graphs are seen as tools that could help students understand the models of situations that they are using and make them into models for other situations later.

The second, described in the literature as algebraic, proposes that there should be a sharp separation between students' intuitive knowledge, in which physical and mathematical knowledge are intertwined, and mathematical knowledge. Students should be led to formalisations early on in instruction and re-establish the connections between mathematical structures and physical knowledge at a later point. Representations using ordinary and decimal fractions and the number line are seen as the tools that can allow students to abstract early on from the physical situations. Students should learn early on to represent equivalences between ordinary fractions (e.g. 2/4 = 4/8), a representation that would provide insight into proportions, and also equivalences between ordinary and decimal fractions (2/4 = 0.5), which would provide insight into the ordering and equivalence of fractions marked on the number line.

Each of these approaches makes assumptions about the significance of students' informal knowledge at the start of the teaching programme. The functional approach assumes that students' informal knowledge can be formalised through instruction and that this will be beneficial to learning. The algebraic approach assumes that students' informal knowledge is an obstacle to students' mathematics learning. There is evidence from a combination of longitudinal and intervention methods, albeit with younger children, that shows that students' knowledge of informal multiplicative reasoning is a causal and positive factor in mathematics learning. Children who scored higher in multiplicative reasoning problems at the start of their first year in school performed significantly better in the government designed and school administered mathematics achievement test than those whose scores were lower. This longitudinal relationship remained significant after the appropriate controls were taken into account. The intervention study provides results that are less clear because the children were taught not only about multiplicative reasoning but also about other concepts considered key to mathematics learning. Nevertheless, children who were at risk for mathematics learning and received teaching that included multiplicative reasoning, along with two other concepts, showed average achievement in the standardised mathematics achievement tests whereas the control group remained in the bottom 20% of the distribution, as predicted by their assessment at the start of school. So, in terms of the assumptions regarding the role of informal knowledge, the functional approach seems to have the edge over the algebraic approach.

These two approaches to instruction also differ in respect to what students need to know to benefit from teaching and what they learn during the course of instruction. Within the functional approach, the tools used in teaching are diagrams, tables, and graphs so it is clear that students need to learn to read graphs in order to be able to use them as tools for thinking about relations between quantities and functions. Research has shown that students have ideas about how to read graphs before instruction and these ideas should be taken into account when graphs are used in the classroom. It is possible to teach students to read graphs and to use them in order to think about relations in the course of instruction about proportions, but much more research is needed to show how students' thinking changes if they do learn to use graphs to analyse the type of relation relevant in specific situations. Within the algebraic approach, it is assumed that students understand the equivalence of fractions without reference to situations. Our review of students' understanding of fractions, summarised in the previous section, shows that this is not trivial so it is necessary to show that students can, in the course of this teaching, learn both about fraction equivalence and proportional relations.

There is no evidence to show how either of these approaches to teaching works in promoting students' progress nor that one of them is more successful than the other. Research that can clarify this issue is urgently needed and could have a major impact in promoting better learning by U.K. students. This is particularly important in view of findings from the international comparisons that show that U.K. students do relatively well in additive reasoning items but comparatively poorly in multiplicative reasoning items.

Understanding space and its representation in mathematics

When children begin to be taught about geometry, they already know a great deal about space, shape, size, distance and orientation, which are the basic subject matter of geometry. They are also quite capable of drawing logical inferences about spatial matters. In fact, their spatial knowledge is so impressive and so sophisticated that one might expect geometry to be an easy subject for them. Why should they have any difficulty at all with geometry if the subject just involves learning how to express this spatial knowledge mathematically?

However, many children do find geometry hard and some children continue to make basic mistakes right through their time at school. There are two main reasons for these well-documented difficulties. One reason is that many of the spatial relations that children must think about and learn to analyse mathematically in geometry classes are different from the spatial relations that they learn about in their pre-school years. The second is that geometry makes great demands on children's spatial imagination. In order to measure length or area or angle, for example, we have to imagine spaces divided into equal units and this turns out to be quite hard for children to learn to do systematically.

Nevertheless, pre-school children's spatial knowledge and spatial experiences are undoubtedly relevant to the geometry that they must learn about later, and it is important for teachers and researchers alike to recognise this. From a very early age children are able to distinguish and remember different shapes, including basic geometrical shapes. Children are able to co-ordinate visual information about size and distance to recognise objects by their actual size, and also to co-ordinate visual shape and orientation information to recognise objects by their actual shapes. In social situations, children quite easily work out what someone else is looking at by extrapolating that person's line of sight often across quite large distances, which is an impressive feat of spatial imagination. Finally, they are highly sensitive not just to the orientation of lines and of objects in their environments, but also to the relation between orientations: for example, young children can, sometimes at least, recognise when a line in the foreground is parallel to a stable background feature.

These impressive spatial achievements must help children in their efforts to understand the geometry that they are taught about at school, but there is little direct research on the links between children's existing informal knowledge about space and the progress that they make when they are eventually taught about geometry. This is a worrying gap, because research of this sort would help teachers to make an effective connection between what their pupils know already and what they have to learn in their initial geometry classes. It would also give us a better understanding of the obstacles that children encounter when they are first taught about geometry.

Some of these obstacles are immediately apparent when children learn about measurement, first of length and then of area. In order to learn how to measure length, children must grasp the underlying logic of measurement and also the role of iterated (i.e. repeated) measurement units, e.g. the unit of 1 cm repeated on a ruler. Using a ruler also involves an active form of one-to-one correspondence, since the child must imagine and impose on the line being measured the same units that are explicit and obvious on the ruler. Research suggests that children do have a reasonable understanding of the underlying logic of measurement by the time that they begin to learn about geometry, but that many have a great deal of difficulty in grasping how to imagine one-to-one correspondence between the iterated units on the ruler and imagined equivalent units on the line that they are measuring. One common mistake is to set the 1 cm rather than the 0 cm point at one end of the line. The evidence suggests that many children apply a poorly understood procedure when they measure length and are not thinking, as they should, of one-to-one correspondence between the units on the ruler and the length being measured. There is no doubt that teachers should think about how to promote children's reflection on measurement procedures. Nunes, Light and Mason (1993), for example, showed that using a broken ruler was one way to promote this.

Measurement of area presents additional problems. One is that area is often calculated from lengths, rather than measured. So, although the measurement is in one kind of unit, e.g. centimetres, the final calculation is in another, e.g. square centimetres. This is what Vergnaud calls a 'product of measures' calculation. Another potential problem is that most calculations of area are multiplicative: with rectangles and parallelograms, one has to multiply the figure's base by height, and with triangles one must calculate base by height and then halve it. There is evidence that many children attempt to calculate area by adding parts of the perimeter, rather than by multiplying. One consequence of the multiplicative nature of area calculations is that doubling a figure's dimensions more than doubles its area. Think of a rectangle with a base of 10 cm and a height of 4 cm, its area is 40 cm^2: if you enlarge the figure by doubling its base and height (20 cm x 8 cm), you quadruple its area (160 cm^2). This set of relations is hard for pupils, and for many adults too, to understand.

The measurement of area also raises the question of relations between shapes. For example the proof that the same base by height rule for measuring rectangles applies to parallelograms as well rests on the demonstration that a rectangle can be transformed into a parallelogram with the same height and base without changing its area. In turn the rule for finding the area of triangles, A = ½ (base x height), is justified by the fact every triangle can be transformed into a parallelogram with the same base and height by doubling that triangle. Thus, rules for measuring area rest heavily on the relations between geometric shapes. Although Wertheimer did some ingenious studies on how children were able to use of the relations between shapes to help them measure the area of some of these shapes, very little research has been done since then on their understanding of this centrally important aspect of geometry.

In contrast, there is a great deal of research on children's understanding of angles. This research shows that children have very little understanding of angles before they are taught about geometry. The knowledge that they do have tends to be quite disconnected because children often fail to see the connection between angles in dissimilar contexts, like the steepness of a slope and how much a person has to turn at a corner. There is evidence that children begin to connect what they know about angles as they grow older: they acquire, in the end, a fairly abstract understanding of angle. There is also evidence, mostly from studies with the programming language Logo, that children learn about angle relatively well in the context of movement.

Children's initial uncertainties with angles contrast sharply to the relative ease with which they adopt the Cartesian framework for plotting positions in any two-dimensional space. This framework requires them to be able to extrapolate imaginary pairs of straight lines, one of which is perpendicular to the vertical axis and the other to the horizontal axis, and then to work out where these imaginary lines will meet, in order to plot specific positions in space. At first sight this might seem an extraordinarily sophisticated achievement, but research suggests that it presents no intellectual obstacle at all to most children. Their success in extrapolating imaginary straight lines and working out their meeting point may stem for their early experiences in social interactions of extrapolating such lines when working out what other people are looking at, but we need longitudinal research to establish whether this is so. Some further research suggests that, although children can usually work out specific spatial positions on the basis of Cartesian co-ordinates, they often find it hard to use these co-ordinates to work out the relation between two or more different positions in space.

We also need research on another possible connection between children's early informal spatial knowledge and how well they learn about geometry later on. We know that very young children tell shapes apart, even abstract geometric shapes, extremely well, but it is also clear that when children begin to learn about geometry they often find it hard to decompose complicated shapes into several simpler component shapes. This is a worrying difficulty because the decomposition of shapes plays an important part in learning about measurement of area and also of angles. More research is needed on how children learn that particular shapes can be broken down into other component shapes.

Overall, research suggests that the relation between the informal knowledge that children build up before they go to school and the progress that they make at school in geometry is a crucial one. Yet, it is a relation on which there is very little research indeed and there are few theories about this possible link as well. The theoretical frameworks that do exist tend to be pragmatic ones. For example, the Institute Freudenthal group assume a strong link between children's preschool spatial knowledge and the progress that they make in learning about geometry later on, and argue that improving children's early understanding of space will have a beneficial effect on their learning about geometry. Yet, there is no good empirical evidence for either of these two important claims.

Algebraic reasoning

Research on learning algebra has considered a range of new ideas that have to be understood in school mathematics: the use and meaning of letters and expressions to represent numbers and variables; operations and their properties; relations, functions, equations and inequalities; manipulation and transformation of symbolic statements. Young children are capable of understanding the use of a letter to take the place of an unknown number, and are also able to construct statements about comparisons between unknown quantities, but algebra is much more than the substitution of letters for numbers and numbers for letters. Letters are used in mathematics in varying ways. They are used as labels for objects that have no numerical value, such as vertices of shapes or for objects that do have numerical value, such as lengths of sides of shapes. They denote fixed constants such as g, e or π and also non-numerical constants such as I and they represent unknowns and variables. Distinguishing between these meanings is usually not taught explicitly, and this lack of instruction might cause children some difficulty: g, for example, can indicate grams, acceleration due to gravity, an unknown in an equation, or a variable in an expression.

Within common algebraic usage, Küchemann (1981) identified six different ways adolescents used letters in the Chelsea diagnostic test instrument (Hart *et al*., 1984). Letters could be evaluated in some way, ignored, used as shorthand for objects or treated as objects used as a specific unknown, as a generalised number, or as a variable. These interpretations appear to be task-dependent, so learners had developed a sense of what sorts of question were treated in what kinds of ways, i.e. generalising (sometimes idiosyncratically) about question-types through familiarity and prior experience.

The early experiences students have in algebra are therefore very important, and if algebra is presented as 'arithmetic with letters' there are many possible confusions. Algebraic statements are about relationships between variables, constructed using operations; they cannot be calculated to find an answer until numbers are substituted, and the same relationship can often be represented in many different ways. The concept of equivalent expressions is at the heart of algebraic manipulation, simplification, and expansion, but this is not always apparent to students. Students who do not understand this try to act on algebraic expressions and equations in ways which have worked in arithmetical contexts, such as trial-and-error, or trying to calculate when they see the equals sign, or rely on learnt rules such as 'BODMAS' which can be misapplied.

Students' prior experience of equations is often associated with finding hidden numbers using arithmetical facts, such as 'what number, times by 4, gives 24?' being expressed as $4p = 24$. An algebraic approach depends on understanding operations or functions and their inverses, so that addition and subtraction are understood as a pair, and multiplication and division are understood as a pair. This was discussed in an earlier section. Later on, roots, exponents and logarithms also need to be seen as related along with other functions and their inverses. Algebraic understanding also depends on understanding an equation as equating two expressions, and solving them as finding out for what values of the variable they are equal. New technologies such as graph-plotters and spreadsheets have made multiple representations available and there is substantial evidence that students who have these tools available over time develop a stronger understanding of the meaning of expressions, and equations, and their solutions, than equivalent students who have used only formal pencil-and-paper techniques.

Students have to learn that whereas the mathematical objects they have understood in primary school can often be modelled with material objects they now have to deal with objects that cannot always be easily related to their understanding of the material world, or to their out-of-school language use. The use of concrete models such as rods of 'unknown' related lengths, tiles of 'unknown' related areas, equations seen as balances, and other diagrammatical methods can provide bridges between students' past experience and abstract relationships and can enable them to make the shift to seeing relations rather than number as the main focus of mathematics. All these metaphors have limitations and eventually, particularly with the introduction of negative numbers, the metaphors they provide break down. Indeed it was this realisation that led to the invention of algebraic notation.

Students have many perceptions and cognitive tendencies that can be harnessed to help them learn algebra. They naturally try to relate what they are offered to what they already know. While this can be a problem if students refer to computational arithmetic, or alphabetic meaning of letters (e.g. a = apples), it can also be useful if they refer to their understanding of relations between quantities and operations and inverses. For example, when students devise their own methods for mental calculation they often use relations between numbers and the concepts of distributivity and associativity.

Students naturally try to generalise when they see repeated behaviour, and this ability has been used successfully in approaches to algebra that focus on expressing generalities which emerge in mathematical exploration. When learners need to express generality, the use of letters to do so makes sense to them, although they still have to learn the precise syntax of their use in order to communicate unambiguously. Students also respond to the visual impact of mathematics, and make inferences based on layout, graphical interpretation and patterns in text; their own mathematical jottings can be structured in ways that relate to underlying mathematical structure. Algebraic relationships represented by graphs, spreadsheets and diagrammatic forms are often easier to understand than when they are expressed in symbols. For example, students who use function machines are more likely to understand the order of operations in inverse functions.

The difficulties learners have with algebra in secondary school are nearly all due to their inability to shift from earlier understandings of arithmetic to the new possibilities afforded by algebraic notation.

- They make intuitive assumptions and apply pragmatic reasoning to a symbol system they do not yet understand.
- They need to grasp the idea that an algebraic expression is a statement about relationships between numbers and operations.
- They may confuse equality with equivalence and try to get answers rather than transform expressions.
- They get confused between using a letter to stand for something they know, and using it to stand for something they do not know, and using it to stand for a variable.
- They may not have a purpose for using algebra, such as expressing a generality or relationship, so cannot see the meaning of what they are doing.

New technologies offer immense possibilities for imbuing algebraic tasks with meaning, and for generating a need for algebraic expression.

The research synthesis sets these observations out in detail and focuses on detailed aspects of algebraic activity that manifest themselves in school mathematics. It also formulates recommendations for practice and research.

Modelling, problem-solving and integrating concepts

Older students' mathematical learning involves situations in which it is not immediately apparent what mathematics needs to be done or applied, nor how this new situation relates to previous knowledge. Learning mathematics includes learning when and how to adapt symbols and meanings to apply them in unfamiliar situations and also knowing when and how to adapt situations and representations so that familiar tools can be brought to bear on them. Students need to learn how to analyse complex situations in a variety of representations, identify variables and relationships, represent these and develop predictions or conclusions from working with representations of variables and relationships. These might be presented graphically, symbolically, diagrammatically or numerically.

In secondary mathematics, students possess not only intuitive knowledge from outside mathematics and outside school, but also a range of quasi-intuitive understandings within mathematics, derived from earlier teaching and generalisations, metaphors, images and strategies that have served them well in the past. In Tall and Vinner's pragmatic theory, (1981) these are called 'concept images', which are a ragbag of personal conceptual, quasi-conceptual, perceptual and other associations that relate to the language of the concept and are loosely connected by the language and observable artefacts associated with the concept. The difference between students' concept images and conventional definitions causes problems when they come to learn new concepts that combine different earlier concepts. They have to expand elementary meanings to understand new abstract concepts, and sometimes these concepts do not fit with the images and models that students know. For example, rules for combining quantities do not easily extend to negative numbers; multiplication as repeated addition does not easily extend to multiplying decimals.

There is little research and theoretical exploration regarding how combinations of concepts are understood by students in general. For example, it would be helpful to know if students who understand the use of letters, ratio, angle, functions, and geometrical facts well have the same difficulties in learning early trigonometry as those whose understanding is more tenuous. Similarly, it would be helpful to know if students whose algebraic manipulation skills are fluent understand quadratic functions more easily, or differently, from students who do not have this, but do understand transformation of graphs.

There is research about how students learn to use and apply their knowledge of functions, particularly in the context of modelling and problem-solving. Students not only have to learn to think about relationships (beyond linear relationships with which they are already familiar), but they also need to think about relations between relations. Our analysis (see Paper 4) suggested that curricula presently do not consider the important task of helping students become aware of the distinctions between quantities and relations; this task is left to the students themselves. It is possible that helping students make this distinction at an earlier age could have a positive impact on their later learning of algebra.

In the absence of specific instructions, students tend to repeat patterns of learning that have enabled them to succeed in other situations over time. Students tend to start on new problems with qualitative judgements based on a particular context, or the visual appearance of symbolic representations, then tend to use additive reasoning, then form relationships by pattern recognition or repeated addition, and then shift to proportional and relational thinking if necessary. The tendency to use addition as a first resort persists as an obstacle into secondary mathematics. Students also tend to check their arithmetic if answers conflict rather than adapting their reasoning by seeing if answers make sense or not, or by analysing what sorts of relations are important in the problem. Pedagogic intervention over time is needed to enable learners to look for underlying structure and, where multiple representations are available (graphs, data, formulae, spreadsheets), students can, over time, develop new habits that focus on covariation of variables. However, they need knowledge and experience of a range of functions to draw on. Students are unlikely to detect an exponential relationship unless they have seen one before, but they can describe changes between nearby values in additive terms. A shift to describing changes in multiplicative terms does not happen naturally.

We hoped to find evidence about how students learn to use mathematics to solve problems when it is not immediately clear what mathematics they should be using. Some evidence in elementary situations has been described in an earlier section, but at secondary level there is only evidence of successful strategies, and not about how students come to have these strategies. In modelling and some other problem-solving situations successful students know how to identify variables and how to form an image of simultaneous variation. Successful students know how to hold one variable still while the change in another is observed. They are also able to draw on a repertoire of known function-types to say more about how the changes in variables are related. It is more common to find secondary students treating each situation as *ad hoc* and using trial-and-adjustment methods which are arithmetically-based. Pedagogic intervention over time is needed to enable them to shift towards seeing relationships, and relations between relations, algebraically and using a range of representational tools to help them do so.

The tendencies described above are specific instances of a more general issue. 'Outside' experiential knowledge is seldom appropriate as a source for meaning in higher mathematics, and students need to learn how to distinguish between situations where earlier and 'outside' understandings are, and are not, going to be helpful. For example, is it helpful to use your 'outside' knowledge about cooking when solving a ratio problem about the size of cakes? In abstract mathematics the same is true: the word 'similar' means something rather vague in everyday speech, but has specific meaning in mathematics. Even within mathematics there are ambiguities. We have to understand, for example, that -40 is greater in magnitude than -4, but a smaller number.

All students generalise inductively from the examples they are given. Research evidence of secondary mathematics reveals many typical problems that arise because of generalising irrelevant features of examples, or over-generalising the domain of applicability of a method, but we found little systematic research to show instances where the ability to generalise contributes positively to learning difficult concepts, except to generate a need to learn the syntax of algebra.

Finally, we found considerable evidence that students do, given appropriate experiences over time, change the ways in which they approach unfamiliar mathematical situations and new concepts. We only found anecdotal evidence that these new ways to view situations are extended outside the mathematics classroom. There is considerable evidence from long-term curriculum studies that the procedures students have to learn in secondary mathematics are learnt more easily if they relate to less formal explorations they have already undertaken. There is evidence that discussion, verbalisation, and explicitness about learning can help students make these changes.

Five common themes across the topics reviewed

In our view, a set of coherent themes cuts across the rich, and at first sight heterogeneous, topics around which we have organised our outline. These themes rise naturally from the material that we have mentioned, and they do not include recent attempts to link brain studies with mathematical education. In our view, knowledge of brain functions is not yet sophisticated enough to account for assigning meaning, forming mathematical relationships or manipulating symbols, which we have concluded are the significant topics in studies of mathematical learning.

In this section, we summarise five themes that emerged as significant across the research on the different topics, summarised in the previous sections.

Number

Number is not a unitary idea that develops conceptually in a linear fashion. In learning, and in mathematical meaning, understanding of number develops in complementary strands, sometimes with discontinuities and changes of meaning. Emphasis on calculation and manipulation with numbers rather than on understanding the underlying relations and mathematical meanings can lead to over-reliance and misapplication of methods.

Most children start school with everyday understandings that can contribute to their early learning of number. They understand 'more' and 'less' without knowing actual quantities, and can compare discrete and continuous quantities of familiar objects. Whole number is the tool which enables them to be precise about comparisons and relations between quantities, once they understand cardinality.

Learning to count and understanding quantities are separate strands of development which have to be experienced alongside each other. This allows comparisons and combinations to be made that are expressed as relations. Counting on its own does not provide for these. Counting on its own also means that the shift from discrete to continuous number is a conceptual discontinuity rather than an extension of meaning.

Rational numbers (we have used 'fraction' and 'rational number' interchangeably in order to focus on their meaning for learners, rather than on their mathematical definitions) arise naturally for children from understanding division in sharing situations, rather than from partitioning wholes. Understanding rational numbers as a way of comparing quantities is fundamental to the development of multiplicative and proportional reasoning, and to applications in geometry, science, and everyday life. This is not the same as saying that children should do arithmetic with rational numbers. The decimal representation does not afford this connection (although it is relatively easy to do additive arithmetic with decimal fractions, as long as the same number of digits appears after the decimal point).

The connection between number and quantity becomes less obvious in higher mathematics, e.g. on the co-ordinate plane the numbers indicate scaled lengths from the axes, but are more usefully understood as values of the variables in a function. Students also have to extend the meaning of number to include negative numbers, infinitesimals, irrationals, and possibly complex numbers. Number has to be abstracted from images of quantity and used as a set of related, continuous, values which cannot all be expressed or depicted precisely. Students also have to be able to handle number-like entities in the form of algebraic terms, expressions and functions. In these contexts, the idea of number as a systematically related set (and subsets) is central to manipulation and transformation; they behave like numbers in relations, but are not defined quantities that can be enumerated. Ordinality of number also has a place in mathematics, in the domain of functions that generate sequences, and also in several statistical techniques.

Successful learning of mathematics includes understanding that number describes quantity; being able to make and use distinctions between different, but related, meanings of number; being able to use relations and meanings to inform application and calculation; being able to use number relations to move away from images of quantity and use number as a structured, abstract, concept.

Logical reasoning plays a crucial part in every branch of mathematical learning

The importance of logic in children's understanding and learning of mathematics is a central theme in our review. This idea is not a new one, since it was also the main claim that Piaget made about children's understanding of mathematics. However, Piaget's theory has fallen out of favour in recent years, and many leading researchers on mathematics learning either ignore or actively dismiss his and his colleagues' contribution to the subject. So, our conclusion about the importance of logic may seem a surprising one but, in our view, it is absolutely inescapable. We conclude that the evidence demonstrates beyond doubt that children rely on logic in learning mathematics and that many of their difficulties in solving mathematical problems are due to failures on their part to make the correct logical move which would have led them to the correct solution.

We have reviewed evidence that four different aspects of logic have a crucial role in learning about mathematics. Within each of these aspects we have been able to identify definite changes over time in children's understanding and use of the logic in question. The four aspects follow.

The logic of correspondence (one-to-one and one-to-many correspondence)

Children must understand one-to-one correspondence in order to learn about cardinal number. Initially they are much more adept at applying this kind of correspondence when they share than when they compare spatial arrays of items. The extension of the use of one-to-one correspondence from sharing to working out the numerical equivalence or non-equivalence of two or more spatial arrays is a vastly important step in early mathematical learning.

One–to-many correspondence, which itself is an extension of children's existing knowledge of one-to-one correspondences, plays an essential, but until recently largely ignored, part in children's learning about multiplication. Researchers and teachers have failed to consider that one-to-many correspondence is a possible basis for children's initial multiplicative reasoning because of a wide-spread assumption that this reasoning is based on children's additive knowledge. However, recent evidence on how to introduce children to multiplication shows that teaching them multiplication in terms of one-to-many correspondence is more effective than teaching them about multiplication as repeated addition.

The logic of inversion

The subject of inversion was also neglected until fairly recently, but it is now clear that understanding that the addition and subtraction of the same quantity leaves the quantity of a set unchanged is of great importance in children's additive reasoning. Longitudinal evidence also shows that this understanding is a strong predictor of children's mathematical progress. Experimental research demonstrates that a flexible understanding of inversion is an essential element in children's geometrical reasoning as well. It is highly likely that children's learning about the inverse relation between multiplication and division is an equally important a part of mathematical learning, but the right research still has to be done on this question. Despite this gap, there is a clear case for giving the concept of inversion a great deal more prominence than it has now in the school curriculum.

The logic of class inclusion and additive composition

Numbers consist of other numbers. One cannot understand what 6 means unless one also knows that sets of 6 are composed of 5 + 1 items, or 4 + 2 items etc. The logic that allows children to work out that every number is a set of combination of other numbers is known as class inclusion. This form of inclusion, which is also referred to as additive composition of number, is the basis of the understanding of ordinal number: every number in the number series is the same as the one that precedes it plus one. It is also the basis for learning about the decade structure: the number 4321 consists of four thousands, three hundreds two tens and one unit, and this can only be properly understood by a child who has thoroughly grasped the additive composition of number. This form of understanding also allows children to compare numbers (7 is 4 more than 3) and thus to understand numbers as a way of expressing relations as well as quantities. The evidence clearly shows that children's ability to use this form of inclusion in learning about number and in solving mathematical problems is at first rather weak, and needs some support.

The logic of transitivity

All ordered series, including number, and also forms of measurement involve transitivity ($a > c$ if $a > b$ and $b > c$: $a = c$ if $a = b$ and $b = c$). Empirical evidence shows that children as young as 5-years of age do to some extent grasp this set of relations, at any rate with continuous quantities like length. However, learning how to use transitive relations in numerical measurements (for example, of area) is an intricate and to some extent a difficult business. Research, including Piaget's initial research on measurement, shows that one powerful reason for children finding it difficult to apply transitive reasoning to measurement successfully is that they often do not grasp the importance of iteration (repeated units of measurement). These difficulties persist through primary school.

One of the reasons why Piaget's ideas about the importance of logic in children's mathematical understanding have been ignored recently is probably the nature of evidence that he offered for them. Although Piaget's main idea was a positive one (children's logical abilities determine their learning about mathematics), his empirical evidence for this idea was mainly negative: it was about children's difficulties with the four aspects of logic that we have just discussed. A constant theme in our review is that this is not the best way to test a causal theory about mathematical learning. We advocate instead a combination of longitudinal research with intervention studies. The results of this kind of research do strongly support the idea that children's logic plays a critical part in their mathematical learning.

Children should be encouraged to reflect on their implicit models and the nature of the mathematical tools

Children need to re-conceptualise their intuitive models about the world in order to access the mathematical models that have been developed in the discipline. Some of the intuitive models used by children lead them to appropriate mathematical problem solving, and yet they may not know why they succeeded. This was exemplified by students' use of one-to-many correspondence in the solution of proportions problems: this schema of action leads to success but students may not be aware of the invariance of the ratio between the variables when the scheme is used to solve problems. Increasing students' awareness of this invariant should improve their mathematical understanding of proportions.

Another example of implicit models that lead to success is the use of distributivity in oral calculation of multiplication and division. Students who know that they can, instead of multiplying a number by 15, multiply it by 10 and then add half of this to the product, can be credited with implicit knowledge of distributivity. It is possible that they would benefit later on, when learning algebra, from the awareness of their use of distribitivity in this context. This understanding of distributivity developed in a context where they could justify it could be used for later learning.

Other implicit models may lead students astray. Fischbein, Deri, Nello and Marino (1985) and Greer (1988) have shown that some implicit models interfere with students' problem solving. If, for example, they make the implicit assumption that in a division the dividend must always be larger than the divisor, they might shift the numbers around in implementing the division operation when the dividend is actually smaller than the divisor. So, when students have developed implicit models that lead them astray, they would also benefit from greater awareness of these implicit models.

The simple fact that students do use intuitive models when they are learning mathematics, whether the teacher recognises the models or not, is a reason for wanting to help students develop an awareness of the models they use. Instruction could and should play a crucial role in this process.

Finally, reflecting on implicit models can help students understand mathematics better and also link mathematics with reality and with other disciplines that they learn in school. Freudenthal (1971) argued that it would be difficult for teachers of other disciplines to tie the bonds of mathematics to reality if these have been cut by the mathematics teacher. In order to tie these bonds, mathematics lessons can explore models that students use intuitively and extend these models to scientific concepts that have been shown to be challenging for students. One of the examples explored in a mathematics lesson designed by Treffers (1991) focuses on the mathematics behind the concept of density. He tells students the number of bicycles owned by people in United Kingdom and in the Netherlands. He also tells them the population of these two countries. He then asks them in which country there are more bicycles. On the basis of their intuitive knowledge, students can easily engage in a discussion that leads to the concept of density: the number of bicycles should be considered in relation to the number of people. A similar discussion might help students understand the idea of population density and of density in physics, a concept that has been shown to be very difficult for students. The discussion of how one should decide which country has more bicycles draws on students' intuitive models; the concept of density in physics extends this model. Streefland and Van den Heuvel-Panhuizen (see Paper 4) suggested that a model of a situation that is understood intuitively can become a model for other situations, which might not be so accessible to intuition. Students' reflection about the mathematics encapsulated in one concept is termed by Treffers horizontal mathematising; looking across concepts and thinking about the mathematics tools themselves leads to vertical mathematising, i.e. a re-construction of the mathematical ideas at a higher level of abstraction. This pragmatic theory about how students' implicit models develop can be easily put to test and could have an impact on mathematics as well as science education.

Mathematical learning depends on children understanding systems of symbols

One of the most powerful contributions of recent research on mathematical learning has come from work on the relation of logic, which is universal, to mathematical symbols and systems of symbols, which are human inventions, and thus are cultural tools that have to be taught. This distinction plays a role in all branches of mathematical learning and has serious implications for teaching mathematics.

Children encounter mathematical symbols throughout their lives, outside school as well as in the classroom. They first encounter them in learning to count. Counting systems with a base provide children with a powerful way of representing numbers. These systems require the cognitive skills involved in generative learning. As it is impossible to memorise a very long sequence of words in a fixed order, counting systems with a base solve this problem: we learn only a few symbols (the labels for units, decades, hundred, thousand, million etc.) by memory and generate the other ones in a rule-based manner. The same is true for the Hindu-Arabic place value system for writing numbers: when we understand how it works, we do not need to memorise how each number is written.

Mathematical symbols are technologies in the sense that they are human-made tools that improve our ability to control and adapt to the environment. Each of these systems makes specific cognitive demands from the learner. In order to understand place-value representation, for example, students' must understand additive composition. If students have explicit knowledge of additive composition and how it works in place-value representation, they are better placed to learn column arithmetic, which should then enable students to calculate with very large numbers; this task is very taxing without written numbers. So the costs of learning to use these tools are worth paying: the tools enable students to do more than they can do without the tools. However, research shows that students should be helped to make connections between symbols and meanings: they can behave as if they understand how the symbols work while they do not understand them completely: they can learn routines for symbol manipulation that remain disconnected from meaning.

This is also true of rational numbers. Children can learn to use written fractions by counting the number of parts into which a whole was cut and writing this below a dash, and counting the number of parts painted and writing this above the dash. However, these symbols can remain disconnected from their logical thinking about division. These disconnections between symbols and meaning are not restricted to writing fractions: they are also observed when students learn to add and subtract fractions and also later when students learn algebraic symbols.

Plotting variables in the Cartesian plane is another use of symbol systems that can empower students: they can, for example, more easily analyse change by looking at graphs than they can by intuitive comparisons. Here, again, research has shown how reading graphs also depends on the interpretations that students assign to this system of symbols.

A recurrent theme in the review of research across the different topics was that the disconnection between symbols and meanings seems to explain many of the difficulties faced by primary school students in learning mathematics. The inevitable educational implication is that teaching aims should include promoting connections between symbols and meaning when symbols are introduced and used in the classroom.

This point is, of course, not new, but it is well worth reinforcing and, in particular, it is well worth remembering in the light of current findings. The history of mathematics education includes the development of pedagogical resources that were developed to help students attribute meaning to mathematical symbols. But some of these resources, like Dienes' blocks and Cuisinaire's rods, are only encountered by students in the classroom; the point we are making here is that students acquire informal knowledge in their everyday lives, which can be used to give meaning to mathematical symbols learned in the classroom. Research in mathematics education over the last five decades or so has helped describe the situations in which these meanings are learned and the way in which they are structured. Curriculum development work that takes this knowledge into account has already started (a major example is the research by members of the Freudenthal Institute) but it is not as widespread as one would expect given the discoveries from past research.

Children need to learn modes of enquiry associated with mathematics

We identify some important mathematical modes of enquiry that arise in the topics covered in this synthesis.

Comparison helps us make new distinctions and create new objects and relations

A cycle of creating and naming new objects through acting on simple objects pervades mathematics, and the new objects can then be related and compared to create higher-level objects. Making additive and multiplicative comparisons is an aspect of understanding relations between quantities and arithmetic. These comparisons are manifested precisely as difference and ratio. Thus difference and ratio arise as two new mathematical ideas, which become new mathematical objects of study and can be represented and manipulated. Comparisons are related to making distinctions, sorting and classifying based on perceptions, and students need to learn to make these distinctions based on mathematical relations and properties, rather than perceptual similarities.

Reasoning about properties and relations rather than perceptions

Many of the problems in mathematics that students find hard occur when immediate perceptions lead to misapplication of learnt methods or informal reasoning. Throughout mathematics, students have to learn to interpret representations before they think about how to respond. They need to think about the relations between different objects in the systems and schemes that are being represented.

Making and using representations

Conventional number symbols, algebraic syntax, coordinate geometry, and graphing methods, all afford manipulations that might otherwise be impossible. Coordinating different representations to explore and extend meaning is a fundamental mathematical skill that is implicit in the use of the number line to represent quantities, for example, the use of graphs to express functions. Equivalent representations, such as for number, algebraic relationships and functions, can provide new insights through comparison and isomorphic analogical reasoning.

Action and reflection-on-action

Learning takes place when we reflect on the effects of actions. In mathematics, actions may be physical manipulation, or symbolic rearrangement, or our observations of a dynamic image, or use of a tool. In all these contexts, we observe what changes and what stays the same as a result of actions, and make inferences about the connections between action and effect. In early mathematics such reflection is usually embedded in children's classroom activity, such as when using manipulatives to model changes in quantity. In later mathematics changes and invariance may be less obvious, particularly when change is implicit (as in a situation to be modelled) or useful variation is hard to identify (as in a quadratic function).

Direct and inverse relations

Direct and inverse relations are discussed in several of our papers. While it may sometimes be easier to reason in a direct manner that accords with action, it is important in all aspects of mathematics to be able to construct and use inverse reasoning. Addition and subtraction must be understood as a pair, and multiplication and division as a pair, rather than as a set of four binary operations. As well as enabling more understanding of relations between quantities, this also establishes the importance of reverse chains of reasoning throughout mathematical problem-solving, algebraic and geometrical reasoning. For example, using reverse reasoning makes it more likely that students will learn the dualism embedded in Cartesian representations; that all points on the graph fulfil the function, and the function generates all points on the graph.

Informal and formal reasoning

At first young children bring everyday understandings into school, and mathematics can allow them to formalise these and make them more precise. On the other hand, intuitions about continuity, approximation, dynamic actions and three-dimensional space might be over-ridden by early school mathematics – yet are needed later on. Mathematics also provides formal tools which do not describe everyday outside experience, but enable students to solve problems in mathematics and in the world which would be unnoticed without a mathematical perspective. In the area of word problems and realistic problems learning when and how to apply informal and formal reasoning is important. Later on, counter-intuitive ideas have to take the place of early beliefs, such as 'multiplication makes things bigger' and students have to be wary of informal, visual and immediate responses to mathematical stimuli.

A recurring issue in the papers is that students find it hard to coordinate *attention on local and global changes*. For example, young children confuse quantifying 'relations between relations' with the original quantities; older children who cannot identify covariation of functions might be able to talk about separate variation of variables; students readily see term-to-term patterns in sequences rather than the generating function; changes in areas are confused with changes in length.

Epilogue

Our aim has been to write a review that summarises our findings from the detailed analysis of a large amount of research. We sought to make it possible for educators and policy makers to take a fresh look at mathematics teaching and learning, starting from the results of research on key understandings, rather than from previous traditions in the organisation of the curriculum. We found it necessary to organise our review around ideas that are already core ideas in the curriculum, such as whole and rational number, algebra and problem solving, but also to focus on ideas that might not be identified so easily in the current curriculum organisation, such as students' understanding of relations between quantities and their understanding of space.

We have tried to make cogent and convincing recommendations about teaching and learning, and to make the reasoning behind these recommendations clear to educationalists. We have also recognised that there are weaknesses in research and gaps in current knowledge, some of which can be easily solved by research enabled by significant contributions of past research. Other gaps may not be so easily solved, and we have described some pragmatic theories that are, or can be, used by teachers when they design instruction. Classroom research, stemming from the exploration of these pragmatic theories, can provide new insights for further research in the future.

Endnotes

1 Details of the search process is provided in Appendix 1. This contains the list of data bases and journals consulted and the total number of papers read although not all of these can be cited in the six papers that comprise this review.

References

Anghileri, J., Beishuizen, M., and Putten, K.V. (2002). From informal strategies to structured procedures: mind the gap! *Educational Studies in Mathematics, 49*, 149–170.

Carey, S. (2004). Bootstrapping and the origins of concepts. *Daedalus, 133*, 59–68.

Cooper, H. M. (1998). *Synthesizing research: a guide for literature reviews*. Thousand Oaks, CA: Sage Publications.

Confrey, J. (2008). *A synthesis of research on rational number reasoning.* Paper presented at the International Conference on Mathematics Education – ICME 11, Monterrey, Mexico.

Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*, 3–17.

Freudenthal, H. (1971). Geometry between the Devil and the Deep Sea. *Educational Studies in Mathematics, 3*, 413–435.

Greer, B. (1988). Nonconservation of multiplication and division: Analysis of a symptom. *Journal of Mathematical Behavior, 7*, 281–298.

Hart. K. (1981) *Children's Understanding of Mathematics 11–16*. London (U.K.): Murray.

Küchemann, D. (1981) Algebra, in K. Hart (Ed.) *Children's Understanding of Mathematics 11–16*, 102–119 London (U.K.); Murray.

Luria, A. R. (1973). *The working brain. An introduction to neuropsychology*. Harmondsworth (U.K.): Penguin.

Noss, R., and Hoyles, C. (1992). Looking back and looking forward. In C. Hoyles and R. Noss (Eds.), *Learning mathematics and LOGO* (pp. 431–470). Cambridge, MA: The MIT Press.

Nunes, T. (1997). Systems of signs and mathematical reasoning. In T. Nunes amd P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 29–44). Hove (UK): Psychology Press.

Nunes, T. (2002). The role of systems of signs in reasoning. In T. Brown and L. Smith (Eds.), *Reductionism and the Development of Knowledge* (pp. 133–158). Mawah, NJ: Lawrence Erlbaum.

Nunes, T., and Bryant, P. (2008). Rational Numbers and Intensive Quantities: Challenges and Insights to Pupils' Implicit Knowledge. *Anales de Psicologia*, in press.

Ruthven, K. (2008). Creating a calculator-aware number curriculum. *Canadian Journal of Science, Mathematics and Technology Education, 3*, 437–450.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495–511). New York, N.Y.: Macmillan.

Thompson, P. W. (1993). Quantitative Reasoning, Complexity, and Additive Structures. *Educational Studies in Mathematics, 25*, 165–208.

Thompson, P. (1994). The Development of the Concept of Speed and Its Relationship to Concepts of Rate. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 181–236). Albany, N.Y.: State University of New York Press.

Treffers, A. (1987). Integrated column arithmetic according to progressive schematisation. *Educational Studies in Mathematics, 18* 125–145.

Treffers, A. (1991). Didactical background of a mathematics programme for primary education. In L. Streefland (Ed.), *Realistic Mathematics Education in Primary School* (pp. 21–56). Utrecht: Freudenthal Institute, Utrecht University.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes and P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 1–28). Hove (UK): Psychology Press.

Vygotsky, L. S. (1981). The genesis of higher mental functions. In J. V. Wertsch (Ed.), *The concept of activity in Soviet Psychology* (pp. 57–93). Amonk, N.Y.: Sharpe.

Wartofsky, M. W. (1979). *Models: representation and the scientific understanding*. Dordrecht: Reidel.

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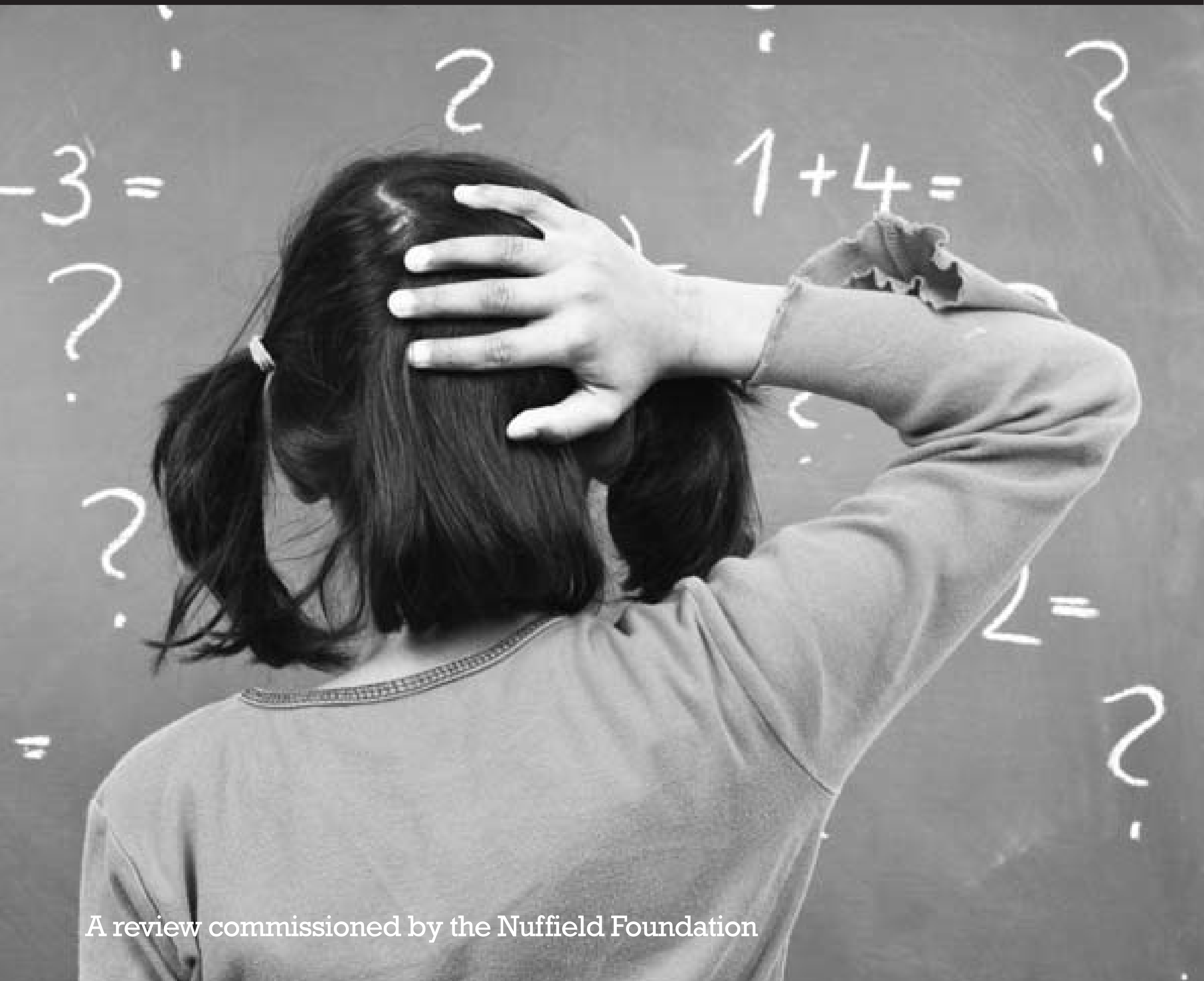
2

Key understandings in mathematics learning

Paper 2: Understanding whole numbers

By Terezinha Nunes and Peter Bryant, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

Contents

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 2: Understanding whole numbers

Headlines

- Whole numbers are used in primary school to represent quantities and relations. It is crucial for children's success in learning mathematics in primary school to establish clear connections between numbers, quantities and relations.

- Using different schemes of action, such as setting objects in correspondence, children can judge whether two quantities are equivalent, and if they are not, make judgements about their order of magnitude. These insights are used in understanding the number system beyond simply producing a string of number words in a fixed order. It takes children some time to make links between their understanding of quantities and their knowledge of number.

- Children start school with varying levels of ability in using different action schemes to solve arithmetic problems in the context of stories. They do not need to know arithmetic facts to solve these problems: they count in different ways depending on whether the problems they are solving involve the ideas of addition, subtraction, multiplication or division.

- Individual differences in the use of action schemes to solve problems predict children's progress in learning mathematics in school.

- Interventions that help children learn to use their action schemes to solve problems lead to better learning of mathematics in school.

- It is considerably more difficult for children to use numbers to represent relations than to represent quantities. Understanding relations is crucial for their further development in mathematics in school.

In children's everyday lives and before they start school, they have experiences of manipulating and comparing quantities. For example, even at age four, many children can share sweets fairly between two recipients by using correspondences: they share giving one-for-you, one-for-me, until there are no sweets left. They do sometimes make mistakes but they know that, when the sharing is done fairly, the two people will have the same amount of sweets at the end. Even younger children know some things about quantities: they know that if you add sweets to a group of sweets, there will be more sweets there, and if you take some away, there will be fewer. However, they might not know that if you add a certain number and take away the same number, there will be just as many sweets as there were before.

At the same time that young children are developing these ideas about quantities, they are often learning to count. They learn to say the sequence of number words in the right order, they know that each object that they are counting must be counted once and only once, and that it does not matter if you count a row of sweets from left to right or from right to left, you should get to the same number.

Four-year-olds are thus amazing learners of mathematics. But they lack one thing which is crucially important: they do not at first make connections between their understanding of quantities and their knowledge of numbers. So if you ask a four-year-old, who just shared some sweets fairly between two dolls, to count the sweets that one doll has and then tell you, without counting, how many sweets the other doll has, the majority (about 60%) will tell you that they do not know. Knowing that the dolls have the same quantity is not sufficient to know that if one has 8 sweets, the other one has 8 sweets also, i.e. has the same number.

Quantities and numbers are not the same thing. We can use numbers as measures of quantities, but we can think about quantities without actually having a measure for them. Until children can understand the connections between numbers and quantities, they cannot use their knowledge of quantities to support their understanding of numbers and *vice versa*. Because the connections between quantities and numbers are many and varied, learning about these connections could take three to four years in primary school.

An important link that children must make between number and quantity is the link between the order of number words in the counting sequence and the magnitude of the quantity represented. How do children come to understand that the any number in the counting sequence is equal to the preceding number plus 1?

Different explanations have been proposed in the literature. One is that they simply see that magnitude increases as they count. But this explanation does not work well: our perception of magnitude is approximate and knowing that any number is equal to its predecessor plus 1 is a very precise piece of knowledge. A second explanation is that children use perception, language and inferences together to reach this understanding. Young children discriminate well, for example, one puppet from two puppets and two puppets from three puppets. Because they know these differences precisely, they put these two pieces of information together, and learn that two is one more than one, and three is one more than two. They then make the inference that all numbers in the counting sequence are equal to the predecessor plus one. But this sort of generalisation could not be stretched into helping children understand that any number is also equal to the last-but-one in the sequence plus 2. This process of putting together perception with language and then generalising is an explanation for only the $n + 1$ idea; it would be much better if we could have a more general explanation of how children understand the connection between quantities and the number sequence.

The third explanation for how children connect heir knowledge of quantities with the magnitude of numbers in the counting sequence is that children's schemes of action play the most important part in this development. The actions of adding and taking away help them understand part–whole relations. When they can link their understanding of part–whole relations with counting, they will understand many things about relations between numbers. A critical change in young children's behaviour when they add two sets is from 'count all' to 'count on'. If they know that they have 5 sweets, and you add 4 to the 5, they could either start from 1 and count all the sweets (count all) or they could point to the 5, and count on from there. 'Count on' is a sign that the children have linked their knowledge of part–whole relations with the counting sequence: they have understood the additive composition number. This explanation works for the relation between a number and its immediate predecessor and any of its predecessors. It is supported by much research that shows that counting on is a sign of abstraction in part–whole relations, which opens the way for children to solve many other problems: they can add a quantity to an invisible set, count coins of different denominations to form a single total, and are ready to learn to use place value to represent numbers in writing.

Adding and subtracting elements to sets also give children the opportunity to understand the inverse relation between addition and subtraction. This insight is not gained in an all-or-nothing fashion: children first apply it only to quantities and later on to number also. The majority of five-year-olds realises that if you add 3 sweets to a set of sweets and then take the *same* sweets away, the number of sweets in the set remains the same. However, many of these children will not realise that if you add 3 sweets to the set and then take 3 *other* sweets away, the number of sweets is still the same. They see that adding and taking away the *same quantity* leaves the original quantity the same but this does not immediately mean to them that adding and taking away the *same number* also leaves the original number the same. Research shows that the step from understanding the inverse relation between addition and subtraction of quantities is a useful start if one wants to teach children about the inverse relation between addition and subtraction of number.

Adding, taking away and understanding part–whole relations form one part of the story of what children know about quantities and numbers in the early years of primary school. They relate to how additive reasoning develops. The other part of the story is surprising to many people: children also know quite a lot about multiplicative reasoning when they start school.

Children use two different schemes of action to solve multiplication and division problems before they are taught about these operations in school: they use one-to-many correspondence and sharing. If five- and six-year-olds are shown, for example, four little houses in a row, told that they should imagine that in each live three dogs, and asked how many dogs live in the street, the majority can say the correct number. Many children will point three times to each house and count in this way until they complete the counting at the fourth house. They are not multiplying: they are solving the problem using one-to-many correspondence. Children can also share objects to recipients and answer problems about division. They do not know the arithmetic operations, but they can use their reasoning to count in different ways and solve the problem. So children manipulate quantities using multiplicative reasoning and solve problems before they learn about multiplication and division in school.

If children are assessed in their understanding of the inverse relation between addition and subtraction, of additive composition, and of one-to-many correspondence in their first year of school, this provides us with a good way of anticipating whether they will have difficulties in learning mathematics in school. Children who do well in these assessments go on to attain better results in mathematics assessments in school. Those who do not do well can improve their prospects through early intervention. Children who received specific instruction on these relations between quantities and how to use them to solve problems did significantly better than a similar group who did not receive such instruction.

Finally, many studies have used story problems to investigate which uses of additive reasoning are easier and which are more difficult for children of primary school age. Two sorts of difficulties have been identified.

The first relates to the need to understand that addition and subtraction are the inverse of each other. One story that requires this understanding is: Ali had some Chinese stamps in his collection and his grandfather gave him 2; now he has 8; how many stamps did he have before his grandfather gave him the 2 stamps? This problem exemplifies a situation in which a quantity increases (the grandfather gave him 2 stamps) but, because the information about the original number in his collection is missing, the problem is not solved by an addition but rather by a subtraction. The problem would also be an inverse problem if Ali had some Chinese stamps in his collection and gave 2 to his grandfather, leaving his collection with 6. In this second problem, there is a decrease in the quantity but the problem has to be solved by an increase in the number, in order to get us back to Ali's collection before he gave 2 stamps away. There is no controversy in the literature: inverse problems are more difficult than direct problems, irrespective of whether the arithmetic operation that is used to solve it is addition or subtraction.

The second difficulty depends on whether the numbers in the problem are all about quantities or whether there is a need to consider a relation between quantities. In the two problems about Ali's stamps, all the numbers refer to quantities. An example of a problem involving relations would be: In Ali's class there are 8 boys and 6 girls; how many more boys than girls in Ali's class? (Or how many fewer girls than boys in Ali's class?). The number 2 here refers neither to the number of boys nor to the number of girls: it refers to the relation (the difference) between number of boys and girls. A difference is not a quantity: it is a relation. Problems that involve relations are more difficult than those that involve quantities. It should not be surprising that relations are more difficult to deal with in numerical contexts than quantities: the majority, if not all, the experiences that children have with counting have to do with finding a number to represent a quantity, because we count things and not relations between things. We can re-phrase problems that involve relations so that all the numbers refer to quantities. For example, we could say that the boys and girls need to find a partner for a dance; how many boys won't be able to find a girl to dance with? There are no relations in this latter problem, all the numbers refer to quantities. This type of problem is significantly easier. So it is difficult for children to use numbers to represent relations. This could be one step that teachers in primary school want to help their children take, because it is a difficult move for every child.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's pre-school knowledge of quantities and counting develops separately.	**Teaching** Teachers should be aware of the importance of helping children make connections between their understanding of quantities and their knowledge of counting.
When children start school, they can solve many different problems using schemes of action in coordination with counting, including multiplication and division problems.	**Teaching** The linear view of development, according to which understanding addition precedes multiplication, is not supported by research. Teachers should be aware of children's mathematical reasoning, including their ability to solve multiplication and division problems, and use their abilities for further learning.
Three logical-mathematical reasoning principles have been identified in research, which seem to be causally related to children's later attainment in mathematics in primary school. Individual differences in knowledge of these principles predict later achievement and interventions reduce learning difficulties.	**Teaching** A greater emphasis should be given in the curriculum to promoting children's understanding of the inverse relation between addition and subtraction, additive composition, and one-to-many correspondence. This would help children who start school at risk for difficulties in learning mathematics to make good progress in the first years. **Research** Long-term longitudinal and intervention studies with large samples are needed before curriculum and policy changes can be proposed. The move from the laboratory to the classroom must be based on research that identifies potential difficulties in scaling up successful interventions.
Children's ability to solve word problems shows that two types of problem cause difficulties for children: those that involve the inverse relation between addition and subtraction and those that involve thinking about relations.	**Teaching** Systematic use of problems involving these difficulties followed by discussions in the classroom would give children more opportunities for making progress in using mathematics in contexts with which they have difficulty. **Research** There is a need for intervention studies designed to promote children's competence in solving problems about relations. Brief experimental interventions have paved the way for classroom-based research but large-scale studies are needed.

Understanding extensive quantities and whole numbers

Counting and reasoning

At school, children's formal learning about mathematics begins with natural numbers (1, 2, …17…103 …525…). Numbers are symbols for quantities: they make it possible for the child to specify single values precisely and also to work out the relations between different quantities. By counting, the child can tell you that there are 20 books in the pile on the teacher's desk (a single quantity), and eventually should be able to work out that there is 1 book for every child in the class if there are 20 children there, or that there are 5 more books than children (a relation between two quantities) if there are 15 children in the class.

Quantities and numbers are not the same. Thompson (1993) suggested that 'a person constitutes a quantity by conceiving of a quality of an object in such a way that he or she understands the possibility of measuring it. Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values' (pp. 165–166).[1]

Children experience and learn about quantities and the relations between them quite independently of learning to count. Similarly, they can learn to count quite independently from understanding quantities and relations between them. We shall argue in this section that the most important task for a child who is learning about natural numbers is to connect these numbers to a good understanding of quantities and relations. The connection should work at two levels.

First, children must realise that their knowledge of quantities and numbers should agree with one another. If Sean has 15 books and Patrick 17, Patrick has more books than Sean. Unless children understand that numbers are a precise way of expressing quantities, the number system will have no meaning for them.

Second, they must realise eventually that the number system enhances their knowledge of quantities in an increasingly powerful way. They may not be able to look at a pile of books and tell without counting that the one with 17 has more books than the one with 15; indeed, the thickness of books varies and the pile of 15 books could well be taller than the pile with 17. By counting they can know which pile has more books. When they know how to count, we can also add and subtract numbers, and work out the exact relations between them. If we understand lots of things about quantities, e.g. how to create equivalent quantities and how their equivalence is changed, but we don't have numbers to represent them, we cannot add and subtract.

In this section, therefore, we will focus on the connections that children make, and sometimes fail to make, between their growing knowledge about quantities and the number system. In many ways this is an unusual thing to do. Most existing accounts of how children learn about number are more restricted. Either they leave out the number system altogether and concentrate instead on children's ability to reason about quantities, or they are strictly confined to how well children count sets of objects.

Piaget's theory (Piaget, 1952) is an example of the first kind of theory. His view that children have to be able to reason logically about quantity in order to understand number and the number system is almost certainly right, but it left out the possibility that learning to count eventually transforms this reasoning in children by making it more powerful and more precise.

In the opposite corner, Gelman's influential theory (Gelman and Gallistel, 1978), which focuses on how children count single sets of objects and has little to say about children's quantitative reasoning, has the serious disadvantage that it by-passes children's reasoning about relations between quantities. In the end, numbers are only important because they allow us to represent quantities and make sense of quantitative relations.

The first part of this section is an account of how children connect numbers with quantity. We will start this account with a detailed list of the connections that they need to make. We argue that children need to make three types of connections between number words and quantities in order to make the most of what they learn when they begin to count: they need to understand cardinality; they need to understand ordinal numbers, and they need to understand the relation between cardinality and addition and subtraction. The second part of this section is an account of how children learn to use numbers to solve problems. We argue that numbers are used to represent quantities but that children must also learn to use them to represent transformations and relations, and that the different meanings that numbers can have affect how easily children solve problems.

Giving meaning to numbers

Young children's dissociation of quantities and numbers

Children may know that two quantities are the same and still not make the inference that the number of objects in one is the same as the number of objects in the other. Conversely, they may know how to count and yet not make use of counting when asked to create two equal sets. We review here briefly research within two different traditions, inspired by Piaget's and Gelman's theories, that shows that young children do not necessarily make a connection between what they know about quantities and what they know about counting.

Equivalence of sets in one-to-one correspondence and its connection to number words

Numbers have both cardinal and ordinal properties. Two sets have the same cardinal value when the items in one set are in one-to-one correspondence with those in the other. There are as many eggcups in a box of six egg-cups as there are eggs in a carton of six eggs, and if there are six people at the breakfast table each will have one of those eggs on its own eggcup to eat. Thus, the eggcups, eggs and people are all in one-to-one correspondence since there is one egg and eggcup for each one person, which means that each of these three sets has the same number.

We shall deal with the ordinal properties of number in a later section. At the moment, all that we need to say is that numbers are arranged in an ordered series.

To return to cardinality, Piaget argued quite reasonably that no one can understand the meaning of 'six' unless he or she also understands the number's cardinal properties, and by this he meant understanding not only that any set of six contains the same number of items as any other set of six but also that that the items in a set of six are in one-to-one correspondence with any other set of six items. So, if we are to pursue the approach of studying the links between children's quantitative reasoning and how they learn about natural numbers, we need to find out how well children understand the principle that sets which are in one-to-one correspondence with each other are equal in quantity, and also how clearly they apply what they understand about one-to-one correspondence to actual numbers like 'six'.

Piaget based his claim that young children have a very poor understanding of one-to-one correspondence on the mistakes that they make when they are shown one set of items (e.g. a row of eggs) and are asked to form another set (e.g. of eggcups) of the same number. Four- and five-year-olds often match the new set with the old one on irrelevant criteria such as two rows' lengths and make no effort to put the rows into one-to-one correspondence. Their ability to establish one-to-one correspondence between sets grows over time: it cannot be taken for granted.

However, even when children do establish a one-to-one correspondence between two sets, they do not necessarily infer that counting the elements in one set tells them how many elements there are in the other set. Piaget (1952) established this in an experiment in which he proposed to buy sweets from the children, using a one-to-one exchange between pence and sweets. For each sweet that the child gave to Piaget, he gave the child a penny. As they exchanged pence and sweets, the child was asked to count how many pence he/she had. Piaget ensured that he stopped this exchange procedure without going over the child's counting range. When he stopped the exchange, he asked the child how many pence the child had. The children were able to answer this without difficulty as they had been counting their coins. He then asked the child how many sweets he had. Piaget reports that some children were unable to make the inference that the number of sweets Piaget had was the same as the number of pence that the child himself/herself had. Unfortunately, Piaget gave no detailed description of how the ability to make this inference related to the children's age.

More recent research, which offers quantitative information, shows that many four-year-olds who do understand one-to-one correspondence well enough to share fairly do not make the inference that equivalent sets have the same number of elements. Frydman and Bryant (1988) asked four-year-old children to share a set of 'chocolates' to two recipients. At this age, children often share things between themselves, and they typically do so on a one-for-A, one-for-B, one-for-A, one-for-B basis. In this study, the children established the correspondence themselves; this contrasts with Piaget's study, where Piaget controlled the exchange of sweets and pence. When the child had done the sharing, the experimenters counted out the number of items that had been given to one recipient, which was six. Having done this, they asked the child how many chocolates had been given to the other recipient.

None of the children immediately made the inference that there were the same number of chocolates in one set as in the other, and therefore that there were also six items in the second set. Instead, every single child began to count the second set. In each case, the experimenter then interrupted the child's counting, and asked him or her if there was any other way of working out the number of items in the second recipient's share. Only 40% of the group of four-year-olds made the correct inference that the second recipient had also been given six chocolates. The failure of more than half of the children is an interesting one. The particular pre-school children who made it knew that the two recipients' shares were equal, and they also knew the number of items in one of the shares. Yet, they did not connect what they knew about the relative quantities to the number symbols. Other children, however, did make this connection, which we think is the first significant step in understanding cardinality. Whether all children will have made this connection by the time that they start learning about numbers and arithmetic at school depends on many factors: for example, the age they start school and their previous experiences with number are related to whether they have taken this important step by then (e.g. socio-economic status related to maths ability at school entry: see Ginsburg, Klein, and Starkey, 1998; Jordan, Huttenlocher, and Levine, 1992; Secada, 1992).

Counting and understanding relations between quantities

Piaget's theory of how children develop an understanding of cardinality was confronted by an alternative theory, by Gelman's nativist view of children's counting and its connection to cardinal number knowledge (Gelman and Gallistel, 1978). Gelman claimed that children are born with a genuine understanding of natural number, and that this makes it possible for them to learn and use the basic principles of counting as soon as they begin to learn the names for numbers. She outlined five basic counting principles. Anyone counting a set of objects should understand that:

- you should count every object once and only once (one-to-one correspondence principle)
- the order in which you count the actual objects (from left-to right, from right to left or from the middle outwards) makes no difference (order irrelevance principle)
- you should produce the number words in a constant order when counting: you cannot count 1-2-3 at one time and 1-3-2 at another (fixed order principle)
- whether the objects in a set are all identical to each other or all quite different has no effect on their number (the abstraction principle)
- the last number that you count is the number of items in the set (cardinal principle).

Each of these principles is justified in the sense that anyone who does not respect them will end up counting incorrectly. A child who produces count words in different orders at different times is bound to make incorrect judgements about the number of items in a set. So will anyone who does not obey the one-to-one principle.

Gelman's original observations of children counting sets of objects, and the results of some subsequent experiments in which children had to spot errors in other protagonists' counting (e.g. Gelman and Meck, 1983), all supported her idea that children obey and apparently understand all five of these principles with small sets of items long before they go to school. The young children's success in counting smaller sets allowed her to dismiss their more frequent mistakes with large sets of items as executive errors rather than failures in understanding. She agued that the children knew the principles of counting and therefore of number, but lacked some of the skills needed to carry them out. This view became known as the 'principles-before-skills hypothesis'.

These observations of Gelman's provoked a great deal of useful further research on children's counting, most of which has confirmed her original results, though with some modifications. For example, five-year-old children do generally count objects in a one-to-one fashion (one number word for each object) but not all of the time (Fuson, 1988). They tend either to miss objects or count some more than once in disorganised arrays. It is now clear that gestures play an important part in helping children keep track during counting (Albilali and DiRusso, 1999) but sometimes they point at some of the objects in a target set without counting them.

Many of the criticisms of Gelman's hypothesis were against her claims that children understand cardinality. Ironically, even critics of Gelman (e.g. Carey, 2004; LeCorre and Carey, 2007)) have in their own research accepted her all too limited definition of understanding cardinality (that it is the realisation that the last number counted represents the number of objects). However, several researchers have criticised her empirical test of cardinality. Gelman had argued that children, who count a set of objects and emphasise the last number ('one-two-three-FOUR') or repeat it ('one-two-three-four- there are four'), understand that this last number represents the quantity of the counted set. However, Fuson (Fuson, and Hall, 1983; Fuson, Richards and Briars, 1982) and Sophian (Sophian, Wood, and Vong, 1995) both made the reasonable argument that emphasising or repeating the last number could just be part of an ill-understood procedure.

Although Gelman's five principles cover some essential aspects of counting, they leave others out. The five principles, and the tools that Gelman devised to study children's understanding of these principles, only apply to what someone must know and do in order to count a single set of objects. They tell us nothing about children's understanding of numerical relations between sets. Piaget's research on number, on the other hand, was almost entirely concerned with comparisons between different quantities, and this has the confusing consequence that when Gelman and Piaget used the same terms, they gave them quite different meanings. For Piaget, understanding cardinality was about grasping that all and only equivalent sets are equal in number: for Gelman it meant understanding that the last number counted represents the number of items in a single set. When Piaget studied one-to-one correspondence, he looked at children's comparisons between two quantities (eggs and egg cups, for example): Gelman's concern with one-to-one correspondence was about children assigning one count word to each item in a set.

Since two sets are equal in quantity if they contain the same number of items and unequal if they do not, one way to compare two sets quantitatively is to count each of them and to compare the two numbers. Another, for much the same reason, is to use one-to-one correspondence: if the sets are in correspondence they are equal; if not, they are unequal. This prompts a question: how soon and how well do children realise that counting sets is a valid way, and sometimes the only feasible valid way, of comparing them quantitatively? Another way of putting the same question is to ask: how soon and how well do children realise that numbers are a measure by which they can compare the quantities of two or more different sets.

Most of the research on this topic suggests that it takes children some time to realise that they can, and often should, count to compare. Certainly many pre-school children seem not to have grasped the connection between counting and comparing even if they have been able to count for more than one year.

One source of evidence comes from the work by Sophian (1988), who asked children to judge whether someone else (a puppet) was counting the right way when asked to do two things. The puppet was faced with two sets of objects, and was asked in some trials to say whether the two sets were equal or not and in others to work out how many items there were on the table altogether. Sometimes the puppet did the right thing, which was to count the two sets separately when comparing them and to count all the items together when working out the grand total. At other times he got it wrong, e.g. counted all the objects as one set when asked to compare the two sets. The main result of Sophian's study was that the pre-school children found it very hard to make this judgement. Most 3-year-olds judged counting each set was the right way to count in both tasks while 4-year-olds judged counting both sets together was the right way to count in both tasks. Neither age group could identify the right way to count reliably.

A second type of study shows that even at school age many children seem not to understand fully the significance of numbers when they make quantitative comparisons. There is, for example, the striking demonstration by Pierre Gréco (1962), a colleague of Piaget's, that children will count two rows of counters, one of which is more spread out and longer than the other, and correctly say that they both have the same number (this one has six, and so does the other) but then will go on to say that there are more counters in the longer row than in the other. A child who makes this mistake understands cardinality in Gelman's sense (i.e. is able to say how many items in the set) but does not know what the word 'six' means in Piaget's sense. Barbara Sarnecka and Susan Gelman (2004) recently replicated this observation. They report that children three- and four-year-olds know that if a set had five objects and you add some to it, it no longer has five objects; however they did not know that equal sets must have the same number word.

Another source of evidence is the observation, repeated in many studies, that children, who can count quite well, nevertheless fail to count the items in two sets that they have been asked to compare numerically (Cowan, 1987; Cowan and Daniels, 1989; Michie, 1984; Saxe, Guberman and Gearhart, 1987); instead they rely on perceptual cues, like length, which of course are unreliable. Children who understand the cardinality of number should understand that they can make the comparison only by counting or using one-to-one correspondence, and yet at the age of five and six years most of them do neither, even when, as in the Cowan and Daniels study, the one-to-one cues are emphasised by lines drawn between items in the two sets that the children were asked to compare.

Finally, the criterion for the cardinality principle has itself been criticised as insufficient to show that children understand cardinality. The criticism is both theoretical and also based on empirical results. From a theoretical standpoint, Vergnaud (2008) pointed out that Gelman's cardinality criterion should actually be viewed as showing that children have some understanding of *ordinal*, not of cardinal, number: Gelman's criterion is indeed based on the *position* of the number word in the counting sequence, because the children use the *last* number word to represent the set. Vergnaud argues that ordinal numbers cannot be added whereas cardinal numbers can. He predicts that children whose knowledge of cardinal number is restricted to Gelman's cardinality principle will not be able to continue counting to answer how many objects are in a set if you add some objects to the set that they have just counted: they will need to count again from one. Research by Siegler and Robinson (1982) and Starkey and Gelman (1982) produced results in line with this prediction: 3-year-olds do not spontaneously count to solve addition problems after counting the first set. Ginsburg, Klein and Starkey (1998) also interpreted such results as indicative of an insufficient development of the concept of cardinality in young children. We return to the definition of cardinality later on, after we have discussed alternative explanations to Gelman's theory of an innate counting principle as the basis for learning about cardinality.

Three further studies will be used here to illustrate that some children who are able to use Gelman's cardinality principle do not seem to have a full grasp of when this principle should be applied; so meeting the criterion for the cardinality principle does not mean understanding cardinality.

Fuson (1988) showed that three-year-old children who seem to understand the cardinality principle continue to use the last number word in the counting sequence to say how many items are in a set even if the counting started from two, rather than from one. Counting in this unusual way should at least lead the children to reject the last word as the cardinal for the set.

Using a similar experimental manoeuvre, Freeman, Antonuccia and Lewis (2000) assessed three- and five-year-olds' rejection of the last word after counting if there had been a mistake in counting. The children participated in a few different tasks, one of which was a task where a puppet counted an array with either 3 or 5 items, but the puppet miscounted, either by counting an item twice or by skipping an item. The children were asked whether the puppet had counted right, and if they said that the puppet had not, they were asked: How many does the puppet think there are? How many are there really? All children had shown that they could count 5 items accurately (2 of 22 could count accurately to 6, another 4 could do so to 7, and the remaining 18 could count items accurately to 10). However, their competence in counting was no assurance that they realised that the puppet's answer was wrong after miscounting: only about one third of the children were able to say that the answer was not right *after* they had detected the error. The children's rejection of the puppet's wrong answer increased with age: 82% of the five-year-olds correctly rejected the puppet's answer in all three trials when a mistake had been made. However, the majority of the children could not say what the cardinal for the set was without recounting: the majority counted the set again in order to answer the question 'how many are there really?' They neither said immediately the next number when the puppet had skipped one nor used the previous number when the puppet double-counted an item. So, quite a few of the younger children passed Gelman's cardinality principle but did not necessarily see that the cardinality principle should not be applied when the counting principles are violated. Most of the older children, who rejected the use of the cardinality principle, did not use it to deduce what the correct cardinal should be; instead, all they demonstrated was that they could replace the wrong routine with the correct one, and then they could say what the number really was. Understanding that the next number is the cardinal for the set if the puppet skipped one item, without having to count again, would have demonstrated that the children have a relatively good grasp of cardinality. Freeman and his colleagues reported that only about one third of the children who detected the puppet's error were able to say what the correct number of items was without recounting. In the subsequent section we return to the importance of knowing what the next number is for the concept of cardinality.

The third study we consider here was by Bermejo, Morales and deOsuna (2004), who argued that if children really understand cardinality, and not just the Gelman's cardinality principle, they should do better than just re-implement the counting in a correct way. For example, they should be able to know how many objects are in a set even if the counting sequence is implemented backwards. If you count a set by saying 'three, two, one', and you reach the last item when you say 'one', you know that there are three objects in the set. If you count backwards from three and the label 'one' does not coincide with the last object, you know that the set does not have three objects. Just like starting to count from two, counting backwards is another way of separating out Gelman's cardinality principle from understanding cardinality: when you count backwards, the *first* number label is the cardinal for the set if there is a one-to-one correspondence between number labels and objects. Bermejo and colleagues showed that four- and six-year-old children who can say that there are three objects in a set when you count forward cannot necessarily say that if you count backwards from four and the last number label is 'two', this does not mean that there are two objects in the set. In fact, many children did not realize that there was a contradiction between the two answers: for them, the set could have three objects if you count one way and two if you count in another way. They also showed that children who were given the opportunity to discuss what the cardinal for the set was when the counting was done backwards showed marked progress in other tasks of understanding cardinality, which included starting to count from other numbers in the counting sequence than the number one, as in Fuson's task. They concluded that reflecting about the use of counting and the different actions involved in achieving a correct counting created opportunities for children to understanding cardinality better.

The evidence that we have presented so far suggests very strongly and remarkably consistently that *learning to count* and *understanding relations between quantities* are two different achievements. On the whole, children can use the procedures for counting long before they realise how counting allows them to measure and compare different quantities, and thus to work out the relations between them. We think that it is only when children establish a connection between what they know about relations between quantities and counting that they can be said to know the meaning of natural numbers.[2]

Summary

1 Natural numbers are a way of representing particular quantities and relations between quantities.

2 When children learn numbers, they must find out not just about the counting sequence and how to count, but also about how the numbers in the counting system represent quantities and relations between them.

3 One basic aspect of this representation is the cardinality of number: all sets with the same number have the same quantity of items in them.

4 Another way of expressing cardinality is to say that all sets with the number are in one-to-one correspondence with each other.

5 There is evidence that young children's first successful experiences with one-to-one correspondence come through sharing; however, even if they succeed in sharing fairly and know the number of items in one set, many do not make the inference that the number of items in the other set is the same.

6 Because of its cardinal properties, number is a measure: one can compare the quantity of items in two different sets by counting each set.

7 Several studies have shown that many children as old as six years are reluctant to count, although they know how to count, when asked to compare sets. They resort to perceptual comparisons instead.

8 This evidence suggests that learning about quantities and learning about numbers develop independently of each other in young children. But in order to understand natural numbers, children must establish connections between quantities and numbers. Thus schools must ensure that children learn not only to count but also learn to establish connections between counting and their understanding of quantities.

Current theories about the origin of children's understanding of the meaning of cardinal number

We have seen that Piaget's theory defines children's understanding of number on the basis of their understanding of relations between quantities; for him, cardinality is not just saying how many items are in sets but grasping that sets in one-to-one correspondence are equivalent in number and vice-versa. He argued that children could only be said to understand numbers if they made a connection between numbers and the relations between quantities that are implied by numbers. He also argued that this connection was established by children as they reflected about the effect of their actions on quantities: setting items in correspondence, adding and taking items away are schemes of action which form the basis for children's understanding of how to compare and to change quantities. Piaget acknowledges that learning to count can accelerate this process of reflection on actions, and so can other forms of social interaction, because they may help the children realise the contradictions that they fall into when they say, for example, that two quantities are different and yet they are labelled by the same number. However, the process that eventually leads to their understanding of the meanings of natural numbers and the implications of these representations is the child's growing understanding of relations between quantities.

Piaget's studies of children's understanding of the relations between quantities involved three different ideas that he considered central to understanding number: understanding equivalence, order, and class-inclusion (which refers to the idea that the whole is the sum of the parts, or that a set with 6 items comprises a set with 5 items plus 1). The methods used in these studies have been extensively criticised, as has the idea that children develop through a sequence of stages that can be easily traced and are closely associated with age. However, to our knowledge the core idea that children come to understand relations between quantities by reflecting upon the results of their actions is still a very important hypothesis in the study of how children learn about numbers. We do not review this vast literature here as there are several collections of papers that do so (see, for example, Steffe, Cobb and Glaserfeld, 1988; Steffe and Thompson, 2000). Later sections of this paper will revisit Piaget's theory and discuss related research.

This is not the only theory about how children come to understand the meaning of cardinal numbers. There are at least two alternative theories which are widely discussed in the literature. One is a nativist theory, which proposes that children have from birth access to an innate, inexact but powerful 'analog' system, whose magnitude increases directly with the number of objects in an array, and they attach the number words to the properties occasioning these magnitudes (Dehaene 1992; 1997; Gallistel and Gelman, 1992; Gelman and Butterworth, 2005; Xu and Spelke, 2000; Wynn, 1992; 1998). This gives all of us from birth the ability to make approximate judgements about numerical quantities and we continue through life to use this capacity. The discriminations that this system allows us to make are much like our discriminations along other continua, such as loudness, brightness and length. One feature of all these discriminations is that the greater the quantities (the louder, the brighter or the longer they are) the harder they are to discriminate (known, after the great 19th century psycho-physicist who meticulously studied perceptual sensitivity, as the 'Weber function'). To quote Carey (2004): 'Tap out as fast as you can without counting (you can prevent yourself from counting by thinking 'the' with each tap) the following numbers of taps: 4, 15, 7, and 28. If you carried this out several times, you'd find the mean number of taps to be 4, 15, 7, and 28, with the range of variation very tight around 4 (usually 4, occasionally 3 or 5) and very great around 28 (from 14 to 40 taps, for example). Discriminability is a function of the absolute numerical value, as dictated by Weber's law' (p. 63). The evidence for this analog system being an innate one comes largely from studies of infants (Xu and Spelke, 2000; McCrink and Wynn, 2004) and to a certain extent studies of animals as well, and is beyond the scope of this review. The evidence for its importance for learning about number and arithmetic comes from studies of developmental or acquired dyscalculia (e.g. Butterworth, Cipolotti and Warrington, 1996; Landerl, Bevana and Butterworth, 2004). However important this basic system may be as a neurological basis for number processing, it is not clear how the link between an analog and imprecise system and a precise system based on counting can be forged: 'ninety' does not mean 'approximately ninety' any more than 'eight' could mean 'approximately eight'. In fact, as reported in the previous section, three- and four-year-olds know that if a set has 6 items and you add one item to it, it no longer has 5 objects: they know that 'six' is not the same as 'approximately six'.

A third well-known theoretical alternative, which starts from a standpoint in agreement with Gelman's theory, is Susan Carey's (2004) hypothesis about three ways of learning about number. Carey accepts Gelman and Gallistel's (1978) limited definition of the cardinality principle but rejects their conclusions about how children first come to understand this principle. Carey argues that initially (by which she means in the first three years of life), very young children can represent number in three different ways (Le Corre and Carey, 2007). The first is the analog system, described in the previous paragraphs. However, although Carey thinks that this system plays a part in people's informal experiences of quantity throughout their lives, she does not seem to assign it a role in children's learning about the counting system, or in any other part of the mathematics that they learn about at school.

In her theory, the second of Carey's three systems, which she calls the 'parallel individuation' system, plays the crucial part in making it possible for children to learn how to connect number with the counting system. This system makes it possible for infants to recognise and represent very small numbers exactly (not approximately like the analog system). The system only operates for sets of 1, 2 and 3 objects and even within this restricted scope there is marked development over children's first three years.

Initially the system allows very young children to recognise sets of 1 as having a distinct quantity. The child understands 1 as a quantity, though he or she does not at first know that the word 'one' applies to this quantity. Later on the child is able to discriminate and recognise – in Carey's words 'to individuate' – sets of 1 and 2 objects, and still later, around the age of three- to four-years, sets of 1, 2 and 3 objects as distinct quantities. In Carey's terms young children progress from being 'one-knowers' to becoming 'two-knowers' and then 'three-knowers'.

During the same period, these children also learn number words and, though their recognition of 1, 2 and 3 as distinct quantities does not in any way depend on this verbal learning, they do manage to associate the right count words ('one', 'two' and 'three') with the right quantities. This association between parallel individuation and the count list eventually leads to what Carey (2004) calls 'bootstrapping': the children lift themselves up by their own intellectual bootstraps. They do so, some time in their fourth or fifth year, and therefore well before they go to school.

This bootstrapping takes two forms. First, with the help of the constant order of number words in the count list, the children begin to learn about the ordinal properties of numbers: 2 always comes after 1 in the count list and is always more numerous than 1, and 3 is more numerous than 2 and always follows 2. Second, since the fact that the count list that the children learn goes well beyond 3, they eventually infer that the number words represent a continuum of distinct quantities which also stretches beyond 'three'. They also begin to understand that the numbers above three are harder to discriminate from each other at a glance than sets of 1, 2 and 3 are, but that they can identify by counting. In Carey's words 'The child ascertains the meaning of 'two' from the resources that underlie natural language quantifiers, and from the system of parallel individuation, whereas she comes to know the meaning of 'five' through the bootstrapping process – i.e., that 'five' means one more than four, which is one more than three – by integrating representations of natural language quantifiers with the external serial ordered count list'. Carey called this new understanding 'enriched parallel individuation' (Carey, 2004; p. 65).

Carey's main evidence for parallel individuation and enriched parallel individuation came from studies in which she used a task, originally developed by Wynn, called 'Give – a number'. In this, an experimenter asks the child to give her a certain number of objects from a set of objects in front of them: 'Could you take two elephants out of the bowl and place them on the table?' Children sometimes put out the number asked for and sometimes just grab objects apparently randomly. Using this task Carey showed that different three-, four- and five-year-old children can be classified quite convincingly as 'one-' 'two-' or 'three-knowers' or as 'counting-principle-knowers'. The one-knowers do well when asked to provide one object but not when asked the other numbers while the two- and three-knowers can respectively provide up to two and three objects successfully. The 'counting-principle-knowers' in contrast count quantities above three or four.

The evidence for the existence of these three groups certainly supports Carey's interesting idea of a radical developmental change from 'knowing' some small quantities to understanding that the number system can be extended to other numbers in the count list. The value of her work is that it shows developmental changes in children's learning about the counting system. These had been by-passed both by Piaget and his colleagues because their theory was about the underlying logic needed for this learning and not about counting itself, and also by Gelman, because her theory about counting principles was about innate or rapidly acquired structures and not about development. However, Carey's explanation of children's counting in terms of enriched parallel individuation suffers the limitation that we have mentioned already: it has no proper measure of children's understanding of cardinality in its full sense. Just knowing that the last number that you counted is the number of the set is not enough.

The third way in which children learn number, according to Carey's theory, is through a system which she called 'set-based quantification': this is heavily dependent on language and particularly on words like 'a' and 'some' that are called 'quantifiers'. Thus far the implications of this third hypothesised system for education are not fully worked out, and we shall not discuss it further.

Carey's theory has been subjected to much criticism for the role that it attributes to induction or analogy in the use of the 'next' principle and to language. Gelman and Butterworth (2005), for example, argue that groups that have very restricted number language still show understanding of larger quantities; their number knowledge is not restricted to small numerosities as suggested in Carey's theory. Rips, Asmuth and Bloomfield (2006; 2008) address it more from a theoretical standpoint and argue that the bootstrapping hypothesis presupposes the very knowledge of number that it attempts to explain. They suggest that, in order to apply the 'next number' principle, children would have to know already that 1 is a set included in 2, 2 in 3, and 3 in 4. If they already know this, then they do not need to use the 'next number' principle to learn about what number words mean.

Which of these approaches is right? We do not think that there is a simple answer. If you hold, as we do, that understanding number is about understanding an ordered set of symbols that represent quantitative relations, Piaget's approach definitely has the edge. Both Gelman's and Carey's theory only address the question of how children give meaning to number words: neither entertains the idea that numbers represent quantities and relations between quantities, and that it is necessary for children to understand this system of relations as well as the fact that the word 'five' represents a set with 5 items in order to learn mathematics. Their research did nothing to dent Piaget's view that children of five years and six years are still learning about very basic relations between quantities, sometimes quite slowly.

Summary

1 Piaget's studies of learning about number concentrated on children's ability to reason logically about quantitative relations, and bypassed their acquisition of the counting system.

2 In contrast many current theories concentrate on children learning to count, and omit children's reasoning about quantitative relations. The most notable omission in these theories is the question of children's understanding of cardinality.

3 Gelman's studies of children's counting, nevertheless, did establish that even very young children systematically obey some basic counting principles when they do count.

Ordinal number

Numbers, as we have noted, come in a fixed order, and this order represents a quantitative series. Numbers are arranged in an ascending scale: 2 is more than 1 and 3 more than 2 and so on. Also the next number in the scale is always 1 more than the number that precedes it. Ordinal numbers indicate the position of a quantity in a series.

Piaget developed much the same argument about ordinal number as about cardinal number. He claimed that children learn to count, and therefore to produce numbers in the right fixed order, long before they understand that this order represents an ordinal series. This claim about children's difficulties with ordinality was based on his experiments on 'seriation' and also on 'transitivity'.

In his 'seriation' experiments, Piaget and his colleagues (Piaget, 1952) showed children a set of sticks all different in length and arranged in order from smallest to largest, and then jumbled them up and asked the child to re-order them in the same way. However, the children were asked to do so not by constructing the visual display all at once, which they would be able to do perceptually and by trial-and-error, but by giving the sticks to the experimenter one by one, in the order that they think they should be placed.

This is a surprisingly difficult task for young children and, at the age range that we are considering here (five- to six-years), children tend to form groups of ordered sticks instead of creating a single ordered series. Even when they do manage to put the sticks into a proper series, they tend then to fail an additional test, which Piaget considered to be the acid test of ordinal understanding: this was to insert another stick which he then gave them into its correct place in the already created series, which was now visible. These difficulties, which are highly reliable and have never been refuted or explained away, are certainly important, but they may not be true of number. Children's reactions to number series may well be different precisely because of the extensive practice that they have with producing numbers in a fixed order.

Recently, however, Brannon (2002) made the striking claim that even one-year-old-children understand ordinal number relations. The most direct evidence that Brannon offered for this claim was a study in which she showed the infant sequences of three cards, each of which depicted a different number of squares. Each three-card sequence constituted either an increasing or a decreasing series. In some sequences the number of squares increased from card to card e.g. 2, 4, 8 and 3, 6, 12: in others the numbers decreased e.g. 16, 8, 4.

Brannon's results suggested that 11-month-old infants could discriminate the two kinds of sequence (after seeing several increasing sequences they were more interested in looking at a decreasing than at yet another increasing sequence, and vice versa), and she concluded that even at this young age children have some understanding of seriation.

However, her task was a very weak test of the understanding of ordinality. It probably shows that children of this age are to some extent aware of the relations 'more' and 'less', but it does not establish that the children were acting on the relation between all three numbers in each sequence.

The point here is that in order to understand ordinality the child must be able to co-ordinate a set of 'more' and 'less' relations. This means understanding that *b* is smaller than *a* and at the same time larger than *c* in an $a > b > c$ series. Piaget was happy to accept that even very young children can see quite clearly that *b* is smaller than a and at another time that it is larger than *c*, but he claimed that in order to form a series children have to understand that intermediary quantities like *b* are *simultaneously* larger than some values and smaller than others. Of course, Brannon did not show whether the young children in her study could or could not grasp these two-way relations.

Piaget's (1921) most direct evidence for children's difficulties with two-way relations came from another kind of task – the transitivity task. The relations between quantities in any ordinal series are transitive. If A > B and B > C, then it follows that A > C, and one can draw this logical conclusion without ever directly comparing A with C. This applies to number as well: since 8 is more than 4 and 4 more than 2, 8 is more than 2.

Piaget claimed that children below the age of roughly eight years are unable to make these inferences because they find it difficult to understand that B can be simultaneously smaller than one quantity (A) and larger than another (C). In his experiments on transitivity Piaget did find that children very rarely made the indirect inference between A and C on the basis of being shown that A > C and B > C, but this was not very strong evidence for his hypothesis because he failed to check the possibility that the children failed to make the inference because they had forgotten the premises – a reason which has nothing directly to do with logic or with reasoning.

Subsequent studies, in which care was taken to check how well the children remembered the premises at the time that they were required to make the A > C inference (Bryant and Trabasso, 1971; Bryant and Kopytynska, 1976 ; Trabasso, 1977) consistently showed that children of five years or older do make the inference successfully, provided that they remember the relevant premises correctly. Young children's success in these tasks throws some doubt on Piaget's claim that they do not understand ordinal quantitative relations, but by and large there is still a host of unanswered questions about children's grasp of ordinality. We shall return to the issue of transitivity in the section on Space and Geometry.

Above all we need a comprehensive set of seriation and transitivity experiments in which the quantities are numbers (discontinuous quantities), and not continuous quantities like the rods of different lengths that have been the staple diet of previous work on these subjects.

Summary

1 The count list is arranged in order of the magnitude of the quantities represented by the numbers. The relations between numbers in this series are transitive: if A > B and B > C , then A > C.

2 Piaget argued that young children find ordinal relations, as well as cardinal relations, difficult to understand. He attributed these difficulties to an inability, on the part of young children, to understand that, in an A > B > C series, B is simultaneously smaller than A and larger than B.

3 Piaget's evidence for this claim came from studies of seriation and transitivity. The difficulties that children have in the seriation experiment, in which they have to construct an ordered series of sticks, are surprising and very striking.

4 However, the criterion for constructing the series in the seriation experiment (different lengths of some sticks) cannot be applied by counting. Therefore, seriation studies do not deal directly with children's understanding of natural number. The question of the seriation of number is still an open one.

Cardinality, additive reasoning and extensive quantities

So far we have discussed how children give meaning to number and how easy or difficult it is for them to make connections between what they understand about quantity and the numbers that they learn when they begin to count. Now we turn to another aspect of cardinal number, its connection with addition and subtraction – or, more generally, with additive reasoning. There are undeniable connections between the concept of cardinality and additive reasoning and we shall explore them in this section, which is about the additive composition of number, and in the subsequent section, which is about the inverse relations between addition and subtraction.

Piaget (1952) included in his definition of children's understanding of number their realisation that a quantity (and its numerical representation) is only changed by addition or subtraction, not by other operations such as spreading the elements or bunching them together. This definition, he indicated, is valid for the domain of extensive quantities, which are measured by the addition of units because the whole is the sum of the parts. If the quantity is made of discrete elements (e.g. a set of coins), the task of measuring it and assigning a number to it is easy: all the children have to do is to count. If the quantity is continuous (e.g. a ribbon), the task of measuring it is more difficult: normally a conventional unit would be applied to the quantity and the number that represents the quantity is the number of iterations of these units. Extensive quantities differ from intensive quantities, which are measured by the ratio between two other quantities. For example, the concentration of a juice is measured by the ratio between amount of concentrate and amount of water used to make the juice. These quantities are considered in Paper 4.

His studies of children's understanding of the conservation of quantities have been criticised on methodological grounds (e.g. Donaldson, 1978; Light, Buckingham and Robbins, 1979; Samuel and Bryant, 1984) but, so far as we know, his idea that children must realise that extensive quantities change either by addition or by subtraction has not been challenged.

Piaget (1952) also made the reasonable suggestion that you cannot understand what 'five' is unless you also know that it is composed of numbers smaller than it. Any set of five items contains a sub-set of 4 items and another sub-set of 1, or one sub-set of 3 and another of 2. A combination of or, in other words, an addition of each of these pairs of sets produces a set of five. This is called the additive composition of number, which is an important aspect of the understanding of relations between numbers.

Piaget used the idea of class-inclusion to describe this aspect of number; others (e.g. Resnick and Ford, 1981) have called it part-whole relations. Piaget's studies consisted in asking children about the quantitative relations between classes, one of which was included in the other. For example, in some tasks children were asked to compare the number of dogs with the number of animals in sets which included other animals, such as cats. For an adult, there is no need to know the actual number of dogs, cats, and animals in such a task: there will be always more animals than dogs because the class of dogs is included in the class of animals. However, some children aged four- to six-years do not necessarily think like adults: if the number of dogs is quite a bit larger than the number of cats, the children might answer that there are more dogs than animals. According to Piaget, this answer which to an adult seems entirely illogical, was the result of the children's difficulties with thinking of the class of dogs as simultaneously included in the class of animals and excluded from it for comparison purposes. Once they mentally excluded the dogs from the set of animals, they could no longer think of the dogs as part of the set of animals: they would then be unable to focus on the fact that the whole (the overall class, animals) is always larger than one part (the included class, dogs).

Piaget and his colleagues (Piaget, 1952; Inhelder, Sinclair and Bovet, 1974) did use a number of conditions to try to eliminate alternative hypotheses for children's difficulties. For example, they asked the children whether in the whole world there would be more dogs or more animals. This question used the same linguistic format but could be answered without an understanding of the necessary relation between a part and a whole: the children could think that there are many types of animals in the world and therefore there say that there are more animals than dogs. Children are indeed more successful in answering this question than the class-inclusion one. Another manipulation Piaget and his colleagues used was to ask the children to circle with a string the dogs and the animals: this had no effect on the children's performance, and they continued to exclude the dogs mentally from the class of animals. The only manipulation that helped the children was to ask the children to first think of the set of animals without separating out the dogs, then replace the dogs with visual representations that marked their inclusion in the class of animals, while the dogs were set in a separate class: the children were then able to create a simultaneous representation of the dogs included in the whole and as a separate part and answer the question correctly. After having answered the question in this situation, some children went on to answer it correctly when other class-inclusion problems were presented (for example, about flowers and roses) without the support of the extra visual signs.

The Piagetian experiments on class-inclusion have been criticised on many grounds: for example, it has been argued that the question the children are asked is an anomalous question because it uses disjunction (dogs *or* animals) when something can be simultaneously a dog and *an* animal (Donaldson, 1978; Markman, 1979). However, Piaget's hypothesis that part-whole relations are an important aspect of number understanding has not been challenged. As discussed in the previous sections, it has been argued (e.g. by Rips, Asmuth and Bloomfield, 2008) that it is most unlikely that a child will understand the ordinality of number until she has grasped the connection between the next number and the plus-one compositions: i.e. that a set of 5 items contains a set of 4 items plus a set of 1, and a set of 4 items is composed of a set of 3 plus 1, and so on.

For exactly the same reasons, the understanding of additive composition of number is essential in any comparison between two numbers. To judge the difference, for example, between 7 and 4, something which as we shall see is not always easy for young children, you need to know that 7 is composed of 4 and 3, which means that 7 is 3 greater than 4.

Of course, even very young children have a great deal of informal experience of quantities increasing or decreasing as a result of additions and subtractions. There is good evidence that pre-schoolers do understand that additions increase and subtractions decrease quantities (Brush, 1978; Cooper, 1984; Klein, 1984) but this does not mean that they realise that the only changes that affect quantity are addition and subtraction. It is possible that their understanding of these changes is qualitative in the sense that it lacks precision. We can take as an example what happens when young children are shown two sets that are unequal and are arranged in one-to-one correspondence, as in Figure 2.1, so that it is possible for the children to see the size of the difference (say one set has 10 objects and the other 7). The experimenter proposes to add to the smaller set fewer items than the difference (i.e. she proposes to add 2 to the set with 7). Some preschoolers judge that the set to which elements are added will become larger than the other. Others think that it is now the same as the other set. It is only at about six- or seven-years that children actually take into account the precise difference between the sets in order to know whether they will be the same or not after the addition of items to the smaller set (Klein, 1984; Blevins-Knabe, Cooper, Mace and Starkey, 1987).

The basic importance of the additive composition of number means that learning to count and learning to add and subtract are not necessarily two successive and separate intellectual steps, as common sense might suggest. At first glance, it seems quite a plausible suggestion that children must understand number and know about the counting system in order to do any arithmetic, like adding and subtracting. It seems simply impossible that they could add 6 and 4 or subtract 4 from 6 without knowing what 6 and 4 mean. However, we have now seen that this link between counting and arithmetic must work in the opposite direction as well, because it is also impossible that children could know what 6 or 4 or any other number mean, or anything about the relations between these numbers, without also understanding something about the additive composition of number.

Given its obvious importance, there is remarkably little research on children's grasp of additive composition of number. The most relevant information, though it is somewhat indirect, comes from the well-known developmental change from 'counting-all' to 'counting-on'. As we have seen, five-year-old children generally know how to count the number of items in a set within the constraints of one-to-one counting. However, their counting is not always economic. If, for example, they are given a set of 7 items which they duly count and then 6 further items are added to this set and the children are asked about the total number in the newly increased set, they tend to count all the 13 items in front of them including the subset that they counted before. Such observations have been replicated many times (e.g. Fuson, 1983; Nunes and Bryant, 1996; Wright, 1994) and have given origin to a widely used analysis of children's progress in understanding cardinality (e.g. Steffe, Cobb and von Glaserfeld, 1988; Steffe, Thompson and Richards, 1982; Steffe, von Glasersfeld, Richards and Cobb, 1983). This counting of all the items is not wrong, of course, but the repeated counting of the initial items is unnecessary. The children could just as well and much more efficiently have counted on from the initial set (not '1, 2, 3,…..13' but '8, 9, 10…..13').

According to Vergnaud (2008), the explanation for children's uneconomical behaviour is conceptual: as referred in the previous section, he argues that their understanding of number may be simply *ordinal* (i.e. what they know is that the last number word represents the set) and so they cannot add because ordinal numbers cannot be added. They can, however, count a new, larger set, and give to it the label of the last number word used in counting.

Studies of young children's reactions to the kind of situation that we have just described have consistently produced two clear results. The first is that young children of around the age of five years consistently count all the items. The second is that between the ages of five and seven years, there is a definite developmental shift from counting-all to counting-on: as children grow older they begin to adopt the more economic strategy of counting-on from the previously counted subset. This new strategy is a definite sign of children's eventual recognition of the additive composition of the new set: they appear to understand that the total number of the new set will contain the original 7 items plus the newly added 6 items. The fact that younger children stick to

Figure 2.1: Two sets in correspondence; the difference between sets is easily seen.

counting-all does not establish that they cannot understand the additive composition of the new set (as is often the case, it is a great deal easier to establish that children do understand some principle than that they do not). However, the developmental change that we have just described does suggest an improvement in children's understanding of additive relations between numbers during their first two years at school.

The study of the connections that children make, or fail to make, between understanding number, additive composition and additive reasoning plainly supports the Piagetian thesis that children give meaning to numbers by establishing relations between quantities though their schemes of action. They do need to understand that addition increases and subtraction decreases the number of items in a set. This forms a foundation for their understanding of the precise way in which the number changes: adding 1 to set *a* creates a number that is equal to *a* + 1. This number can be seen as a whole that includes the parts *a* and 1. Instead of relying on the 'next number' induction or analogy, children use addition and the logic of part-whole to understand numbers.

Summary

1 In order to understand number as an ordinal series, children have to realise that numbers are composed of combinations of smaller numbers.

2 This realisation stems from their progressive understanding of how addition affects number: at first they understand that addition increases the number of items in a set without being precise about the extent of this increase but, as they co-ordinate their knowledge of addition with their understanding of part-whole relations, they can also become more precise about additive composition.

3 Young children's tendency to count-all rather than to count-on suggests either that they do not understand the additive composition of number or that their grasp of additive composition is too weak for them to take advantage of it.

4 Their difficulties suggest that children should be taught about additive composition, and therefore about addition, as they learn about the counting system.

The decade structure and additive composition

Additive composition and the understanding of number and counting are linked for another important reason. The power and the effectiveness of counting rest largely on the invention of base systems, and these systems depend on additive composition. The base-10 system, which is now widespread, frees us from having to remember long strings of numbers, as indeed any base system does. In English, once we know the simple rules for the decimal system and remember the number words for 1 to 20, for the decades, and then for a hundred, a thousand and a million, we can generate most of the natural numbers that we will ever need to produce with very little effort or difficulty. The link between understanding additive composition and adopting a base system is quite obvious. Base systems rest on the additive composition of number and the decade structure is in effect a clear reminder that 'fourteen' is a combination of 10 and 4, and 'thirty-five' of three 10s and 5.

Additive composition is the basic concept that underlies any counting system with a base, oral or written. This includes of course the Hindu-Arabic place-value system that we use to write numbers. For example, the decimal system explicitly represents the fact that all the numbers between 10 and a 100 must be a combination of one or more decades and a number less than 10: 17 is a combination of 10 and 7 and 23 a combination of two 10s and 3. The digits express the additive composition of any number from 10 on: e.g. in 23, 2 represents two 10s which are added to 3, which represents three units.

Additive composition is also at the root of our ability to count money using coins and notes of different denominations. When we have, for example, one 10p and five 1p coins, we can only count the 10p and the 1p coins together if we understand about additive composition.

The data from the 'shop task', a test devised by Nunes and Schliemann (1990), suggest that initially children find it hard to combine denominations in this way. In the shop task children are shown a set of toys in a 'shop', are given some (real or artificial) money and are asked to choose a toy that they would like to buy. Then the experimenter asks them to pay a certain sum for their choice. Sometimes the child can pay for this with coins of one denomination only: for example, the experimenter charges a child 15p for a toy car and the child has that number of pence to make the purchase or the charge is 30p and the child can pay with three 10p coins. In other trials, the child can pay only by combining denominations: the car costs 15p and the child, having fewer 1p coins than that, must pay with the combination of a 10p and five 1p coins. Although the values that the children are asked to pay when they use only 10p coins are larger than those they pay when using combinations of different values, children can count in tens (ten, twenty, thirty etc.) using simple correspondences between the counting labels and the coins. This task does not require understanding additive composition. So Nunes and Schliemann predicted that the mixed denomination trials would be significantly more difficult than the other trials in the task. They found that the mixed denominations trials were indeed much harder for the children than the single denomination trials and that there was a marked improvement between the ages of five and seven years in children's performance in the combined denomination trials. This work was originally carried out in Brazil and the results have been confirmed in other research in the United Kingdom (Krebs, Squire and Bryant, 2003; Nunes *et al*., 2007).

A fascinating observation in this task is that children don't change from being unable to carry out the additive composition to counting on from ten as they add 1p coins to the money they are counting. The show the same count-all behaviour that they show when they have a set of objects and more object are added to the set. However, as there are no visible 1p coins within the 10p, they point to the 10p ten times as they count, or they lift up 10 fingers and say 'ten', and only then go on to count 'eleven, twelve, thirteen etc.' This repeated pointing to count invisible objects has been documented also by Steffe and his colleagues (e.g. Steffe, von Glasersfeld, Richards and Cobb, 1983), who interpreted it, as we do, as a significant step in coordinating counting with a more mature understanding of cardinality.

In a recent training study (Nunes *et al*., 2007), we encouraged children who did not succeed in the shop-task to use the transition behaviour we had observed, and asked them to show us ten with their fingers; we then pointed to their fingers and the 10p coin and asked the children to say how much there was in each display; finally, we encouraged them to go on and count the money. Our study showed that some children seemed to be able to grasp the idea of additive composition quite quickly after this demonstration and others took some time to do so, but all children benefited significantly from brief training sessions using this procedure.

Since children appear to be finding out about the additive nature of the base-10 system and at the same time (their first two years at school) about the additive composition of number in general, one can reasonably ask what the connection between these two is. One possibility is that children must gain a full understanding of the additive composition of number before they can understand the decade structure. Another is that instruction about the decade structure is children's first entrée to additive composition. First they learn that 12 is a combination of 10 and 2 and then they extend this knowledge to other combinations (e.g. 12 is also a combination of 8 and 4). The results of a recent study by Krebs, Squire and Bryant (2003), in which the same children were given the shop task and counting all/counting on tasks, favour the second hypothesis. All the children who consistently counted on (the more economic strategy) also did well on the shop task, but there were some children who scored well in the shop task but nevertheless tended not to count on. However, no child scored well in the counting all/on task but poorly in the shop task. This pattern suggests that the cues present in the language help children learn about the decimal system first and then extend their new understanding of additive composition to combinations that do not involve decades.

Summary

1 The decimal system is a good example of an invented and culturally transmitted mathematical tool. It enhances our power to calculate and frees us from having to remember extended sequences of number.

2 Once we know the rules for the decade system and the names of the different classes and orders (tens, hundreds, thousands etc.), we can use the system to count by generating numbers ourselves.

3 However, the system also makes some quite difficult intellectual demands. Children find it hard at first to combine different denominations, such as tens and ones.

4 Teachers should be aware that the ability to combine denominations rests on a thorough grasp of additive composition.

5 There is some evidence that experience with the structure of the decimal system may enhances children's understanding of additive composition. There is also evidence that it is possible to use money to provoke children's progress in understanding additive composition.

The inverse relation between addition and subtraction

The research we have considered so far suggests that by the age of six or seven children understand quite a lot about number: they understand equivalence well enough to know that if two sets are equivalent they can infer the number that describes one by counting the other; they understand that addition and subtraction are the operations that change the number in a set; they understand additive composition and what must be added to one set to make it equivalent to the other; they understand that they can count on if you add more elements to a set; and they understand about ordinal number and can make transitive inferences. However, there is an insight about how addition and subtraction affect the number of elements in a set that we still need to consider. This is the insight that addition is the inverse of subtraction and vice versa, and thus that equal additions and subtractions cancel each other out: 27 + 19 – 19 = 27 and 27 – 19 + 19 = 27.

It is easy to see that one cannot understand either addition or subtraction or even number fully without also knowing about the inverse relation of each of these operations to the other. It is absolutely essential when adding and subtracting to understand that these are reversible actions. Otherwise one will not understand that one can move along the number scale in two opposite directions – up and down.

The understanding of any inverse relation should, according to Piaget (1952), be particularly hard for young children, since in his theory young children are not able to carry out 'reversible' thought processes. Children in the five- to eight-year range do not see that if 4 + 8 = 12, therefore 12 – 8 = 4 because they do not realise that the original addition (+8) is cancelled out by the inverse subtraction (–8). This claim is a central part of Piaget's theory about children's arithmetical learning, but he never tested it directly, even though it would have been quite easy to do so.

In one of his last publications, Piaget and Moreau (2001) did report an ingenious, but rather too complicated, study of the inverse relations between addition and subtraction and also between multiplication and division. They asked children, aged from six- to ten-years, to choose a number but not to tell them what this was. Then they asked the child first to add 3 to this number, next to double the sum and then to add 5 to the result of the multiplication. Next, they asked the child what the result was, and went on to tell him or her what was the number that s/he chose to start with. Finally the experimenters asked each child to explain how they had managed to work out what this initial number was.

Piaget and Moreau reported that this was a difficult task. The youngest children in the sample did not understand that the experimenters had performed the inverse operations, subtracting where the child had added and dividing where s/he had multiplied. The older children did show some understanding that this was how the experimenters reached the right number, but did not understand that the order of the inverse operations was important. The experimenters accounted for the younger children's difficulties by arguing that these children had failed to understand the adult's use of inversion (equal additions and subtractions and equal multiplications and divisions) because they did not understand the principle of inversion.

This was a highly original study but Piaget and Moreau's conclusions from it can be questioned. One alternative explanation for the children's difficulties is that they may perfectly have understood the inverse relations between the different operations, but they may still not have been able to work out how the adult used them to solve the problem. The children, also, had to deal with two kinds of inversion (addition-subtraction and multiplication-division) in order to explain the adult's correct solution, and so their frequent failures to produce a coherent explanation may have been due to their not knowing about one of the inverse relations, e.g. between multiplication and division, even though they were completely at home with the other, e.g. between addition and subtraction.

Nevertheless, some following studies seemed to confirm that young school-children are often unaware aware that inverse transformations cancel each other out in $a + b – b$ sums. In two studies (Bisanz and Lefevre, 1990; Stern, 1992), the vast majority of the younger children did no better with inverse $a + b – b$ sums in which they could take advantage of the inversion principle than with control $a + b – c$ sums where this was not possible. For example, Stern reported that only 13% of the seven-year-old children and 48% of the nine-year-olds in her study used the inversion principle consistently, when some of the problems that they were given were $a + b - b$ sums and others $a + b - c$ sums.

This overall difficulty was confirmed in a further study by Siegler and Stern (1998), who gave eight-year-old German children inversion problems in eight successive sessions. Their aim was to see whether the children improved in their use of the inversion principle to solve problems. The children were also exposed to other traditional scholastic problems (e.g. $a + b - c$), which could not be solved by using the inverse principle. In the last of the eight sessions, Siegler and Stern also gave the children control problems, which involved sequences such as $a + b + b$, so that the inversion principle was not appropriate for solution. The experimenters recorded how well children distinguished the problems that could be solved through the inversion principle from those that had to be solved in some other way.

The study showed that the children who were given lots of inversion problems in the first seven sessions tended to get better at solving these problems over these sessions, but in the final session in which the children were given control as well as inversion problems they often, quite inappropriately, overgeneralised the inversion strategy to the control sums: they would give a as the answer in $a + b + b$ control problems as well as in inverse $a + b - b$ problems. Their relatively good performance with the inversion problems in the previous sessions, therefore, was probably not the result of an increasing understanding of inversion. They seem to have learned some lower-level and totally inadequate strategy, such as 'if the first number (a) is followed by another number (b) which is then repeated, the answer must be a'.

The pervasive failures of the younger school children in these studies to take advantage of the inversion principle certainly suggest that it is extremely difficult for them to understand and to learn how to use this principle, as Piaget first suggested. However, in all these tasks the problems were presented either verbally or in written form. Other studies, which employed sets of physical objects, paint a different picture. (Bryant, Christie and Rendu, 1999; Rasmussen, Ho and Bisanz, 2003). For example, Bryant *et al*. used sets of bricks to present five- and six-year-old children with $a + b - b$ inversion problems and $a + a - b$ control problems. In this particular task young children did a great deal better with the inversion problems than with the control problems, which is good evidence that they were using the inversion principle when they could. In the same study the children were also given equivalent inversion and control problems as verbal sums (27 + 14 – 14): they used the inversion principle much less often in this task than in the task with bricks, a result which resonates well with Hughes' (1981) discovery that pre-school children are much more successful at working out the results of additions and subtractions in problems that involve concrete objects than in abstract, verbal sums.

The fact that young children are readier to use the inversion principle in concrete than in abstract problems suggests that they may learn about inversion initially through their actions with concrete material. Bryant *et al*. raised this possibility, and they also made a distinction between two levels in the understanding of the inverse relation between addition and subtraction. One is the level of *identity*: when identical stuff is added to and then subtracted from an object, the final state of this object is the same as the initial state. Young children have many informal experiences of inverse transformations at this level. A child gets his shirt dirty (mud is added to it) and then it is cleaned (mud is subtracted) and the shirt is as it was before. At meal-times various objects (knives, forks etc.) are put on the dining room table and then subtracted when the meal is over; the table top is as empty after the meal, as it was before.

Note that understanding the inversion of identity may not involve quantity. The child can understand that, if the same (or identical) stuff is added and then removed, the *status quo* is restored without having to know anything about the quantity of the stuff.

The other possible level is the understanding of the inversion of *quantity*. If I have 10 sweets and someone gives me 3 more and then I eat 3, I have the same number left as at the start, and it doesn't matter whether the 3 sweets that I ate are the same 3 sweets as were given to me or different ones. Provided that I eat the same number as I was given, the quantitative *status quo* is now restored.

In a second study, again using toy bricks, Bryant *et al*. established that five- and six-year-old children found problems, called identity problems, in which exactly the same bricks were added to and then subtracted from the initial set (or vice versa), easier than other problems, called quantity problems, in which the same number of bricks was added and then subtracted (or vice versa), but the actual bricks subtracted were quite different from the bricks that had been added before. Bryant *et al*. also found a greater improvement with age in children's performance in quantity inversion problems than in identity inversion problems. These results point to a developmental hypothesis: children's understanding of the inversion of identity precedes, and may provide the basis for, their understanding of the inversion of quantity. First they understand that adding and subtracting the same stuff restores the physical *status quo*. Then they extend this knowledge to quantity, realising now that adding and subtracting the same quantity restores the quantitative *status quo*, whether the addend and subtrahend are the same stuff or not.

However, the causal determinants of learning about inversion might vary between children. Certainly there are many reports of substantial individual differences within the same age groups in the understanding of the inversion principle. Many of the seven- and nine-year-olds in Bisanz and LeFevre's study (1990) used the inversion principle to solve appropriate problems but over half of them did not. Over half of the ten-year-olds tested in Stern's (1992) original study did take advantage of the principle, but around 40% seemed unable to do so.

Recent work by Gilmore (Gilmore and Bryant, 2006; Gilmore and Papadatou-Pastou, 2008) suggests that the underlying pattern of these individual differences might take a more complex and also a more interesting form than just a dichotomy between those who do and those who do not understand the inversion principle. She used cluster analysis with samples of six- to eight-year-olds who had been given inversion and control problems (again the control problems had to be solved through calculation), and consistently found three groups of children. One group appeared to have a clear understanding of inversion and good calculation skills as well; these children did better in the inversion than in the control problems, but their scores in the control problems were also relatively high. Another group consisted of children who seemed to have little understanding of inversion and whose calculation skills were weak as well. The remaining group of children had a good understanding of inversion, but their calculation skills were weak: in other words, these children did better in the inversion than in the control problems, but their scores in the control problems were particularly low. Thus, the discrepancy between knowing about inversion and knowing how to calculate went one way but not the other. Gilmore identified a group of children who could use the inversion principle and yet did not calculate well, but she found no evidence at all for the existence a group of children who could calculate well but were unable to use the inversion principle. Children, therefore, do not have to be good at adding and subtracting in order to understand the relation between these two operations. On the contrary, they may need to understand the inverse relation before they can learn to add and subtract efficiently.

How can knowledge of inversion facilitate children's ability to calculate? Our answer to this is only hypothetical at this stage, but it is worth examining here. If children understand well the principle of inversion, they may use their knowledge of number facts more flexibly, and thus succeed in more problems where calculation is required than children who cannot use their knowledge flexibly. For example, if they know that $9 + 7 = 16$ and understand inversion, they can use this knowledge to answer two more questions: $16 - 9 = ?$ and $16 - 7 = ?$ Similarly, the use of 'indirect addition' to solve difficult subtraction problems depends on knowing and using the inverse relation between addition and subtraction. One must understand inversion to be able to see, for example, that an easy way to solve the subtraction $42 - 39$ is to convert it into an addition: the child can count up from 39 to 42, find that this is 3, and will then know that $42 - 39$ must equal 3. In our view, no one could reason this way without also understanding the inverse relation between addition and subtraction.

If this hypothesis is correct, it has fascinating educational implications. Children spend much time at home and in school practising number facts, perhaps trying to memorise them as if they were independent of each other. However, a mixture of learning about number facts and about mathematical principles that help them relate one number fact to others, such as inversion, could provide them with more flexible knowledge as well as more interesting learning experiences. So far as we know, there is no direct evidence of how instruction that focuses both on number facts and principles works in comparison with instruction that focuses only on number facts. However, there is some preliminary evidence on the role of inversion in facilitating children's understanding of the relation between the sum $a + b = c$ and $c - b$ (or $- a$) $= ?$

Some researchers have called this 'the complement' question and analysed its difficulty in a quite direct way by telling children first that $a + b = c$ and then immediately asking them the $c - a = ?$ question (Baroody, Ginsburg and Waxman, 1983; Baroody, 1999; Baroody and Tiilikainen, 2003; Resnick, 1983; Putnam, de Bettencourt and Leinhardt, 1990). These studies established that the step from the first to the second sum is extremely difficult for children in their first two or three years at school, and most of them fail to take it. Only by the age of about eight years do a majority of children use the information from the addition to solve the subtraction, and even at this age many children continue to make mistakes. Would they be able to do better if their understanding of the inverse relation improved?

The study by Siegler and Stern (1998) described earlier on, with eight-year-olds, seems to suggest that it is not that easy to improve children's understanding of the inverse relation between addition and subtraction: after solving over 100 inversion problems, distributed over 7 days, the children did very poorly in using it selectively; i.e. using it when it was appropriate, and not using it when it was not appropriate. However, the method that they used had several characteristics, which may not have facilitated learning. First, the problems were all presented simply as numbers written on cards, with no support of concrete materials or stories. Second, the children were encouraged to answer correctly and also quickly, if possible, but they did not receive any feedback on whether they were correct. Finally, they were asked to explain how they had solved the problem, but if they indicated that they had used the inverse relation to solve it, they were neither told that this was a good idea nor asked to think more about it if they had used it inappropriately. In brief, it was not a teaching study.

Recently we completed two studies on teaching children about the inverse relation between addition and subtraction (Nunes, Bryant, Hallett, Bell and Evans, 2008). Our aims were to test whether it is possible to improve children's understanding of the inverse relation and to see whether they would improve in solving the complement problem after receiving instruction on inversion.

One of the studies was with eight-year-olds, i.e. children of the same age as those who participated in the Siegler and Stern study. Our study was considerably briefer, as it involved a pre-test, two teaching sessions, and a post-test. In the pre-test the children answered inverse problems ($a + b - b$), control problems ($a + b - c$) and complement problems ($a + b = c$; $c - a/b = ?$). During the training, they only worked on inversion problems. So if the taught groups improved significantly on the complement problems, this would have to be a consequence of realising the relevance of the inverse principle to this type of problem.

For the teaching phase, the children were randomly assigned to one of three groups: a Control group, who only received practice in calculation; a Visual Demonstration group and a Verbal Calculator group, both receiving instruction on the inverse relation. The form of the instruction varied between the two groups.

The Visual Demonstration group was taught with the support of concrete materials, and started with a series of trials that took advantage of the identity inversion. First the children counted the number of bricks in a row of Unifix bricks, which was subsequently hidden under a cloth so that no counting was possible after that. Next, the experimenter added some bricks to the row and subtracted others. The child was then asked how many bricks were left under the cloth. The number of bricks added and subtracted was either the same or differed by one; this required the children to attend during all trials, as the answer was not in all examples the same number as before the additions and subtractions, but they could still use the inverse principle easily because the difference of one did not make the task too different from an exact inversion trial. When they had given their answer, they received feedback and explained how they had found the answer. If they had not used the inversion principle, they were encouraged to think about it (e.g. How many were added? How many were taken away? Would the number be the same or different?).

The Verbal Calculator group received the same number of trials but no visual demonstration. After they had provided their answer, they were encouraged to repeat the trial verbally as they entered the operations into a calculator and checked the answer. Thus they would be saying, for example, 'fourteen plus eight minus eight is' and looking at the answer.

As explained, we had three types of problems in the pre- and post-test: inversion, control and complement problems, which were transfer problems for our intervention group, as they had not learned about these directly during the training. We did not expect the groups to differ in the control trials, as the amount of experience they had between pre- and post-test was limited, but we expected the experimental teaching groups to perform significantly better than the Control group in the inversion and transfer problems.

The results were clear:

- Both taught groups made more progress than the Control group from the pre-test to the post-test in the inversion problems.
- The Visual Demonstration group made more progress than the Control group in the transfer problems; the Verbal Calculator group's improvement did not differ from the improvement in the Control group in the transfer problems.
- The children's performance did not improve significantly in the control problems in any of the groups.

Thus with eight-year-olds both Visual and Verbal methods can be used to promote children's reflection about the inverse relation between addition and subtraction. Although the two methods did not differ when directly compared to each other, they differed when compared with the fixed-standard provided by the control group: the Visual Demonstration method was effective in promoting transfer from the types of items used in the training to new types of items, of a format not presented during the training, and the Verbal method did not.

In our second teaching study, we worked with much younger children, whose mean age was just five years. We carried out the study using the same methods, with a pre-test, two teaching sessions, and a post-test, but this time all the children were taught using the Visual Demonstration method. Because the children were so young, we did not use complement problems to assess transfer, but we included a delayed post-test, given to the children about three weeks after they had completed the training in order to see whether the effects of the intervention, if any, would remain significant at a later date without further instruction.

The intervention showed significant effects for the children in one school but not for the children in the other school; the effects persisted until the delayed post-test was given. Although we cannot be certain, we think that the difference between the schools was due to the fact that in the school where the intervention did not have a significant effect we were unable to find a quite room to work with the children without interruptions and the children had difficulty in concentrating.

The main lesson from this second study was that it is possible for this intervention to work with such young children and for the effects to last without further instruction, but it is not certain that it will do so.

Finally, we need to consider whether knowing about inversion is really as important as we have claimed here. Two studies support this claim. The first was by Stern (2005). She established in a longitudinal study that German children's performance in inversion tasks, which they solved in their second year at school, significantly predicted their performance in an algebra assessment given about 15 years later, when they were 23 years old and studying in university. In fact, the brief inversion task that she gave to the children had a higher correlation with their performance in the algebra test than the IQ test given at about the same time as the inversion task. Partialling out the effect of IQ from the correlation between the inversion and the algebra tests did not affect this predictive relation between the inversion task and the algebra test.

The second study was by our own team (Nunes *et al*., 2007). It combined longitudinal and intervention methods to test whether the relation between reasoning principles and mathematics learning is a causal one. The participants in the longitudinal study were tested in their first year in school. In the second year, they completed the mathematics achievement tests administered by the teachers and designed centrally in the United Kingdom. The gap between our assessment and the mathematics achievement test was about 14 months. One of the components of our reasoning test was an assessment of children's understanding of the inverse relation between addition and subtraction; the others were additive composition (assessed by the shop task) and correspondence (in particular, one-to-many correspondence). We found that children's performance in the reasoning test significantly predicted their mathematics achievement even after controlling for age, working memory, knowledge of mathematics at school entry, and general cognitive ability. We did not report the specific connection between the inversion problems and the children's mathematics achievement in the original paper, so we report it here. We used a fixed-order regression analysis so that the connection between the inversion task and mathematics achievement could be considered after controlling for the children's age, general cognitive ability and working memory. The inversion task remained a significant predictor of the children's mathematics achievement, and explained 12% extra variance. This is a really remarkable result: 6 inversion problems given about 14 months before the mathematics achievement test made a significant contribution to predicting children's achievement after such stringent controls.

Our study also included an intervention component. We identified children who were underperforming in the logical assessment for their age at the beginning of their first year in school and created a control and an intervention group. The control children received no intervention and the intervention group received instruction on the reasoning principles for one hour a week for 12 weeks during the time their peers were participating in mathematics lessons. So they had no extra time on maths but specialised instruction on reasoning principles. We then compared their performance in the state-designed mathematics achievement tests with that of the control group. The intervention group significantly outperformed the control group. The mean for the control group in the mathematics assessment was at the 28th percentile using English norms; the intervention group's mean was just above the 50th percentile, i.e. above the mean. So a group of children who seemed at risk for difficulties with mathematics caught up through this intervention. In the intervention study it is not possible to separate out the effect of inversion; the children received instruction on three reasoning principles that we considered of great importance as a basis for their learning. It would be possible to carry out separate studies of how each of the three reasoning principles that we taught the children affects their mathematics performance but we did not consider this a desirable approach, as our view is that each one of them is central to children's mathematics learning.

The combination of longitudinal and intervention methods in the analysis of the causes of success and difficulties in learning to read is an approach that was extremely successful (Bradley and Bryant, 1983). The study by Nunes *et al.*, (2007) shows that this combination of methods can also be used successfully in the analysis of how children learn mathematics. However, three caveats are called for here. First, the study involved relatively small samples: a replication with a larger sample is highly recommended. Second, it is our view that it is also necessary to attempt to replicate the results of the intervention in studies carried out in the classroom. Experimental studies, such as ours, provide a proof of existence: they show that it is possible to accomplish something under controlled conditions. But they do not show that it is possible to accomplish the same results in the classroom. The step from the laboratory to the classroom must be carefully considered (see Nunes and Bryant, 2006, for a discussion of this issue). Finally, it is clear to us that developmental processes that describe children's development when they do not have any special educational needs (they do not have brain deficiencies, for example, and have hearing and sight within levels that grant them access to information normally accessed by children) may need further analysis when we want to understand the development of children who do have special educational needs. We exemplify here briefly the situation of children with severe or profound hearing loss. The vast majority of deaf children are born to hearing parents (about 90%), who may not know how to communicate with their children without much additional learning. Mathematics learning involves logical reasoning, as we have argued, and also involves learning conventional representations for numbers. Knowledge of numbers can be used to accelerate and promote children's reflections about their schemes of action, and this takes place through social interaction. Parents teach children a lot about counting before they go to school (Schaeffer, Eggleston and Scott, 1974; Young-Loveridge, 1989) but the opportunities for these informal learning experiences may be restricted for deaf children. They would enter school with less knowledge of counting and less understanding of the relations between addition, subtraction, and number. This does not mean that they have to develop their understanding of numbers in a different way from hearing children, but it does mean that they may need to learn in a much more carefully planned environment so that their learning opportunities are increased and appropriate for their visual and language skills. In brief, there may be special children whose mathematical development requires special attention. Understanding their development may or may not shed light on a more general theory of mathematics learning.

Summary

1 The inversion principle is an essential part of additive reasoning: one cannot understand either addition or subtraction unless one also understands their relation to each other.

2 Children probably first recognise the inverse relation between adding and subtracting the identical stuff. We call this the inversion of identity.

3 The understanding of the inversion of quantity is a step-up. It means understanding that a quantity stays the same if the same number of items is added to it and subtracted from it, even though the added and subtracted items are different from each other.

4 The inversion of quantity is more difficult for young children to understand, but in tasks that involve concrete objects, many children in the five- to seven-year age range do grasp this form of inversion to some extent.

5 There are however strong individual differences among children in this form of understanding. Children in the five- to eight-year range fall into three main groups. Those who are good a calculating and also good at using the inversion principle, those who are weak in both things, and those who are good at using the inversion principle, but weak in calculating.

6 The evidence suggests that children's understanding of the inversion principle plays an important causal role in their progress in learning about mathematics. Children's understanding of inversion is a good predictor of their mathematical success, and improving this understanding has the result of improving children's mathematical knowledge in general.

Additive reasoning and problem solving

In this section we continue to analyse children's ability to solve additive reasoning problems. Additive reasoning refers to reasoning used to solve problems where addition or subtraction are the operations used to find a solutions. We prefer to use this expression, rather than addition and subtraction problems, because it is often possible to solve the same problem either by addition or by subtraction. For example, if you buy something that costs £35, you may pay with two £20-notes. You can calculate your change by subtraction (40 – 35) or by addition (35 + 5). So, problems are not addition or subtraction problems in themselves, but they can be defined by the type of reasoning that they require, additive reasoning.

Although preschoolers' knowledge of addition and subtraction is limited, as we argued in the previous section, it is clear that their initial thinking about these two arithmetical operations is rooted in their everyday experiences of seeing quantities being combined with, or taken away from, other quantities. They find purely numerical problems like 'what is 2 and 1 more?' a great deal more difficult than problems that involve concrete situations, even when these situations are described in words and left entirely to the imagination (Ginsburg, 1977; Hughes, 1981; 1986; Levine, Jordan and Huttenlocher, 1992).

The type of knowledge that children develop initially seems to be related to two types of action: putting more elements in a set (or joining two sets) and taking out elements from one set (or separating two sets). These schemes of action are used by children to solve arithmetic problems when they are presented in the context of stories.

By and large, three main kinds of story problem have been used to investigate children's additive reasoning:

• the *Change problem* ('Bill had eight apples and then he gave three of them away. How many did he have left?').

• the *Combine problem* ('Jane has three dolls and Mary has four. How many do they have altogether?').

• the *Compare problem* ('Sam has five books and Sarah has eight. How many more books does Sarah have than Sam?').

A great deal of research (e.g. Brown, 1981; Carpenter, Hiebert and Moser, 1981; Carpenter and Moser, 1982; De Corte and Verschaffel, 1987; Kintsch and Greeno, 1985; Fayol, 1992; Ginsburg, 1977; Riley, Greeno and Heller, 1983; Vergnaud, 1982) has shown that in general, the Change and Combine problems are much easier than the Compare problems. The most interesting aspect of this consistent pattern of results is that problems that are solved by the same arithmetic operation – or in other words, by the same sum – can differ radically in how difficult they are.

Usually pre-school children do make the appropriate moves in the easiest Change and Combine problems: they put together and count up (counting on or counting all) and separate and count the relevant set to find the answer. Very few pre-school children seem to know addition and subtraction facts, and so they succeed considerably more if they have physical objects (or use their fingers) in order to count. Research by Carpenter and Moser (1982) gives an indication of how pre-school children perform in the simpler problems. These researchers interviewed children (aged about four to five years) twice before they had had been given any instruction about arithmetic in school; we give here the results of each of these interviews, as there is always some improvement worth noting between the testing occasions.

For Combine problems (given two parts, find the whole), 75% and 82% of the answers were correct when the numbers were small and 50% and 71% when the numbers were larger; only 13% of the responses with small numbers were obtained through the recall of number facts and this was the largest percentage of recall of number facts observed in their study. For Change problems (Tim had 11 candies; he gave 7 to Martha; how many did he have left?), the pre-schoolers were correct 42% and 61% with larger numbers (Carpenter and Moser do not report the figures for smaller numbers) at each of the two interviews; only 1% of recall of number facts is reported. So, pre-school children can do relatively well on simple Change and Combine problems before they know arithmetic facts; they do so by putting sets together or by separating them and counting.

This classification of problems into three types – Combine, Change and Compare – is not sufficient to describe story problem-solving. In a Change problem, for example, the story might provide the information about the initial state and the change (Tim had 11 candies; he gave 7 to Martha); the child is asked to say what the final state is. But it is also possible to provide information, for example, about the transformation and the end state (Tim had some candies; he gave Martha 7 and he has 4 left) and ask the child to say what the initial state was (how many did he have before he gave candies to Martha?). This sort of analysis has resulted in more complex classifications, which consider which information is given and which information must be supplied by the children in the answer. Stories that describe a situation where the *quantity decreases*, as in the example above, but have a missing initial state can most easily be solved by an *addition*. The conflict between the decrease in quantity and the operation of addition can be solved if the children understand the inverse relation between subtraction and addition: by adding the number that Tim still has and the number he gave away, one can find out how many candies he had before.

Different analyses of word problems have been proposed (e.g. Briars and Larkin, 1984; Carpenter and Moser, 1982; Fuson, 1992; Nesher, 1982; Riley, Greeno and Heller, 1983; Vergnaud, 1982). We focus here on some aspects of the analysis provided by Gérard Vergnaud, which allows for the comparison of many different types of problems and can also be used to help understand the level of difficulty of further types of additive reasoning problems, involving directed numbers (i.e. positive and negative numbers).

First, Vergnaud distinguishes between *numerical* and *relational* calculation. *Numerical calculation* refers to the arithmetic operations that the children carry out to find the answer to a problem: in the case of additive reasoning, addition and subtraction are the relevant operations. *Relational calculation* refers to the operations of thought that the child must carry out in order to handle the relations involved in the problem. For example, in the problem 'Bertrand played a game of marbles and lost 7 marbles. After the game he had 3 marbles left. How many marbles did he have before the game?', the relational calculation is the realisation that the solution requires using the inverse of subtraction to go from the final state to the initial state and the numerical calculation would be 7 + 3.

Vergnaud proposes that children perform these relational calculations in an implicit manner: to use his expression, they rely on 'theorems in action'. The children may say that they 'just know' that they have to add when they solve the problem, and may be unable to say that the reason for this is that addition is the inverse of subtraction. Vergnaud reports approximately twice as many correct responses by French pre-school children (aged about five years) to a problem that involves no relational calculation (about 50% correct in the problem: Pierre had 6 marbles. He played a game and lost 4; how many did he have after the game?) than to the problem above (about 26% correct responses), where we are told how many marbles Bertrand lost and asked how many he had before the game.

Vergnaud also distinguished three types of meanings that can be represented by natural numbers: quantities (which he calls measures), transformations and relations. This distinction has an effect on the types of problems that can be created starting from the simple classification in three types (change, combine and compare) and their level of difficulty.

First, consider the two problems below, the first about combining a quantity and a transformation and the second about combining two transformations.

- Pierre had 6 marbles. He played one game and lost 4 marbles. How many marbles did he have after the game?
- Paul played two games of marbles. He won 6 in the first game and lost 4 in the second game. What happened, counting the two games together?

French children, who were between pre-school and their fourth year in school, consistently performed better on the first than on the second type of problem, even though the same arithmetic calculation (6-4) is required in both problems. By the second year in school, when the children are about seven years old, they achieve about 80% correct responses in the first problem, and they only achieve a comparable level of success two years later in the second problem. So, combining transformations is more difficult than combining a quantity and a transformation.

Brown (1981) confirmed these results with English students in the age range 11 to 16. In her task, students are shown a sign-post that indicates that Grange is 29 miles to the west and Barton is 58 miles to the east; they are asked how do they work out how far they need to drive to go from Grange to Barton. There were eight choices of operations connecting these two numbers for the students to indicate the correct one. The rate of correct responses to this problem was 73%, which contrasts with 95% correct responses when the problem referred to a union of sets (a combine problem).

The children found problems even more difficult when they needed to de-combine transformations than when they had to combine them. Here is an example of a problem with which they needed to de-combine two transformations, because the story provides the result of combining operations and the question that must be answered is about the state of affairs before the combination took place.

- Bruno played two games of marbles. He played the first and the second game. In the second game he lost 7 marbles. His final result, with the two games together, was that he had won 3 marbles. What happened in the first game?

This de-combination of transformations was still very difficult for French children in the fourth year in school (age about nine years): they attained less than 50% correct responses.

Vergnaud's hypothesis is that when children combine transformations, rather than quantities, they have to go beyond natural numbers: they are now operating in the domain of whole numbers. Natural numbers are counting numbers. You can certainly count the number of marbles that Pierre had before he started the game, count and take away the marbles that he lost in the second game, and say how many he had left at the end. In the case of Paul's problem, if you count the marbles that he won in the first game, you need to count them as 'one more, two more, three more etc.': you are actually not counting marbles but the relation between the number that he now has to the number he had to begin with. So if the starting point in a problem that involves transformations is not known, the transformations are now relations. Of course, children who do solve the problem about Paul's marbles may not be fully aware of the difference between a transformation and a relation, and may succeed exactly because they overlook this difference. This point is discussed in Paper 4, when we consider in detail how children think about relations.

Finally, problems where children are asked to quantify *relations* are usually difficult as well:

- Peter has 8 marbles. John has 3 marbles. How many more marbles does Peter have than John?

The question in this problem is neither about a quantity (i.e. John's or Peter's marbles) nor about a transformation (no-one lost or got more marbles): it is about the relation between the two quantities. Although most pre-school children can say that Peter has more marbles, the majority cannot quantify the relation (or the difference) between the two. The best known experiments that demonstrate this difficulty were carried out by Hudson (1983) in the United States. In a series of three experiments, he showed the children some pictures and asked them two types of question:

- Here are some birds and some worms. How many more birds than worms?

- Here are some birds and some worms. The birds are racing to get a worm. How many birds won't get worms?

The first question asks the children to quantify the relation between the two sets, of worms and birds; the second question asks the children to imagine that the sets were matched and quantify the set that has no matching elements. The children in the first year of school (mean age seven years) attained 64% correct responses to the first question and 100% to the second question; in nursery school (mean age four years nine months) and kindergarten (mean age 6 years 3 months), the rate of correct responses was, respectively, 17% and 25% to the first question and 83% and 96% to the second question.

It is, of course, difficult to be completely certain that the second question is easier because the children are asked a question about quantity whereas the first question is about a relation. The reason for this ambiguity is that two things have to change at the same time for the story to be different: in order to change the target of the question, so that it is either a quantity or a relation, the language used in the problem also varies: in the first problem, the word 'more' is used, and in the second it is not.

Hudson included in one of his experiments a pre-test of children's understanding of the word 'more' (e.g. Are there more red chips or more white chips?') and found that they could answer this question appropriately. He concluded that it was the linguistic difficulty of the 'How many more…?' question that made the problem difficult, not simply the difficulty of the word 'more'. We are not convinced by his conclusion and think that more research about children's understanding of how to quantify relations is required. Stern (2005), on the other hand, suggests that both explanations are relevant: the linguistic form is more difficult and quantifying relations is also more difficult than using numbers to describe quantities.

In the domain of directed numbers (i.e. positive and negative numbers), it is relatively easier to study the difference between attributing numbers to quantities and to relations without asking the 'how many more' question. Unfortunately, studies with larger sample, which would allow for a quantitative comparison in the level of difficulty of these problems, are scarce. However, some indication that quantifying relations is more difficult for students is available in the literature.

Vergnaud (1982) pointed out that relationships between people could be used to create problems that do not contain the question 'how many more'. Among others, he suggested the following example.

- Peter owes 8 marbles to Henry but Henry owes 6 marbles to Peter. What do they have to do to get even?

According to his analysis, this problem involves a composition of relations.

Marthe (1979) compared the performance of French students in the age range 11 to 15 years in two problems involving such composition of relations with their performance in two problems involving a change situation (i.e. quantity, transformation, quantity). In order to control for problem format, all four problems had the structure $a + x = b$, in which x shows the place of the unknown. The problems used large numbers so that students had to go through the relational calculation in order to determine the numerical calculation (with small numbers, it is possible to work in an intuitive manner, sometimes starting from a hypothetical amount and adjusting the starting point later to make it fit). An example of a problem type using a composition of relations is shown below.

- Mr Dupont owes 684 francs to Mr. Henry. But Mr Henry also owes money to Mr Dupont. Taking everything into account, Mr Dupont must give back 327 francs to Mr Henry. What amount did Mr Henry owe to Mr Dupont?

Marthe did find that problems about relations were quite a bit more difficult than those about quantities and transformations; there was a difference of 20% between the rates of correct responses for the younger children and 10% for the older children. However, the most important effect in these problems seemed to be whether the students had to deal with numbers that had the same or different signs: problems with same signs were consistently easier than those with different signs.

In summary, different researchers have argued that it is one thing to learn to use numbers to represent quantities and a quite different one to use numbers to quantify relations. Relations are more abstract and more challenging for students. Thompson (1993) hypothesises that learning to quantify and think about numbers as measures of relations is a crucial step that students must take in order to understand algebra. We are completely sympathetic to this hypothesis, but we think that the available evidence is a bit thin.

More than two decades ago, Dickson, Brown and Gibson (1984) reviewed research on additive reasoning and problem solving, and pointed out how difficult it is to come to firm conclusions when no single study has covered the variety of problems that any theoretical model would aim to compare. We have to piece the evidence together from diverse studies, and of course samples vary across different locations and cohorts. In the last decade research on additive reasoning has received less attention in research on mathematics education than before. Unfortunately, this has left some questions with answers that are, at best, based on single studies with limited numbers of students. It is time to use a new synthesis to re-visit these questions and seek for unambiguous answers within a single research programme.

Summary

1 In word problems children are told a brief story which ends in an arithmetical question. These problems are widely used in school textbooks and also as a research tool.

2 There are three main kinds of word problem: Combine, Change and Compare.

3 Vergnaud argued that the crucial elements in these problems were Quantities (measures), Transformations and Relations. On the whole, problems that involve Relations are harder than those involving Transformations.

4 However, other factors also affect the level of difficulty in word problems. Any change in sign is often hard for children to handle: when the story is about an addition but the solution is to subtract, as in missing addend problems, children often fail to use the inverse operation.

5 Overall the extreme variability in the level of difficulty of different problems, even when these demand exactly the same mathematical solution (the same simple additions or subtractions) confirms the view that there is a great deal more to arithmetical learning than knowing how to carry out particular operations.

6 Research on word problems supports a different approach, which is that arithmetical learning depends on children making a coherent connection between quantitative relations and the appropriate numerical analysis.

Overall conclusions and educational implications

- Learning about quantities and numbers are two different matters: children can understand relations between quantities and not know how to make inferences about the numbers that are used to represent the quantities; they might also learn to count without making a connection between counting and what it implies for the relations between quantities.

- Some ideas about quantities are essential for understanding number: equivalence between quantities, their order of magnitude, and the part-whole relations implicit in determining the number of elements in a set.

- These core ideas, in turn, require that children come to understand yet other logical principles: transitive relations in equivalence and order, which operations change quantities and which do not, and the inverse relation between addition and subtraction. These notions are central to understanding numbers and how they represent quantities; children who have a good grasp of them learn mathematics better in school. Children who have difficulties with these ideas and do not receive support to come to grips with them are at risk for difficulties in learning mathematics, but these difficulties can be prevented to a large extent if they receive appropriate instruction.

- There is no question that word problems give us a valuable insight into children's reasoning about addition and subtraction. They demonstrate that there is a great deal more to understanding these operations than just learning how to add and subtract. Children's solutions do depend on their ability to reason about the relations between quantities in a logical manner. There is no doubt about these conclusions, even if there is need for further research to pin down some of the details.

- Learning to count and to use numbers to represent quantities is an important element in this developmental process. Children can more easily reason about the relation between addition, subtraction, and number when they know how to represent quantities by counting. But this is not a one-way relation: it is by adding, subtracting, and understanding the inverse relation between these operations that children understand additive composition and learn to solve additive reasoning problems.

- The major implication from this review is that schools should take very seriously the need to include in the curriculum instruction that promotes reflection about relations between quantities, operations, and the quantification of relations.

- These reflections should not be seen as appropriate only for very young children: when natural numbers start to be used to represent relations, directed numbers become a new domain of activity for children to re-construct their understanding of additive relations. The construction of a solid understanding of additive relations is not completed in the first years of primary school: some problems are still difficult for students at the age of 15.

Endnotes

1 Gelman and Butterworth (2005) make a similar distinction between numerosity and the representation of number: 'we need to distinguish possession of the concept of numerosity itself (knowing that any set has a numerosity that can be determined by enumeration) from the possession of rerepresentations (in language) of particular numerosities' (pp. 6). However, we adopt here the term 'quantities' because it has an established definition and use in the context of children's learning of mathematics.

2 It is noted here that evidence from cases studies of acquired dyscalculia (a cognitive disorder affecting the ability to solve mathematics problems observed in patients after neurological damage) is consistent with the idea that understanding quantities and number knowledge can be dissociated: calculation may be impaired and conservation of quantities may be intact in some patients whereas in others calculation is intact and conceptual knowledge impaired (Mittmair-Delazer, Sailer and Benke, 1995). Dissociations between arithmetic skills and the meaning of numbers were extensively described by McCloskey (1992) in a detailed review of cases of acquired dyscalculia.

References

Alibali, M. W., & DiRusso, A. A. (1999). The function of gesture in learning to count: More than keeping track. *Cognitive Development, 14*, 37-56.

Baroody, A. J., Ginsburg, H.P., & Waxman, B. (1983). Children's use of mathematical structure. *Journal for Research in Mathematics Education, 14*, 156-168.

Baroody, A. J., & Tiilikainen, S. H. (2003). Two perspectives on addition development. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills* (pp. 75-126). Erlbaum: Mahwah, New Jersey.

Bermejo, V., Morales, S., & deOsuna, J. G. (2004). Supporting children's development of cardinality understanding. *Learning and Instruction, 14* 381-398.

Bisanz, J., & Lefevre, J.-A. (1990). Strategic and non-strategic processing in the development of mathematical cognition. In D. J. Bjorklund (Ed.), *Children's strategies: Contemporary views of cognitive development* (pp. 213-244).

Blevins-Knabe, B., Cooper, R. G., Mace, P. G., & Starkey, P. (1987). Preschoolers sometimes know less than we think: The use of quantifiers to solve addition and subtraction tasks. *Bulletin of the Psychometric Society, 25*, 31-34.

Bradley, L., & Bryant, P. E. (1983). Categorising sounds and learning to read -a causal connection. *Nature, 301*, 419-521.

Brannon, E. M. (2002). The development of ordinal numerical knowledge in infancy. *Cognition, 83*, 223-240.

Briars, D. J., & Larkin, J. H. (1984). An integrated model of skills in solving elementary arithmetic word problems. *Cognition and Instruction, 1*, 245-296.

Brown, M. (1981). Number operations. In K. Hart (Ed.), *Children's Understanding of Mathematics: 11-16* (pp. 23-47). Windsor, UK: NFER-Nelson.

Brush, L. R. (1978). Preschool children's knowledge of addition and subtraction. *Journal for Research in Mathematics Education, 9*, 44-54.

Bryant, P. E., & Kopytynska, H. (1976). Spontaneous measurement by young children. *Nature, 260*, 773.

Bryant, P. E., & Trabasso, T. (1971). Transitive inferences and memory in young children. *Nature, 232*, 456-458.

Bryant, P., Christie, C., & Rendu, A. (1999). Children's understanding of the relation between addition and subtraction: Inversion, identity and decomposition. *Journal of Experimental Child Psychology, 74*, 194-212.

Butterworth, B., Cipolotti, L., & Warrington, E. K. (1996). Short-term memory impairments and arithmetical ability. *Quarterly Journal of Experimental Psychology, 49A*, 251-262.

Carey, S. (2004). Bootstrapping and the origin of concepts. *Daedalus, 133*(1), 59-69.

Carpenter, T. P., & Moser, J. M. (1982). The development of addition and subtraction problem solving. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 10-24). Hillsdale (NJ): Lawrence Erlbaum.

Carpenter, T. P., Hiebert, J., & Moser, J. M. (1981). Problem structure and first grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education, 12*, 27-39.

Cooper, R. G. (1984). Early number development: discovering number space with addition and subtraction. In C. Sophian (Ed.), *The origins of cognitive skill* (pp. 157-192). Hillsdale, NJ: Lawrence Erlbaum Ass.

Cowan, R. (1987). When do children trust counting as a basis for relative number judgements? *Journal of Experimental Child Psychology, 43*, 328-345.

Cowan, R., & Daniels, H. (1989). Children's use of counting and guidelines in judging relative number. *British Journal of Educational Psychology, 59*, 200-210.

De Corte, E., & Verschaffel, L. (1987). The effect of semantic structure on first graders' solution strategies of elementary addition and subtraction word problems. *Journal for Research in Mathematics Education, 18*, 363-381.
Dehaene, S. (1992). Varieties of numerical abilities. *Cognition, 44* 1-42.
Dehaene, S. (1997). *The Number Sense*. London: Penguin.
Dickson, L., Brown, M., & Gibson, O. (1984). *Children learning mathematics: A teacher's guide to recent research*. Oxford: The Alden Press.
Donaldson, M. (1978). *Children's minds*. London: Fontana.
Fayol, M. (1992). From number to numbers in use: solving arithmetic problems. In J. Bideaud, C. Meljac & J.-P. Fischer (Eds.), *Pathways to Number* (pp. 209-218). Hillsdale, NJ: LEA.
Freeman, N. H., Antonuccia, C., & Lewis, C. (2000). Representation of the cardinality principle: early conception of error in a counterfactual test. *Cognition, 74*, 71-89.
Frydman, O., & Bryant, P. E. (1988). Sharing and the understanding of number equivalence by young children. *Cognitive Development, 3*, 323-339.
Fuson, K. (1992). Research on whole number addition and subtraction. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and learning* (pp. 243-275). New York: Macmilla Publishing Company.
Fuson, K. C. (1988). *Children's Counting and Concepts of Number*. New York: Springer Verlag.
Fuson, K. C., Richards, J., & Briars, D. J. (1982). The acquisition and elaboration of the number word sequence. In C. J. Brainerd (Ed.), *Children's logical and mathematical cognition* (pp. 33-92). New York: Springer Verlag.
Fuson, K., & Hall, J. W. (1983). The Acquisition of Early Number Word Meanings: A Conceptual Analysis and Review. In H. P. Ginsburg (Ed.), *The Development of Mathematical Thinking* (pp. 50-109). New York: Academic Press.
Gallistel, C. R., & Gelman, R. (1992). Preverbal and verbal counting and computation. *Cognition, 44*, 43-74.
Gelman, R., & Butterworth, B. (2005). Number and language: how are they related? *Trends in Cognitive Sciences, 9*, 6-10.
Gelman, R., & Gallistel, C. R. (1978). *The child's understanding of number*. Cambridge, Mass: Harvard University Press.
Gelman, R., & Meck, E. (1983). Preschoolers' counting: Principles before skill. *Cognition, 13*, 343-359.
Gilmore, C., & Bryant, P. (2006). Individual differences in children's understanding of inversion and arithmetical skill. *British Journal of Educational Psychology, 76*, 309-331.
Gilmore, C., & Papadatou-Pastou, M. (2009). Patterns of Individual Differences in Conceptual Understanding and Arithmetical Skill: A Meta-Analysis. In special issue on "Young Children's Understanding and Application of the Addition-Subtraction inverse Principle". *Mathematical Thinking and Learning, in press*.
Ginsburg, H. (1977). *Children's Arithmetic: The Learning Process*. New York: Van Nostrand.
Ginsburg, H. P., Klein, A., & Starkey, P. (1998). The Development of Children's Mathematical Thinking: Connecting Research with Practice. In W. Damon, I. E. Siegel & A. A. Renninger (Eds.), *Handbook of Child Psychology. Child Psychology in Practice* (Vol. 4, pp. 401-476). New York: John Wiley & Sons.
Gréco, P. (1962). Quantité et quotité: nouvelles recherches sur la correspondance terme-a-terme et la conservation des ensembles. In P. Gréco & A. Morf (Eds.), *Structures numeriques elementaires: Etudes d'Epistemologie Genetique Vol 13* (pp. 35-52). Paris.: Presses Universitaires de France.
Hudson, T. (1983). Correspondences and numerical differences between sets. *Child Development, 54*, 84-90.
Hughes, M. (1981). Can preschool children add and subtract? *Educational Psychology, 3*, 207-219.
Hughes, M. (1986). *Children and number*. Oxford: Blackwell.
Inhelder, B., Sinclair, H., & Bovet, M. (1974). *Learning and the development of cognition*. London: Routledge and Kegan Paul.
Jordan, N., Huttenlocher, J., & Levine, S. (1992). Differential calculation abilities in young children from middle- and low-income families. *Developmental Psychology, 28*, 644-653.
Kintsch, W., & Greeno, J. G. (1985). Understanding and solving word arithmetic problems. *Psychological Review, 92*, 109-129.
Klein, A. (1984). The early development of arithmetic reasoning: Numerative activities and logical operations. *Dissertation Abstracts International, 45*, 131-153.
Krebs, G., Squire, S., & Bryant, P. (2003). Children's understanding of the additive composition of number and of the decimal structure: what is the relationship? *International Journal of Educational Research, 39*, 677-694.

Landerl, K., Bevana, A., & Butterworth, B. (2004). Developmental dyscalculia and basic numerical capacities: A study of 8-9 year old students. *Cognition 93*, 99-125.

Le Corre, M., & Carey, S. (2007). One, two, three, nothing more: an investigation of the conceptual sources of verbal number principles. *Cognition, 105*, 395-438.

Light, P. H., Buckingham, N., & Robbins, A. H. (1979). The conservation task as an interactional setting. *British Journal of Educational Psychology, 49*, 304-310.

Markman, E. M. (1979). Classes and collections: conceptual organisation and numerical abilities. *Cognitive Psychology, 11*, 395-411.

Marthe, P. (1979). *Additive problems and directed numbers.* Paper presented at the Proceedings of the Annual Meeting of the International Group for the Psychology of Mathematics Education, Warwick, UK.

McCrink, K., & Wynn, K. (2004). Large number addition and subtraction by 9-month-old infants. *Psychological Science, 15*, 776-781.

Michie, S. (1984). Why preschoolers are reluctant to count spontaneously. *British Journal of Developmental Psychology, 2*, 347-358.

Mittmair-Delazer, M., Sailer, U., & Benke, T. (1995). Impaired arithmetic facts but intact conceptual knowledge - A single-case study of dyscalculia. *Cortex, 31*, 139-147.

Nesher, P. (1982). Levels of description in the analysis of addition and subtraction word problems. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), *Addition and subtraction*. Hillsdale,NJ: Lawrence Erlbaum Ass.

Nunes, T., & Bryant, P. (2006). *Improving Literacy through Teaching Morphemes*: Routledge.

Nunes, T., & Schliemann, A. D. (1990). Knowledge of the numeration system among pre-schoolers. In L. S. T. Wood (Ed.), *Transforming early childhood education: International perspectives* (pp. pp.135-141). Hillsdale, NJ: Lawrence Erlbaum.

Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. N. (2007). The Contribution of Logical Reasoning to the Learning of Mathematics in Primary School. *British Journal of Developmental Psychology, 25*, 147-166.

Nunes, T., Bryant, P., Hallett, D., Bell, D., & Evans, D. (2008). Teaching Children about the Inverse Relation between Addition and Subtraction. *Mathematical Thinking and Learning, in press*.

Piaget, J. (1921). Une forme verbale de la comparaison chez l'enfant. *Archives de Psychologie, 18*, 141-172.

Piaget, J. (1952). *The Child's Conception of Number*. London: Routledge & Kegan Paul.

Piaget, J., & Moreau, A. (2001). The inversion of arithmetic operations (R. L. Campbell, Trans.). In J. Piaget (Ed.), *Studies in Reflecting Abstraction* (pp. 69-86). Hove: Psychology Press.

Putnam, R., deBettencourt, L. U., & Leinhardt, G. (1990). Understanding of derived fact strategies in addition and subtraction. *Cognition and Instruction, 7*, 245-285.

Rasmussen, C., Ho, E., & Bisanz, J. (2003). Use of the mathematical principle of inversion in young children. *Journal of Experimental Child Psychology, 85*, 89-102.

Resnick, L., & Ford, W. W. (1981). *The Psychology of Mathematics for Instruction*. Hillsdale, NJ: Lawrence Erlbaum Ass.

Resnick, L. B. (1983). A Developmental theory of Number Understanding. In H. P. Ginsburg (Ed.), *The Development of Mathematical Thinking* (pp. 110-152). New York: Academic Press.

Riley, M., Greeno, J. G., & Heller, J. I. (1983). Development of children's problem solving ability in arithmetic. In H. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153-196). New York: Academic Press.

Rips, L. J., Asmuth, J., & Bloomfield, A. (2006). Giving the boot to the bootstrap: How not to learn the natural numbers. *Cognition, 101*, B51-B60.

Rips, L. J., Asmuth, J., & Bloomfield, A. (2008). Discussion. Do children learn the integers by induction? *Cognition, 106* 940-951.

Samuel, J., & Bryant, P. E. (1984). Asking only one question in the conservation experiment. *Journal of Child Psychology and Psychiatry, 25*, 315-318.

Sarnecka, B. W., & Gelman, S. A. (2004). Six does not just mean a lot: preschoolers see number words as specific. *Cognition, 92* 329-352.

Saxe, G., Guberman, S. R., & Gearhart, M. (1987). Social and developmental processes in children's understanding of number. *Monographs of the Society for Research in Child Development, 52*, 100-200.

Schaeffer, B., Eggleston, V. H., & Scott, J. L. (1974). Number development in young children. *Cognitive Psychology, 6*, 357-379.

Secada, W. G. (1992). Race, ethnicity, social class, language, and achievement in mathematics. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 623-660). New York: Macmillan.

Siegler, R. S., & Robinson, M. (1982). The development of numerical understanding. In H. W. Reese & L. P. Lipsitt (Eds.), *Advances in child development and behavior*. New York: Academic Press.

Siegler, R. S., & Stern, E. (1998). Conscious and unconscious strategy discoveries: a microgenetic analysis. *Journal of Experimental Psychology-General, 127*, 377-397.

Sophian, C. (1988). Limitations on preschool children's knowledge about counting: using counting to compare two sets. *Developmental Psychology, 24*, 634-640.

Sophian, C., Wood, A. M., & Vong, C. I. (1995). Making numbers count: the early development of numerical inferences. *Developmental Psychology, 31*, 263-273.

Spelke, E. S. (2000). Core knowledge. *American Psychologist, 55*, 1233-1243.

Starkey, P., & Gelman, R. (1982). The development of addition and subtraction abilities prior to formal schooling in arithmetic. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), *Addition and subtraction: a cognitive perspective* (pp. 99-115). Hillsdale,NJ: Elrbaum.

Steffe, L. P., & Thompson, P. W. (2000). *Radical constructivism in action : building on the pioneering work of Ernst von Glasersfeld*. New York Falmer.

Steffe, L. P., Cobb, P., & Glaserfeld, E. v. (1988). *Construction of arithmetical meanings and strategies*. London Springer-Verlag.

Steffe, L. P., Thompson, P. W., & Richards, J. (1982). Children's Counting in Arithmetical Problem Solving. In T. P. Carpenter, J. M. Moser & T. A. Romberg (Eds.), *Addition and Subtraction: A Cognitive Perspective* (pp. 83-96). Hillsdale, NJ: Lawrence Erlbaum Associates.

Steffe, L. P., von Glasersfeld, E., Richards, J., & Cobb, P. (1983). *Children's Counting Types: Philosophy, Theory and Application.* New York: Praeger.

Stern, E. (1992). Spontaneous used of conceptual mathematical knowledge in elementary school children. *Contemporary Educational Psychology, 17*, 266-277.

Stern, E. (2005). Pedagogy - Learning for Teaching. *British Journal of Educational Psychology, Monograph Series, II, 3*, 155-170.

Thompson, P. W. (1993). Quantitative Reasoning, Complexity, and Additive Structures. *Educational Studies in Mathematics, 3*, 165-208.

Trabasso, T. R. (1977). The role of memory as a system in making transitive inferences. In R. V. Kail & J. W. Hagen (Eds.), *Perspectives on the development of memory and cognition*. Hillsdale, NJ: Lawrence Erlbaum Ass.

Vergnaud, G. (1982). A classification of cognitive tasks and operations of thought involved in addition and subtraction problems. In T. P. Carpenter, J. M. Moser & R. T. A (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 60-67). Hillsdale (NJ): Lawrence Erlbaum.

Vergnaud, G. (2008). The theory of conceptual fields. *Human Development, in press*.

Wynn, K. (1992). Evidence against empiricist accounts of the origins of numerical knowledge. *Mind and Language, 7*, 315-332.

Wynn, K. (1998). Psychological foundations of number: numerical competence in human infants. *Trends in Cognitive Science, 2*, 296-303.

Xu, F., & Spelke, E. (2000). Large number discrimination in 6-month-old infants. *Cognition, 74*, B1-B11.

Young-Loveridge, J. M. (1989). The relationship between children's home experiences and their mathematical skills on entry to school. *Early Child Development and Care, 1989* (43), 43 - 59.

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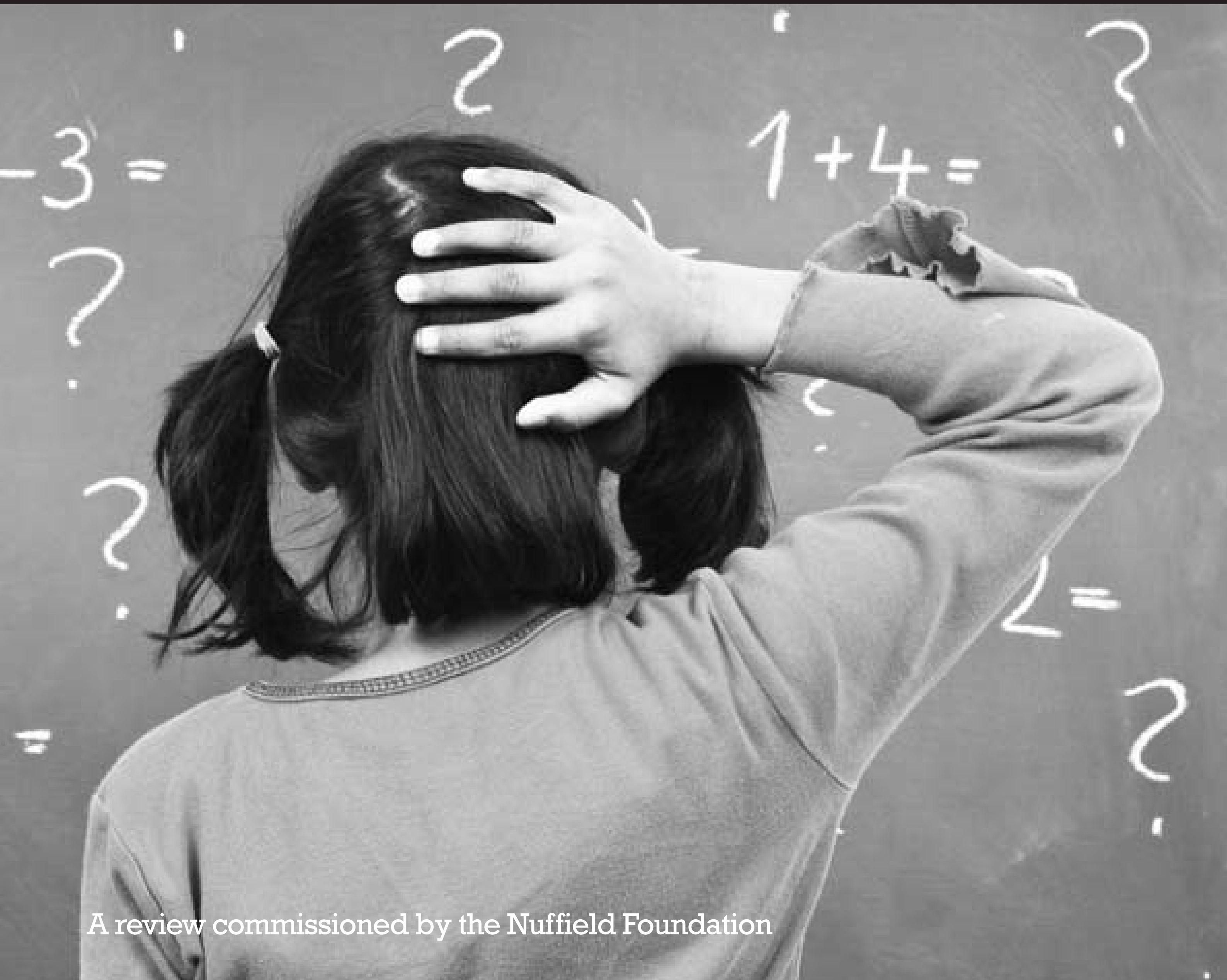
3

Key understandings in mathematics learning

Paper 3: Understanding rational numbers and intensive quantities

By Terezinha Nunes and Peter Bryant, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

Contents

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 3: Understanding rational numbers and intensive quantities

Headlines

- Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. As with whole numbers, children need to make connections between quantities and their representations in fractions in order to be able to use fractions meaningfully.

- There are two types of situation in which fractions are used in primary school. The first involves measurement: if you want to represent a quantity by means of a number and the quantity is smaller than the unit of measurement, you need a fraction – for example, a half cup or a quarter inch. The second involves division: if the dividend is smaller than the divisor, the result of the division is represented by a fraction. For example, when you share 3 cakes among 4 children, each child receives ¾ of a cake.

- Children use different schemes of action in these two different situations. In division situations, they use correspondences between the units in the numerator and the units in the denominator. In measurement situations, they use partitioning.

- Children are more successful in understanding equivalence of fractions and in ordering fractions by magnitude in situations that involve division than in measurement situations.

- It is crucial for children's understanding of fractions that they learn about fractions in both types of situation: most do not spontaneously transfer what they learned in one situation to the other.

- When a fraction is used to represent a quantity, children need to learn to think about how the numerator and the denominator relate to the value represented by the fraction. They must think about direct and inverse relations: the larger the numerator, the larger the quantity but the larger the denominator, the smaller the quantity.

- Like whole numbers, fractions can be used to represent quantities and relations between quantities, but in primary school they are rarely used to represent relations. Older students often find it difficult to use fractions to represent relations.

There is little doubt that students find fractions a challenge in mathematics. Teachers often say that it is difficult to teach fractions and some think that it would be better for everyone if children were not taught about fractions in primary school. In order to understand fractions as numbers, students must be able to know whether two fractions are equivalent or not, and if they are not, which one is the bigger number. This is similar to understanding that 8 sweets is the same number as 8 marbles and that 8 is more than 7 and less than 9, for example. These are undoubtedly key understandings about whole numbers and fractions. But even after the age of 11 many students have difficulty in knowing whether two fractions are equivalent and do not know how to order some fractions. For example, in a study carried out in London, students were asked to paint 2/3 of figures divided in 3, 6 and 9 equal parts. The majority solved the task correctly when the figure was divided into 3 parts but 40% of the 11- to 12-year-old students could not solve it when the figure was divided into 6 or 9 parts, which meant painting an equivalent fraction (4/6 and 6/9, respectively).

Fractions are used in primary school to represent quantities that cannot be represented by a single whole number. If the teaching of fractions were to be omitted from the primary school curriculum, children would not have the support of school learning to represent these quantities. We do not believe that it would be best to just forget about teaching fractions in primary school because research shows that children have some informal knowledge that could be used as a basis for learning fractions. Thus the question is not whether to teach fractions in primary school but what do we know about their informal knowledge and how can teachers draw on this knowledge.

There are two types of situation in which fractions are used in primary school: measurement and division situations.

When we measure anything, we use a unit of measurement. Often the object we are measuring cannot be described only with whole units, and we need fractions to represent a part of the unit. In the kitchen we might need to use a ½ cup of milk and when setting the margins for a page in a document we often need to be precise and define the margin as, for example, as 3.17 cm. These two examples show that, when it comes to measurement, we use two types of notation, ordinary and decimal notation. But regardless of the notation used, we could not accurately describe the quantities in these situations without using fractions. When we speak of ¾ of a chocolate bar, we are using fractions in a measurement situation: we have less than one unit, so we need to describe the quantity using a fraction.

In division situations, we need a fraction to represent a quantity when the dividend is smaller than the divisor. For example, if 3 cakes are shared among 4 children, it is not possible for each one to have a whole cake, but it is still possible to carry out the division and to represent the amount that each child receives using a number, ¾. It would be possible to use decimal notation in division situations too, but this is rarely the case. The reason for preferring ordinary fractions in these situations is that there are two quantities in division situations: in the example, the number of cakes and the number of children. An ordinary fraction represents each of these quantities by a whole number: the dividend is represented by the numerator, the divisor by the denominator, and the operation of division by the dash between the two numbers.

Although these situations are so similar for adults, we could conclude that it is not necessary to distinguish between them, however, research shows that children think about the situations differently. Children use different schemes of action in each of these situations.

In measurement situations, they use partitioning. If a child is asked to show ¾ of a chocolate, the child will try to cut the chocolate in 4 equal parts and mark 3 parts. If a child is asked to compare ¾ and 6/8, for example, the child will partition one unit in 4 parts, the other in 8 parts, and try to compare the two. This is a difficult task because the partitioning scheme develops over a long period of time and children have to solve many problems to succeed in obtaining equal parts when partitioning. Although partitioning and comparing the parts is not the only way to solve this problem, this is the most likely solution path tried out by children, because they draw on their relevant scheme of action.

In division situations, children use a different scheme of action, correspondences. A problem analogous to the one above in a division situation is: there are 4 children sharing 3 cakes and 8 children sharing 6 identical cakes; if the two groups share the cakes fairly, will the children in one group get the same amount to eat as the children in the other group? Primary school pupils often approach this problem by establishing correspondences between cakes and children. In this way they soon realise that in both groups 3 cakes will be shared by 4 children; the difference is that in the second group there are two lots of 3 cakes and two lots of 4 children, but this difference does not affect how much each child gets.

From the beginning of primary school, many children have some informal knowledge about division that could be used to understand fractional quantities. Between the ages of five and seven years, they are very bad at partitioning wholes into equal parts but can be relatively good at thinking about the consequences of sharing. For example, in one study in London 31% of the five-year-olds, 50% of the six-year-olds and 81% of the seven-year-olds understood the inverse relation between the divisor and the shares resulting from the division: they knew that the more recipients are sharing a cake, the less each one will receive. They were even able to articulate this inverse relation when asked to justify their answers. It is unlikely that they had at this time made a connection between their understanding of quantities and fractional representation; actually, it is unlikely that they would know how to represent the quantities using fractions.

The lack of connection between students' understanding of quantities in division situations and their knowledge about the magnitude of fractions is very clearly documented in research. Students who have no doubt that recipients of a cake shared between 3 people will fare better than those of a cake shared between 5 people may, nevertheless, say that 1/5 is a bigger fraction than 1/3 because 5 is a bigger number than 3. Although they understand the inverse relation in the magnitude of quantities in a division situation, they do not seem to connect this with the magnitude of fractions. The link between their understanding of fractional quantities and fractions as numbers has to be developed through teaching in school.

There is only one well-controlled experiment which compared directly young children's understanding of quantities in measurement and division situations. In this study, carried out in Portugal, the children were six- to seven-years-old. The context of the problems in both situations was very similar: it was about children eating cakes, chocolates or pizzas. In the measurement problems, there was no sharing, only partitioning. For example, in one of the measurement problems, one girl had a chocolate bar which was too large to eat in one go. So she cut her chocolate in 3 equal pieces and ate 1. A boy had an identical bar of chocolate and decided to cut his into 6 equal parts, and eat 2. The children were asked whether the boy and the girl ate the same amount of chocolate. The analogous division problem was about 3 girls sharing one chocolate bar and 6 boys sharing 2 identical chocolate bars. The rate of correct responses in the partitioning situation was 10% for both six- and seven-year-olds and 35% and 49%, respectively, for six- and seven-year-olds in the division situation.

These results are relevant to the assessment of variations in mathematics curricula. Different countries use different approaches in the initial teaching of fraction, some starting from division and others from measurement situations. There is no direct evidence from classroom studies to show whether one starting point results in higher achievement in fractions than the other. The scarce evidence from controlled studies supports the idea that division situations provide children with more insight into the equivalence and order of quantities represented by fractions and that they can learn how to connect these insights about quantities with fractional representation. The studies also indicate that there is little transfer across situations: children who succeed in comparing fractional quantities and fractions after instruction in division situations do no better in a post-test when the questions are about measurement situations than other children in a control group who received no teaching. The converse is also true: children taught in measurement situations do no better than a control group in division situations.

A major debate in mathematics teaching is the relative weight to be given to conceptual understanding and procedural knowledge in teaching. The difference between conceptual understanding and procedural knowledge in the teaching of fractions has been explored in many studies. These studies show that students can learn procedures without understanding their conceptual significance. Studies with adults show that knowledge of procedures can remain isolated from understanding for a long time: some adults who are able to implement the procedure they learned for dividing one fraction by another admit that they have no idea why the numerator and the denominator exchange places in this procedure. Learners who are able to co-ordinate their knowledge of procedures with their conceptual understanding are better at solving problems that involve fractions than other learners who seem to be good at procedures but show less understanding than expected from their knowledge of procedures. These results reinforce the idea that it is very important to try to make links between children's knowledge of fractions and their understanding of fractional quantities.

Finally, there is little, if any, use of fractions to represent relations between quantities in primary school. Secondary school students do not easily quantify relations that involve fractions. Perhaps this difficulty could be attenuated if some teaching about fractions in primary school involved quantifying relations that cannot be described by a single whole number.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's knowledge of fractional quantities starts to develop before they are taught about fractions in school.	**Teaching** Teachers should be aware of children's insights regarding quantities that are represented by fractions and make connections between their understanding of these quantities and fractions.
There are two types of situation relevant to primary school teaching in which quantities cannot be represented by a single whole number: measurement and division.	**Teaching** The primary school curriculum should include the study of both types of situation in the teaching of fr actions. Teachers should be aware of the different types of reasoning used by children in each of these situations. **Research** Evidence from experimental studies with larger samples and long-term interventions in the classroom are needed to establish whether division situations are indeed a better starting point for teaching fractions.
Children do not easily transfer their understanding of fractions from division to measurement situations and *vice-versa*.	**Teaching** Teachers should consider how to establish links between children's understanding of fractions in division and measurement situations. **Research** Investigations on how links between situations can be built are needed to support curriculum development and classroom teaching.
Many students do no make links between their conceptual understanding of fractions and the procedures that they are taught to compare and operate on fractions in school.	**Teaching** Greater attention may be required in the teaching of fractions to creating links between procedures and conceptual understanding. **Research** There is a need for longitudinal studies designed to clarify whether this separation between procedural and conceptual knowledge does have important consequences for further mathematics learning.
Fractions are taught in primary school only as representations of quantities.	**Teaching** Consideration should be given to the inclusion of situations in which fractions are used to represent relations. **Research** Given the importance of understanding and representing relations numerically, studies that investigate under what circumstances primary school students can use fractions to represent relations between quantities are urgently needed.

Understanding rational numbers and intensive quantities

Introduction

Rational numbers, like natural numbers, can be used to represent quantities. There are some quantities that cannot be represented by a natural number, and to represent these quantities, we must use rational numbers. We cannot use natural numbers when the quantity that we want to represent numerically:

- is smaller than the unit used for counting, irrespective of whether this is a natural unit (e.g. we have less than one banana) or a conventional unit (e.g. a fish weighs less than a kilo)
- involves a ratio between two other quantities (e.g. the concentration of orange juice in a jar can be described by the ratio of orange concentrate to water; the probability of an event can be described by the ratio between the number of favourable cases to the total number of cases).[1]

The term 'fraction' is often identified with situations where we want to represent a quantity smaller than the unit. The expression 'rational number' usually covers both sorts of examples. In this paper, we will use the expressions 'fraction' and 'rational number' interchangeably. Fractions are considered a basic concept in mathematics learning and one of the foundations required for learning algebra (Fennell, Faulkner, Ma, Schmid, Stotsky, Wu et al. (2008); so they are important for representing quantities and also for later success in mathematics in school.

In the domain of whole numbers, it has been known for some time (e.g. Carpenter and Moser, 1982; Ginsburg, 1977; Riley, Greeno and Heller, 1983) that it is important for the development of children's mathematics knowledge that they establish connections between the numbers and the quantities that they represent. There is little comparable research about rational numbers (but see Mack, 1990), but it is reasonable to expect that the same hypothesis holds: children should learn to connect quantities that must be represented by rational numbers with their mathematical notation. However, the difficulty of learning to use rational numbers is much greater than the difficulty of learning to use natural numbers. This paper discusses why this is so and presents research that shows when and how children have significant insights into the complexities of rational numbers.

In the first section of this paper, we discuss what children must learn about rational numbers and why these might be difficult for children once they have learned about natural numbers. In the second section we describe research which shows that these are indeed difficult ideas for students even at the end of primary school. The third section compares children's reasoning across two types of situations that have been used in different countries to teach children about fractions. The fourth section presents a brief overview of research about children's understanding of intensive quantities. The fifth section considers whether children develop sound understanding of equivalence and order of magnitude of fractions when they learn procedures to compare fractions. The final section summarises our conclusions and discusses their educational implications.

What children must know in order to understand rational numbers

Piaget's (1952) studies of children's understanding of number analysed the crucial question of whether young children can understand the ideas of *equivalence* (cardinal number) and *order* (ordinal number) in the domain of natural numbers. He also pointed out that learning to count may help the children to understand both equivalence and order. All sets that are represented by the same number are equivalent; those that are represented by a different number are not equivalent. Their order of magnitude is the same as the order of the number labels we use in counting, because each number label represents one more than the previous one in the counting string.

The understanding of *equivalence* in the domain of fractions is also crucial, but it is not as simple because language does not help the children in the same way. Two fractional quantities that have different labels can be equivalent, and in fact there is an infinite number of equivalent fractions: 1/3, 2/6, 6/9, 8/12 etc. are different number labels but they represent equivalent quantities. Because rational numbers refer, although often implicitly, to a whole, it is also possible for two fractions that have the same number label to represent different quantities: 1/3 of 12 and 1/3 of 18 are not representations of equivalent quantities.

In an analogous way, it is not possible simply to transfer knowledge of *order* from natural to rational numbers. If the common fraction notation is used, there are two numbers, the numerator and the denominator, and both affect the order of magnitude of fractions, but they do so in different ways. If the denominator is constant, the larger the numerator, the larger is the magnitude of the fraction; if the numerator is constant, the larger the denominator, the smaller is the fraction. If both vary, then more knowledge is required to order the fractions, and it is not possible to tell which quantity is more by simply looking at the fraction labels.

Rational numbers differ from whole numbers also in the use of *two numerical signs* to represent a single quantity: it is the relation between the numbers, not their independent values, that represents the quantity. Stafylidou and Vosniadou (2004) analysed Greek students' understanding of this form of numerical representation and observed that most students in the age range 11 to 13 years did not seem to interpret the written representation of fractions as involving a multiplicative relation between the numerator and the denominator: 20% of the 11-year-olds, 37% of the 12-year-olds and 48% of the 13-year-olds provided this type of interpretation for fractions. Many younger students (about 38% of the 10-year-olds in grade 5) seemed to treat the numerator and denominator as independent numbers whereas others (about 20%) were able to conceive fractions as indicating a part-whole relation but many (22%) are unable to offer a clear explanation for how to interpret the numerator and the denominator.

Rational numbers are also different from natural numbers in their *density* (see, for example, Brousseau, Brousseau and Warfield, 2007; Vamvakoussi and Vosniadou, 2004): there are no natural numbers between 1 and 2, for example, but there is an infinite number of fractions between 1 and 2. This may seem unimportant but it is this difference that allows us to use rational numbers to represent quantities that are smaller than the units. This may be another source of difficulty for students.

Rational numbers have another property which is not shared by natural numbers: every non-zero rational number has a *multiplicative inverse* (e.g. the inverse of 2/3 is 3/2). This property may seem unimportant when children are taught about fractions in primary school, but it is important for the understanding of the division algorithm (i.e. we multiply the fraction which is the dividend by the inverse of the fraction that is the divisor) and will be required later in school, when students learn about algebra. Booth (1981) suggested that students often have a limited understanding of inverse relations, particularly in the domain of fractions, and this becomes an obstacle to their understanding of algebra. For example, when students think of fractions as representing the number of parts into which a whole was cut (denominator) and the number of parts taken (numerator), they find it very difficult to think that fractions indicate a division and that it has, therefore, an inverse.

Finally, rational numbers have *two common written notations*, which students should learn to connect: 1/2 and 0.5 are conceptually the same number with two different notations. There isn't a similar variation in natural number notation (Roman numerals are sometimes used in specific contexts, such as clocks and indices, but they probably play little role in the development of children's mathematical knowledge). Vergnaud (1997) hypothesized that different notations afford the understanding of different aspects of the same concept; this would imply that students should learn to use both notations for rational numbers. On the one hand, the common fractional notation 1/2 can be used to help students understand that fractions are related to the operation of division, because this notation can be interpreted as '1 divided by 2'. The connection between fractions and division is certainly less explicit when the decimal notation 0.5 is used. It is reasonable to expect that students will find it more difficult to understand what the multiplicative inverse of 0.5 is than the inverse of 1/2, but unfortunately there seems to be no evidence yet to clarify this.

On the other hand, adding 1/2 and 3/10 is a cumbersome process, whereas adding the same numbers in their decimal representation, 0.5 and 0.3, is a simpler matter. There are disagreements regarding the order in which these notations should be taught and the need for students to learn both notations in primary school (see, for example, Brousseau, Brousseau and Warfield, 2004; 2007), but, to our knowledge, no one has proposed that one notation should be the only one used and that the other one should be banned from mathematics classes. There is no evidence on whether children find it easier to understand the concepts related to rational numbers when one notation is used rather than the other.

Students' difficulties with rational numbers

Many studies have documented students' difficulties both with understanding equivalence and order of magnitude in the domain of rational numbers (e.g. Behr, Harel, Post and Lesh, 1992; Behr, Wachsmuth, Post and Lesh, 1984; Hart, 1986; Hart, Brown, Kerslake, Küchermann and Ruddock, 1985; Kamii and Clark, 1995; Kerslake, 1986). We illustrate here these difficulties with research carried out in the United Kingdom.

The difficulty of *equivalence* questions varies across types of tasks. Kerslake (1986) noted that when students are given diagrams in which the same shapes are divided into different numbers of sections and asked to compare two fractions, this task is relatively simple because it is possible to use a perceptual comparison. However, if students are given a diagram with six or nine divisions and asked to mark 2/3 of the shape, a large proportion of them fail to mark the equivalent fractions, 4/6 and 6/9. Hart, Brown, Kerslake, Küchermann and Ruddock (1985), working with a sample of students (N =55) in the age range 11 to 13 years, found that about 60% of the 11- to 12-year-olds and about 65% of the 12- to 13-year-olds were able to solve this task. We (Nunes, Bryant, Pretzlik and Hurry, 2006) gave the same item more recently to a sample of 130 primary school students in Years 4 and 5 (mean ages, respectively, 8.6 and 9.6 years). The rate of correct responses across these items was 28% for the children in Year 4 and 49% for the children in Year 5. This low percentage of correct answers could not be explained by a lack of knowledge of the fraction 2/3: when the diagram was divided into three sections, 93% of the students in the study by Hart *el al.* (1985) gave a correct answer; in our study, 78% of the Year 4 and 91% of the Year 5 students' correctly shaded 2/3 of the figure.

This quantitative information is presented here to illustrate the level of difficulty of these questions. A different approach to the analysis of how the level of difficulty can vary is presented later, in the third section of this paper.

Students often have difficulty in *ordering* fractions according to their magnitude. Hart *et al.* (1985) asked students to compare two fractions with the same denominator (3/7 and 5/7) and two with the same numerator (3/5 and 3/4). When the fractions have the same denominator, students can respond correctly by considering the numerators only and ordering them as if they were natural numbers. The rate of correct responses in this case is relatively high but it does not effectively test students' understanding of rational numbers. Hart *et al.* (1985) observed approximately 90% correct responses among their students in the age range 11 to 13 years and we (Nunes *et al.*, 2006) found that 94% of the students in Year 4 and 87% of the students in Year 5 gave correct responses. In contrast, when the numerator was the same and the denominator varied (comparing 3/5 and 3/4), and the students had to consider the value of the fractions in a way that is not in agreement with the order of natural numbers, the rate of correct responses was considerably lower: in the study by Hart *et al.*, approximately 70% of the answers were correct, whereas in our study the percent of correct responses were 25% in Year 4 and 70% among in Year 5.

These difficulties are not particular to U.K. students: they have been widely reported in the literature on equivalence and order of fractions (for examples in the United States see Behr, Lesh, Post and Silver, 1983; Behr, Wachsmuth, Post and Lesh 1984; Kouba, Brown, Carpenter, Lindquist, Silver and Swafford, 1988).

Difficulties in comparing rational numbers are not confined to fractions. Resnick, Nesher, Leonard, Magone, Omanson and Peled (1989) have shown that students have difficulties in comparing decimal fractions when the number of places after the decimal point differs. The samples in their study were relatively small (varying from 17 to 38) but included students from three different countries, the United States, Israel and France, and in three grade levels (4^{th} to 6^{th}). The children were asked to compare pairs of decimals such as 0.5 and 0.36, 2.35 and 2.350, and 4.8 and 4.63. The rate of correct responses varied between 36% and 52% correct, even though all students had received instruction on decimals. A more recent study (Lachance and Confrey, 2002) of 5^{th} grade students (estimated age approximately 10 years) who had received an introduction to decimal fractions in the previous year showed that only about 43% were able to compare decimal fractions correctly.[2] Rittle-Johnson, Siegler and Alibali (2001) confirmed students' difficulties when comparing the magnitude of decimals: the rate of correct responses by the students ($N = 73$; 5^{th} grade; mean age 11 years 8 months) in their study was 19%.

In conclusion, the very basic ideas about *equivalence* and *order* of fractions by magnitude, without which we could hardly say that the students have a good sense for what fractions represent, seems to elude many students for considerable periods of time. In the section that follows, we will contrast two situations that have been used to introduce the concept of fractions in primary school in order to examine the question of whether children's learning may differ as a function of these differences between situations.

Children's schemas of action in division situations

Mathematics educators and researchers may not agree on many things, but there is a clear consensus among them on the idea that rational numbers are numbers in the domain of quotients (Brousseau, Brousseau and Warfield, 2007; Kieren, 1988; 1993; 1994; Ohlsson, 1988): that is, numbers defined by the operation of division. So, it seems reasonable to seek the origin of children's understanding of rational numbers in their understanding of division.[3]

Our hypothesis is that in division situations children can develop some insight into the equivalence and order of quantities in fractions; we will use the term fractional quantities to refer to these quantities. These insights can be developed *even in the absence of knowledge of representations for fractions*, either in written or in oral form. Two schemes of action that children use in division have been analysed in the literature: partitioning and correspondences (or dealing).

Behr, Harel, Post and Lesh (1992; 1993) pointed out that fractions represent quantitites in a different way across two types of situation. The first type is the *part-whole situation*. Here one starts with a single quantity, the whole, which is divided into a certain number of parts (*y*), out of which a specified number is taken (*x*); the symbol *x/y* represents this quantity in terms of part-whole relations. *Partitioning* is the scheme of action that children use in part-whole tasks. The most common type of fraction problem that teachers give to children is to ask them to partition a whole into a fixed number of parts (the denominator) and show a certain fraction with this denominator. For example, the children have to show what 3/5 of a pizza is.[4]

The second way in which fractions represent quantities is in *quotient situations*. Here one starts with two quantities, *x* and *y*, and treats *x* as the dividend and *y* as the divisor, and by the operation of *division* obtains a single quantity *x/y*. For example, the quantities could be 3 chocolates (*x*) to be shared among 5 children (*y*). The fractional symbol *x/y* represents both the division (3 divided by 5) and the quantity that each one will receive (3/5). A quotient situation calls for the use of *correspondences* as the scheme of action: the children establish correspondences between portions and recipients. The portions may be imagined by the children, not actually drawn, as they must be when the children are asked to partition a whole and show 3/5.[5]

When children use the scheme of partitioning in part-whole situations, they can gain insights about quantities that could help them understand some principles relevant to the domain of rational numbers. They can, for example, reason that, the more parts they cut the whole into, the smaller the parts will be. This could help them understand how fractions are ordered. If they can achieve a higher level of precision in reasoning about partitioning, they could develop some understanding of the equivalence of fractions: they could come to understand that, if they have twice as many parts, each part would be halved in size. For example, you would eat the same amount of chocolate after cutting one chocolate bar into two parts and eating one part as after cutting it into four parts and eating two, because the number of parts and the size of the parts compensate for each other precisely. It is an empirical question whether children attain these understandings in the domain of whole numbers and extend them to rational numbers.

Partitioning is the scheme that is most often used to introduce children to fractions in the United Kingdom, but it is not the only scheme of action relevant to division. Children use correspondences in quotient situations when the dividend is one quantity (or measure) and the divisor is another quantity. For example, when children share out chocolate bars to a number of recipients, the dividend is in one domain of measures – the number of chocolate bars – and the divisor is in another domain – the number of children. The difference between partitioning and correspondence division is that in partitioning there is a single whole (i.e. quantity or measure) and in correspondence there are two quantities (or measures).

Fischbein, Deri, Nello and Marino (1985) hypothesised that children develop implicit models of division situations that are related to their experiences. We use their hypothesis here to explore what sorts of implicit models of fractions children may develop from using the partitioning or the correspondence scheme in fractions situations. Fischbein and colleagues suggested, for example, that children form an implicit model of division that has a specific constraint: the dividend must be larger than the divisor. We ourselves hypothesise that this implicit model is developed only in the context of partitioning. When children use the correspondence scheme, precisely because there are two domains of measures, young children readily accept that the dividend can be smaller than the divisor: they are ready to agree that it is perfectly possible to share one chocolate bar among three children.

At first glance, the difference between these two schemes of action, partitioning and correspondence, may seem too subtle to be of interest when we are thinking of children's understanding of fractions. Certainly, research on children's understanding of fractions has not focused on this distinction so far. However, our review shows that it is a crucial distinction for children's learning, both in terms of what insights each scheme of action affords and in terms of the empirical research results.

There are at least four differences between what children might learn from using the partitioning scheme or the scheme of correspondences.

- The first is the one just pointed out: that, when children set two measures in correspondence, there is no necessary relation between the size of the dividend and of the divisor. In contrast, in partitioning children form the implicit model that the sum of the parts must not be larger than the whole. Therefore, it may be easier for children to develop an understanding of improper fractions when they form correspondences between two fields of measures than when they partition a single whole. They might have no difficulty in understanding that 3 chocolates shared between 2 children means that each child could get one chocolate plus a half. In contrast, in partitioning situations children might be puzzled if they are told that someone ate 3 parts of a chocolate divided in 2 parts.

- A second possible difference between the two schemes of action may be that, when using correspondences, children can reach the conclusion that the way in which partitioning is carried out does not matter, as long as the correspondences between the two measures are 'fair'. They can reason, for example, that if 3 chocolates are to be shared by 2 children, it is not necessary to divide all 3 chocolates in half, and then distribute the halves; giving a whole chocolate plus a half to each child would accomplish the same fairness in sharing. It was argued in the first section of this paper that this an important insight in the domain of rational numbers: different fractions can represent the same quantity.

- A third possible insight about quantities that can be obtained from correspondences more easily than from partitioning is related to ordering of quantities. When forming correspondences, children may realize that there is an inverse relation between the divisor and the quotient: the more people there are to share a cake, the less each person will get: Children might achieve the corresponding insight about this inverse relation using the scheme of partitioning: the more parts you cut the whole into, the smaller the parts. However, there is a difference between the principles that children would need to abstract from each of the schemes. In partitioning, they need to establish a within-quantity relation (the more parts, the smaller the parts) whereas in correspondence they need to establish a between-quantity relation (the more children, the less cake). It is an empirical matter to find out whether or not it is easier to achieve one of these insights than the other.

- Finally, both partitioning and correspondences could help children to understand something about the equivalence between quantities, but the reasoning required to achieve this understanding differs across the two schemes of action. When setting chocolate bars in correspondence with recipients, the children might be able to reason that, if there were twice as many chocolates and twice as many children, the shares would be equivalent, even though the dividend and the divisor are different. This may be easier than the comparable reasoning in partitioning. In partitioning, understanding equivalence is based on *inverse* proportional reasoning (twice as many pieces means that each piece is half the size) whereas in contexts where children use the correspondence scheme, the reasoning is based on a *direct* proportion (twice as many chocolates and twice as many children means that everyone still gets the same).

This exploratory and hypothetical analysis of how children can reach an understanding of equivalence and order of fractions when using partitioning or correspondences in division situations suggests that the distinction between the two schemas is worth investigating empirically. It is possible that the scheme of correspondences affords a smoother transition from natural to rational numbers, at least as far as understanding equivalence and order of fractional quantities is concerned.

We turn now to an empirical analysis of this question. The literature about these schemes of action is vast but this paper focuses on research that sheds light on whether it is possible to find continuities between children's understanding of quantities that are represented by natural numbers and fractional quantities. We review research on correspondences first and then research on partitioning.

Children's use of the correspondence scheme in judgements about quantities

Piaget (1952) pioneered the study of how and when children use the correspondence scheme to draw conclusions about quantities. In one of his studies, there were three steps in the method.

- First, Piaget asked the children to place one pink flower into each one of a set of vases;

- next, he removed the pink flowers and asked the children to place a blue flower into each one of the same vases;

- then, he set all the flowers aside, leaving on the table only the vases, and asked the children to take from a box the exact number of straws required if they wanted to put one flower into each straw.

Without counting, and only using correspondences, five- and six-year old children were able to make inferences about the equivalence between straws and flowers: by setting two straws in correspondence with each vase, they constructed a set of straws equivalent to the set of vases. Piaget concluded that the children's judgements were based on 'multiplicative equivalences' (p. 219) established by the use of the correspondence scheme: the children reasoned that, if there is a 2-to-1 correspondence between flowers and vases and a 2-to-1 correspondence between straws and vases, the number of flowers and straws must be the same.

In Piaget's study, the scheme of correspondence was used in a situation that involved ratio but not division. Frydman and Bryant (1988) carried out a series of studies where children established correspondences between sets in a division situation which we have described in more detail in Paper 2, Understanding whole numbers. The studies showed that children aged four often shared pretend sweets fairly, using a one-for-you one-for-me type of procedure. After the children had distributed the sweets, Frydman and Bryant asked them to count the number of sweets that one doll had and then deduce the number of sweets that the other doll had. About 40% of the four-year-olds were able to make the necessary inference and say the exact number of sweets that the second doll had; this proportion increased with age. This result extends Piaget's observations that children can make equivalence judgements not only in multiplication but also in division problems by using correspondence.

Frydman and Bryant's results were replicated in a number of studies by Davis and his colleagues (Davis and Hunting, 1990; Davis and Pepper, 1992; Pitkethly and Hunting, 1996), who refer to this scheme of action as 'dealing'. They used a variety of situations, including redistribution when a new recipient comes, to study children's ability to use correspondences in division situations and to make inferences about equality and order of magnitude of quantities. They also argue that this scheme is basic to children's understanding of fractions (Davis and Pepper, 1992).

Correa, Nunes and Bryant (1998) extended these studies by showing that children can make inferences about quantities resulting from a division not only when the divisors are the same but also when they are different. In order to circumvent the possibility that children feel the need to count the sets after division because they think that they could have made a mistake in sharing, Bryant and his colleagues did not ask the children to do the sharing: the sweets were shared by the experimenter, outside the children's view, after the children had seen that the number of sweets to be shared was the same.

There were two conditions in this study: same dividend and same divisor versus same dividend and different divisors. In the same dividend and same divisor condition, the children should be able to conclude for the equivalence between the sets that result from the division; in the same dividend and different divisor condition, the children should conclude that the more recipients there are, the fewer sweets they receive; i.e. in order to answer correctly, they would need to use the inverse relation between the divisor and the result as a principle, even if implicitly.

About two-thirds of the five-year-olds, the vast majority of the six-year-olds, and all the seven-year-olds concluded that the recipients had equivalent shares when the dividend and the divisor were the same. Equivalence was easier than the inverse relation between divisor and quotient: 34%, 53% and 81% of the children in these three age levels, respectively, were able to conclude that the more recipients there are, the smaller each one's share will be. Correa (1994) also found that children's success in making these inferences improved if they solved these problems after practising sharing sweets between dolls; this indicates that thinking about how to establish correspondences improves their ability to make inferences about the relations between the quantities resulting from sharing.

In all the previous studies, the dividend was composed of discrete quantities and was larger than the divisor. The next question to consider is whether children can make similar judgements about equivalence when the situations involve continuous quantities and the dividend is smaller than the divisor: that is, when children have to think about fractional quantities.

Kornilaki and Nunes (2005) investigated this possibility by comparing children's inferences in division situations in which the quantities were discrete and the dividends were larger than the divisors to their inferences in situations in which the quantities were continuous and dividends smaller than the divisors. In the discrete quantities tasks, the children were shown one set of small toy fishes to be distributed fairly among a group of white cats and another set of fishes to be distributed to a group of brown cats; the number of fish was always greater than the number of cats. In the continuous quantities tasks, the dividend was made up of fish-cakes, to be distributed fairly among the cats: the number of cakes was always smaller than the number of cats, and varied between 1 and 3 cakes, whereas the number of cats to receive a portion in each group varied between 2 and 9. Following the paradigm devised by Correa, Nunes and Bryant (1998), the children were neither asked to distribute the fish nor to partition the fish cakes. They were asked whether, after a fair distribution in each group, each cat in one group would receive the same amount to eat as each cat in the other group.[6]

In some trials, the number of fish (dividend) and cats (divisor) was the same; in other trials, the dividend was the same but the divisor was different. So in the first type of trials the children were asked about equivalence after sharing and in the second type the children were asked to order the quantities obtained after sharing.

The majority of the children succeeded in all the items where the dividend and the divisor were the same: 62% of the five-year-olds, 84% of the six-year-olds and all the seven-year-olds answered all the questions correctly. When the dividend was the same and the divisors differed, the rate of success was 31%, 50% and 81%, respectively, for the three age levels. There was no difference in the level of success attained by the children with discrete versus continuous quantities.

In almost all the items, the children explained their answers by referring to the type of relation between the dividends and the divisors: same divisor, same share or, with different divisors, the more cats receiving a share, the smaller their share. The use of numbers as an explanation for the relative size of the recipients' shares was observed in 6% of answers by the seven-year-olds when the quantities were discrete and less often than this by the younger children. Attempts to use numbers to speak about the shares in the continuous quantities trials were practically non-existent (3% of the seven-year-olds' explanations). Thus, the analysis of justifications supports the idea that the children were reasoning about relations between quantities rather than using counting when they made their judgments of equivalence or ordered the quantities that would be obtained after division.

This study replicated the previous findings, which we have mentioned already, that young children can use correspondences to make inferences about equivalences and also added new evidence relevant to children's understanding of fractional quantities: many young children who have never been taught about fractions used correspondences to order fractional quantities. They did so successfully when the division would have resulted in unitary fractions and also when the dividend was greater than 1 and the result would not be a unitary fraction (e.g. 2 fish cakes to be shared by 3, 4 or 5 cats).

A study by knowledge Mamede, Nunes and Bryant (2005) confirmed that children can make inferences about the order of magnitude of fractions in sharing situations where the dividend is smaller than the divisor (e.g. 1 cake shared by 3 children compared to 1 cake shared by 5 children). She worked with Portuguese children in their first year in school, who had received no school instruction about fractions. Their performance was only slightly weaker than that of British children: 55% of the six-year-olds and 71% of the seven-year-olds were able to make the inference that the larger the divisor, the smaller the share that each recipient would receive.

These studies strongly suggest that children can learn principles about the relationship between dividend and divisor from experiences with sharing when they establish correspondences between the two domains of measures, the shared quantities and the recipients. They also suggest that children can make a relatively smooth transition from natural numbers to rational numbers when they use correspondences to understand the relations between quantities. This argument is central to Streefland's (1987; 1993; 1997) hypothesis about what is the best starting point for teaching fractions to children and has been advanced by others also (Davis and Pepper, 1992; Kieren, 1993; Vergnaud, 1983).

This research tell an encouraging story about children's understanding of the logic of division even when the dividend is smaller than the divisor, but there is one further point that should be considered in the transition between natural and rational numbers. In the domain of rational numbers there is an infinite set of equivalences (e.g. 1/2 = 2/4 = 3/6 etc) and in the studies that we have described so far the children were only asked to make equivalence judgements when the dividend and the divisor in the equivalent fractions were the same. Can they still make the inference of equivalence in sharing situations when the dividend and the divisor are different across situations, but the dividend-divisor ratio is the same?

Nunes, Bryant, Pretzlik, Bell, Evans and Wade (2007) asked British children aged between 7.5 and 10 years, who were in Years 4 and 5 in school, to make comparisons between the shares that would be received by children in sharing situations where the dividend and divisor were different but their ratio was the same. Previous research (see, for example, Behr, Harel, Post and Lesh, 1992; Kerslake, 1986) shows that children in these age levels have difficulty with the equivalence of fractions. The children in this study had received some instruction on fractions: they had been taught about halves and quarters in problems about partitioning. They had only been taught about one pair of equivalent fractions: they were taught that one half is the same as two quarters. In the correspondence item in this study, the children were presented with two pictures: in the first, a group of 4 girls was going to share fairly 1 pie; in the second, a group of 8 boys was going to share fairly 2 pies that were exactly the same as the pie that the girls had. The question was whether each girl would receive the same share as each boy. The overall rate of correct responses was 73% (78% in Year 4 and 70% in Year 5; this difference was not significant). This is an encouraging result: the children had only been taught about halves and quarters; nevertheless, they were able to attain a high rate of correct responses for fractional quantities that could be represented as 1/4 and 2/8.

In the studies reviewed so far the children were asked about quantities that resulted from division and always included two domains of measures; thus the children's correspondence reasoning was engaged in these studies. However, they did not involve asking the children to represent these quantities through fractions. The final study reviewed here is a brief teaching study (Nunes, Bryant, Pretzlik, Evans, Wade and Bell, 2008), where the children were taught to represent fractions in the context of two domains of measures, shared quantities and recipients, and were asked about the equivalence between fractions. The types of arguments that the children produced to justify the equivalence of the fractions were then analyzed and compared to the insights that we hypothesized would emerge in the context of sharing from the use of the correspondence scheme.

Brief teaching studies are of great value in research because they allow the researchers to know what understandings children can construct if they are given a specific type of guidance in the interaction with an adult (Cooney, Grouws and Jones, 1988; Steffe and Tzur, 1994; Tzur, 1999; Yackel, Cobb, Wood, Wheatley and Merkel, 1990). They also have compelling ecological validity: children spend much of their time in school trying to use what they have been taught to solve new mathematics problems. Because this study has only been published in a summary form (Nunes, Bryant, Pretzlik and Hurry, 2006), some detail is presented here.

The children ($N = 62$) were in the age range from 7.5 to 10 years, in Years 4 and 5 in school. Children in Year 4 had only been taught about half and quarters and the equivalence between half and two quarters; children in Year 5 had been taught also about thirds. They worked with a researcher outside the classroom in small groups (12 groups of between 4 and 6 children, depending on the class size) and were asked to solve each problem first individually, and then to discuss their answers in the group. The sessions were audio- and video-recorded. The children's arguments were transcribed *verbatim;* the information from the video-tapes was later coordinated with the transcripts in order to help the researchers understand the children's arguments.

In this study the researchers used problems developed by Streefland (1990). The children solved two of his sharing tasks on the first day and an equivalence task on the second day of the teaching study. The tasks were presented in booklets with pictures, where the children also wrote their answers. The tasks used in the first day were:

- Six girls are going to share a packet of biscuits. The packet is closed; we don't know how many biscuits are in the packet. (a) If each girl received one biscuit and there were no biscuits left, how many biscuits were in the packet? (b) If each girl received a half biscuit and there were no biscuits left, how many biscuits were in the packet? (c) If some more girls join the group, what will happen when the biscuits are shared? Do the girls now receive more or less each than the six girls did?
- Four children will be sharing three chocolates. (a) Will each child be able to get one bar of chocolate? (b) Will each child be able to get at least a half bar of chocolate? (c) How would you share the chocolate? (The booklets contained a picture with three chocolate bars and four children and the children were asked to show how they would share the chocolates) Write what fraction each one gets.

After the children had completed these tasks, the researcher told them that they were going to practice writing fractions which they had not yet learned in school. The children were asked to write 'half' with numerical symbols, which they knew already. The researcher taught the children to write fractions that they had not yet learned in school in order to help the children re-interpret the meaning of fractions. The numerator was to be used to represent the number of items to be divided, the denominator should represent the number of recipients, and the dash between them two numbers should represent the sign for division (for a discussion of children's interpretation of fraction symbols in this situation, see Charles and Nason, 2000, and Empson, Junk, Dominguez and Turner, 2005).

The equivalence task, presented on the second day, was:

- Six children went to a pizzeria and ordered two pizzas to share between them. The waiter brought one first and said they could start on it because it would take time for the next one to come. (a) How much will each child get from the first pizza that the waiter brought? Write the fraction that shows this. (b) How much will each child get from the second pizza? Write your answer. (c) If you add the two pieces together, what fraction of a pizza will each child get? You can write a plus sign between the first fraction and the second fraction, and write the answer for the share each child gets in the end. (d) If the two pizzas came at the same time, how could they share it differently? (e) Are these fractions (the ones that the children wrote for answers c and d equivalent?

According to the hypotheses presented in the previous section, we would expect children to develop some insights into rational numbers by thinking about different ways of sharing the same amount. It was expected that they would realize that:

1 it is possible to divide a smaller number by a larger number

2 different fractions might represent the same amount

3 twice as many things to be divided and twice as many recipients would result in equivalent amounts

4 the larger the divisor, the smaller the quotient.

The children's explanations for why they thought that the fractions were equivalent provided evidence for all these insights, and more, as described below.

It is possible to divide a smaller number by a larger number

There was no difficulty among the students in attempting to divide 1 pizza among 6 children. In response to part a of the equivalence problem, all children wrote at least one fraction correctly (some children wrote more than one fraction for the same answer, always correctly).

In response to part c, when the children were asked how they could share the 2 pizzas if both pizzas came at the same time and what fraction would each one receive, some children answered 1/3 and others answered 2/12 from each pizza, giving a total share of 4/12. The latter children, instead of sharing 1 pizza among 3 girls, decided to cut each pizza in 12 parts: i.e. they cut the sixths in half.

Different fractions can represent the same amount

The insight that different fractions can represent the same amount was expressed in all groups. For example, one child said that, 'They're the same amount of people, the same amount of pizzas, and that means the same amount of fractions. It doesn't matter how you cut it.' Another child said, 'Because it wouldn't really matter when they shared it, they'd get that [3 girls would get 1 pizza], and then they'd get that [3 girls would get the other pizza], and then it would be the same.' Another child said, 'It's the same amount of pizza. They might be different fractions but the same amount [this child had offered 4/12 as an alternative to 2/6].' Another child said: 'Erm, well basically just the time doesn't make much difference, the main thing is the number of things.'

When the dividend is twice as large and the divisor is also twice as large, the result is an equivalent amount

The principle that when the dividend is twice as large and the divisor is also twice as large, the result is an equivalent amount was expressed in 11 of the 12 groups. For example, one child said, 'It's half the girls and half the pizzas; three is a half of six and one is a half of two.' Another child said, 'If they have two pizzas, then they could give the first pizza to three girls and then the next one to another three girls. (…) If they all get one piece of that each, and they get the same amount, they all get the same amount'.

So all three ideas we thought that could appear in this context were expressed by the children. But two other principles, which we did not expect to observe in this correspondence problem, were also made explicit by the children.

The number of parts and size of parts are inversely proportional

The principle that the number of parts and size of parts are inversely proportional was enunciated in 8 of the 12 groups. For example, one child who cut the pizzas the second time around in 12 parts each said, 'Because it's double the one of that [total number of pieces] and it's double the one of that [number of pieces for each], they cut it twice and each is half the size; they will be the same'. Another child said, 'Because one sixth and one sixth is actually a different way in fractions [from 1 third] and it doubled [the number of pieces] to make it [the size of the piece] littler, and halving [the number of pieces] makes it [the size of the piece] bigger, so I halved it and it became one third'.

The fractions show the same part-whole relation

The reasoning that the fractions show the same part-whole relation, which we had not expected to emerge from the use of the correspondence scheme, was enunciated in only one group (out of 12), initially by one child, and was then reiterated by a second child in her own terms. The first child said, 'You need three two sixths to make six [6/6 – he shows the 6 pieces marked on one pizza], and you need three one thirds to make three (3/3 – shows the 3 pieces marked on one pizza). [He wrote the computation and continued] There's two sixths, add two sixths three times to make six sixths. With one third, you need to add one third three times to make three thirds.'

To summarize: this brief teaching experiment was carried out to elicit discussions between the children in situations where they could use the correspondence scheme in division. The first set of problems, in which they are asked about sharing discrete quantities, created a background for the children to use this scheme of action. The researchers then helped them to construct an interpretation for written fractions where the numerator is the dividend, the denominator is the divisor, and the line indicates the operation of division. This interpretation did not replace their original interpretation of number of parts taken from the whole; the two meanings co-existed and appeared in the children's arguments as they explained their answers. In the subsequent problems, where the quantity to be shared was continuous and the dividend was smaller than the divisor, the children had the opportunity to explore the different ways in which continuous quantities can be shared. They were not asked to actually partition the pizzas, and some made marks on the pizzas whereas others did not. The most salient feature of the children's drawings was that they were not concerned with partitioning per se, even when the parts were marked, but with the correspondences between pizzas and recipients. Sometimes the correspondences were carried out mentally and expressed verbally and sometimes the children used drawings and gestures which indicated the correspondences.

Other researchers have identified children's use of correspondences to solve problems that involve fractions, although they did not necessarily use this label in describing the children's answers. Empson (1999), for example, presented the following problem to children aged about six to seven years (first graders in the USA): 4 children got 3 pancakes to share; how many pancakes are needed for 12 children in order for the children to have the same amount of pancake as the first group? She reported that 3 children solved this problem by partitioning and 3 solved it by placing 3 pancakes in correspondence to each group of 4 children. Similar strategies were reported when children solved another problem that involved 2 candy bars shared among 3 children.

Kieren (1993) also documented children's use of correspondences to compare fractions. In his problem, the fractions were not equivalent: there were 7 recipients and 4 items in Group A and 4 recipients and 2 items in Group B. The children were asked how much each recipient would get in each group and whether the recipients in both groups would get the same amount. Kieren presents a drawing by an eight-year-old, where the items are partitioned in half and the correspondences between the halves and the recipients are shown; in Group A, a line without a recipient shows that there is an extra half in that group and the child argues that there should be one more person in Group A for the amounts to be the same. Kieren termed this solution 'corresponding or 'ratiolike' thinking' (p. 54).

Conclusion

The scheme of correspondences develops relatively early: about one-third of the five-year-olds, half of six-year-olds and most seven-year-olds can use correspondences to make inferences about equivalence and order in tasks that involve fractional quantities. Children can use the scheme of correspondences to:

- establish equivalences between sets that have the same ratio to a reference set (Piaget, 1952)
- re-distribute things after having carried out one distribution (Davis and Hunting, 1990; Davis and Pepper, 1992; Davis and Pitkethly, 1990; Pitkethly and Hunting, 1996);
- reason about equivalences resulting from division both when the dividend is larger or smaller than the divisor (Bryant and colleagues: Correa, Nunes and Bryant, 1994; Frydman and Bryant, 1988; 1994; Empson, 1999; Nunes, Bryant, Pretzlik and Hurry, 2006; Nunes, Bryant, Pretzlik, Bell, Evans and Wade, 2007; Mamede, Nunes and Bryant, 2005);
- order fractional quantities (Kieren, 1993; Kornilaki and Nunes, 2005; Mamede, 2007).

These studies were carried out with children up to the age of ten years and all of them produced positive results. This stands in clear contrast with the literature on children's difficulties with fractions and prompts the question whether the difficulties might stem from the use of partitioning as the starting point for the teaching of fractions (see also Lamon, 1996; Streefland, 1987). The next section examines the development of children's partitioning action and its connection with children's concepts of fractions.

Children's use of the scheme of partitioning in making judgements about quantities

The scheme of partitioning has been also named subdivision and dissection (Pothier and Sawada, 1983), and is consistently defined as the process of dividing a whole into parts. This process is understood not as the activity of cutting something into parts in any way, but as a process that must be guided from the outset by the aim of obtaining a pre-determined number of equal parts.

Piaget, Inhelder and Szeminska (1960) pioneered the study of the connection between partitioning and fractions. They spelled out a number of ideas that they thought were necessary for children to develop an understanding of fractions, and analysed them in partitioning tasks. The motivation for partitioning was sharing a cake between a number of recipients, but the task was one of partitioning. They suggested that 'the notion of fraction depends on two fundamental relations: the relation of part to whole (…) and the relation of part to part' (p. 309). Piaget and colleagues identified a number of insights that children need to achieve in order to understand fractions:

1 the whole must be conceived as divisible, an idea that children under the age of about two do not seem to attain

2 the number of parts to be achieved is determined from the outset

3 the parts must exhaust the whole (i.e. there should be no second round of partitioning and no remainders)

4 the number of cuts and the number of parts are related (e.g. if you want to divide something in 2 parts, you should use only 1 cut)

5 all the parts should be equal

6 each part can be seen as a whole in itself, nested into the whole but also susceptible of further division

7 the whole remains invariant and is equal to the sum of the parts.

Piaget and colleagues observed that children rarely achieved correct partitioning (sharing a cake) before the age of about six years. There is variation in the level of success depending on the shape of the whole (circular areas are more difficult to partition than rectangles) and on the number of parts. A major strategy in carrying out successful partitioning was the use of successive divisions in two: so children are able to succeed in dividing a whole into fourths before they can succeed with thirds. Successive halving helped the children with some fractions: dividing something into eighths is easier this way. However, it interfered with success with other fractions: some children, attempting to divide a whole into fifths, ended up with sixths by dividing the whole first in halves and then subdividing each half in three parts.

Piaget and colleagues also investigated children's understanding of their seventh criterion for a true concept of fraction, i.e. the conservation of the whole. This conservation, they argued, would require the children to understand that each piece could not be counted simply as one piece, but had to be understood in its relation to the whole. Some children aged six and even seven years failed to understand this, and argued that if someone ate a cake cut into 1/2 + 2/4 and a second person ate a cake cut into 4/4, the second one would eat more because he had four parts and the first one only had three. Although these children would recognise that if the pieces were put together in each case they would form one whole cake, they still maintained that 4/4 was more than 1/2 + 2/4. Finally, Piaget and colleagues also observed that children did not have to achieve the highest level of development in the scheme of partitioning in order to understand the conservation of the whole.

Children's difficulties with partitioning continuous wholes into equal parts have been confirmed many times in studies with pre-schoolers and children in their first years in school (e.g. Hiebert and Tonnessen, 1978; Hunting and Sharpley, 1988 *b*,) observed that children often do not anticipate the number of cuts and fail to cut the whole extensively, leaving a part of the whole un-cut. These studies also extended our knowledge of how children's expertise in partitioning develops. For example, Pothier and Sawada (1983) and Lamon (1996) proposed more detailed schemes for the analysis of the development of partitioning schemes and other researchers (Hiebert and Tonnessen, 1978; Hunting and Sharpley, 1988 *a* and *b*; Miller, 1984; Novillis, 1976) found that the difficulty of partitioning discrete and continuous quantities is not the same, as hypothesized by Piaget. Children can use a procedure for partitioning discrete quantities that is not applicable to continuous quantities: they can 'deal out' the discrete quantities but not the continuous ones. Thus they perform significantly better with the former than the latter. This means that the transition from discrete to continuous quantities in the use of partitioning is not difficult, in contrast to the smooth transition noted in the case of the correspondence scheme.

These studies showed that the scheme of partitioning continuous quantities develops slowly, over a longer period of time. The next question to consider is whether partitioning can promote the understanding of equivalence and ordering of fractions once the scheme has developed.

Many studies investigated children's understanding of equivalence of fractions in partitioning contexts (e.g. Behr, Lesh, Post and Silver, 1983; Behr, Wachsmuth, Post and Lesh, 1984; Larson, 1980; Kerslake, 1986), but differences in the methods used in these studies render the comparisons between partitioning and correspondence studies ambiguous. For example, if the studies start with a representation of the fractions, rather than with a problem about quantities, they cannot be compared to the studies reviewed in the previous section, in which children were asked to think about quantities without necessarily using fractional representation. We shall not review all studies but only those that use comparable methods.

Kamii and Clark (1995) presented children with identical rectangles and cut them into fractions using different cuts. For example, one rectangle was cut horizontally in half and the second was cut across a diagonal. The children had the opportunity to verify that the rectangles were the same size and that the two parts from each rectangle were the same in size. They asked the children: if these were chocolate cakes, and the researcher ate a part cut from the first rectangle and the child ate a part cut from the second, would they eat the same amount? This method is highly comparable to the studies by Kornilaki and Nunes (2005) and by Mamede (2007), where the children do not have to carry out the actions, so their difficulty with partitioning does not influence their judgements. They also use similarly motivated contexts, ending in the question of whether recipients would eat the same amount. However, the question posed by Kamii and Clark draws on the child's understanding of partitioning and the relations between the parts of the two wholes because each whole corresponds to a single recipient.

The children in Kamii's study were considerably older than those in the correspondence studies: they were in the fifth or sixth year in school (approximately 11 and 12 years). Both groups of children had been taught about equivalent fractions. In spite of having received instruction, the children's rate of success was rather low: only 44% of the fifth graders and 51% of the sixth graders reasoned that they would eat the same amount of chocolate cake because these were halves of identical wholes.

Kamii and Clark then showed the children two identical wholes, cut one in fourths using a horizontal and a vertical cut, and the other in eighths, using only horizontal cuts. They discarded one fourth from the first 'chocolate cake', leaving three fourths be eaten, and asked the children to take the same amount from the other cake, which had been cut into eighths, for themselves. The percentage of correct answers was this time even lower: 13% of the fifth graders and 32% of the sixth graders correctly identified the number of eighths required to take the same amount as three fourths.

Recently, we (Nunes and Bryant, 2004) included a similar question about halves in a survey of English children's knowledge of fractions. The children in our study were in their fourth and fifth year (mean ages eight and a half and nine and a half, respectively) in school. The children were shown pictures of a boy and a girl and two identical rectangular areas, the 'chocolate cakes'. The boy cut his cake along the diagonal and the girl cut hers horizontally. The children were asked to indicate whether they ate the same amount of cake and, if not, to mark the child who ate more. Our results were more positive than Kamii and Clark's: 55% of the children in year four (eight and a half year olds) in our study answered correctly. However, these results are weak by comparison to children's rate of correct responses when the problem draws on their understanding of correspondences. In the Kornilaki and Nunes study, 100% of the seven-year-olds (third graders) realized that two divisions that have the same dividend and the same divisor result in equivalent shares. Our results with fourth graders, when both the dividend and the divisor were different, still shows a higher rate of correct responses when correspondences are used: 78% of the fourth graders gave correct answers when comparing one fourth and two eighths.

In the preceding studies, the students had to think about the quantities ignoring their perceptual appearance. Hart *et al.* (1985) and Nunes *et al.*

(2004) presented students with verbal questions, which did not contain drawings that could lead to incorrect conclusions based on perception. In both studies, the children were told that two boys had identical chocolate bars; one cut his into 8 parts and ate 4 and the other cut his into 4 parts and ate 2. Combining the results of these two studies, it is possible to see how the rate of correct responses changed across age: 40% at ages 8 to 9 years, 74% at 10 to11 years, 60% at 11 to 12 years, and 64% at 12 to 13 years. The students aged 8 to 10 were assessed by Nunes *et al.* and the older ones by Hart *et al*. These results show modest progress on the understanding of equivalence questions presented in the context of partitioning even though the quantities eaten were all equivalent to half.

Mamede (2007) carried out a direct comparison between children's use of the correspondence and the partitioning scheme in solving equivalence and order problems with fractional quantities. In this well-controlled study, she used story problems involving chocolates and children, similar pictures and mathematically identical questions; the division scheme relevant to the situation was the only variable distinguishing the problems. In correspondence problems, for example, she asked the children: in one party, three girls are going to share fairly one chocolate cake; in another party, six boys are going to share fairly two chocolate cakes. The children were asked to decide whether each boy would eat more than each girl, each girl would eat more than each boy, or whether they would have the same amount to eat. In the partitioning problems, she set the following scenario: This girl and this boy have identical chocolate cakes; the cakes are too big to eat at once so the girl cuts her cake in 3 identical parts and eats one and the boy cuts his cake in 6 identical parts and eats 2. The children were asked whether the girl and the boy ate the same amount or whether one ate more than the other. The children (age range six to seven) were Portuguese and in their first year in school; they had received no instruction about fractions.

In the correspondence questions, the responses of 35% of the six-year-olds and 49% of the seven-year-olds were correct; in the partitioning questions, 10% of the answers of children in both age levels were correct. These highly significant differences suggest that the use of correspondence reasoning supports children's understanding of equivalence between fractions whereas partitioning did not seem to afford the same insights.

Finally, it is important to compare students' arguments for the equivalence and order of quantities represented by fractions in teaching studies where partitioning is used as the basis for teaching. Many teaching studies that aim at promoting students' understanding of fractions through partitioning have been reported in the literature (e.g., Behr, Wachsmuth, Post and Lesh, 1984; Brousseau, Brousseau and Warfield, 2004; 2007; Empson, 1999; Kerslake, 1986; Olive and Steffe, 2002; Olive and Vomvoridi, 2006; Saenz-Ludlow, 1994; Steffe, 2002). In most of these studies, students' difficulties with partitioning are circumvented either by using pre-divided materials (e.g. Behr, Wachsmuth, Post and Lesh, 1984) or by using computer tools where the computer carries out the division as instructed by the student (e.g. Olive and Steffe, 2002; Olive and Vomvoridi, 2006).

Many studies combine partitioning with correspondence during instruction, either because the researchers do not use this distinction (e.g. Saenz-Ludlow, 1994) or because they wish to construct instruction that combines both schemes in order to achieve a better instructional program (e.g. Brousseau, Brousseau and Warfield, 2004; 2007). These studies will not be discussed here. Two studies that analysed student's arguments focus the instruction on partitioning and are presented here.

The first study was carried out by Berhr, Wachsmuth, Post and Lesh (1984). The researchers used objects of different types that could be manipulated during instruction (e.g. counters, rectangles of the same size and in different colours, pre-divided into fractions such as halves, quarters, thirds, eighths) but also taught the students how to use algorithms (division of the denominator by the numerator to find a ratio) to check on the equivalence of fractions. The students were in fourth grade (age about 9) and received instruction over 18 weeks. Behr *et al.* provided a detailed analysis of children's arguments regarding the ordering of fractions. In summary, they report the following insights after instruction.

- When ordering fractions with the same numerator and different denominators, students seem to be able to argue that there is an inverse relation between the number of parts into which the whole was cut and the size of the parts. This argument appears either with explicit reference to the numerator ('there are two pieces in each, but the pieces in two fifths are smaller.' p. 328) or without it ('the bigger the number is, the smaller the pieces get.' p. 328).

- A third fraction can be used as a reference point when two fractions are compared: three ninths is less than three sixths because 'three ninths is … less than half and three sixths is one half' (p. 328). It is not clear how the students had learned that 3/6 and 1/2 are equivalent but they can use this knowledge to solve another comparison.

- Students used the ratio algorithm to verify whether the fractions were equivalent: 3/5 is not equivalent to 6/8 because 'if they were equal, three goes into six, but five doesn't go into eight.' (p. 331).

- Students learned to use the manipulative materials in order to carry out perceptual comparisons: 6/8 equals 3/4 because 'I started with four parts. Then I didn't have to change the size of the paper at all. I just folded it, and then I got eight.' (p. 331).

Behr *et al.* report that, after 18 weeks of instruction, a large proportion of the students (27%) continued to use the manipulatives in order to carry our perceptual comparisons; the same proportion (27%) used a third fraction as a reference point and a similar proportion (23%) used the ratio algorithm that they had been taught to compare fractions.

Finally, there is no evidence that the students were able to understand that the number of parts and size of parts could compensate for each other precisely in a proportional manner. For example, in the comparison between 6/8 and 3/4 the students could have argued that there were twice as many parts when the whole was cut into 8 parts in comparison with cutting in 4 parts, so you need to take twice as many (6) in order to have the same amount.

In conclusion, students seemed to develop some insight into the inverse relation between the divisor and the quantity but this only helped them when the dividend was kept constant: they could not extend this understanding to other situations where the numerator and the denominator differed.

The second set of studies that focused on partitioning was carried out by Steffe and his colleagues (Olive and Steffe, 2002; Olive and Vomvoridi, 2006; Steffe, 2002). Because the aim of much of the instruction was to help the children learn to label fractions or compose fractions that would be appropriate for the label, it is not possible to extract from their reports the children's arguments for equivalence of fractions.

However, one of the protocols (Olive and Steffe, 2002) provides evidence for the student's difficulty with improper fractions, which, we hypothesise, could be a consequence of using partitioning as the basis for the concept of fractions. The researcher asked Joe to make a stick 6/5 long. Joe said that he could not because 'there are only five of them'. After prompting, Joe physically adds one more fifth to the five already used, but it is not clear whether this physical action convinces him that 6/5 is mathematically appropriate. In a subsequent example, Joe labels a stick made with 9 sticks, which had been defined as 'one seventh' of an original stick, 9/7, but according to the researchers 'an important perturbation' remains. Joe later counts 8 of a stick that had been designated as 'one seventh' but doesn't use the label 'eight sevenths'. When the researcher proposes this label, he questions it: 'How can it be EIGHT sevenths?' (Olive and Steffe, 2002, p. 426). Joe later refused to make a stick that is 10/7, even though the procedure is physically possible. Subsequently, on another day, Joe's reaction to another improper fraction is: 'I still don't understand how you could do it. *How can a fraction be bigger than itself?* (Olive and Steffe, 2002, p. 428; emphasis in the original).

According to the researchers, Joe only sees that improper fractions are acceptable when they presented a problem where pizzas were to be shared by people. When 12 friends ordered 2 slices each of pizzas cut into 8 slices, Joe realized immediately that more than one pizza would be required; the traditional partitioning situation, where one whole is divided into equal parts, was transformed into a less usual one, where two wholes are required but the size of the part remains fixed.

This example illustrates the difficulty that students have with improper fractions in the context of partitioning but which they can overcome by thinking of more than one whole.

Conclusion

Partitioning, defined as the action of cutting a whole into a pre-determined number of equal parts, shows a slower developmental process than correspondence. In order for children to succeed, they need to anticipate the solution so that the right number of cuts produces the right number of equal parts and exhausts the whole. Its accomplishment, however, does not seem to produce immediate insights into equivalence and order of fractional quantities. Apparently, many children do not see it as necessary that halves from two identical wholes are equivalent, even if they have been taught about the equivalence of fractions in school.

In order to use this scheme of action as the basis for learning about fractions, teaching schemes and researchers rely on pre-cut wholes or computer tools to avoid the difficulties of accurate partitioning. Students can develop insight into the inverse relation between the number of parts and the size of the parts through the partitioning scheme but there is no evidence that they realize that if you cut a whole in twice as many parts each one will be half in size. Finally, improper fractions seem to cause uneasiness to students who have developed their conception of fractions in the context of partitioning; it is important to be aware of this uneasiness if this is the scheme chosen in order to teach fractions.

Rational numbers and children's understanding of intensive quantities

In the introduction, we suggested that rational numbers are necessary to represent quantities that are measured by a relation between two other quantities. These are called intensive quantities and there are many examples of such quantities both in everyday life and in science. In everyday life, we often mix liquids to obtain a certain taste. If you mix fruit concentrate with water to make juice, the concentration of this mixture is described by a rational number: for example, 1/3 concentrate and 2/3 water. Probability is an intensive quantity that is important both in mathematics and science and is measured as the number of favourable cases divided by the number of total cases.[7]

The conceptual difficulties involved in understanding intensive quantities are largely similar to those involved in understanding the representation of quantities that are smaller than the whole. In order to understand intensive quantities, students must form a concept that takes two variables simultaneously into account and realise that there is an inverse relation between the denominator and the quantity represented.

Piaget and Inhelder described children's thinking about intensive quantities as one of the many examples of the development of the scheme of proportionality, which they saw as one of the hallmarks of adolescent thinking and formal operations. They devoted a book to the analysis of children's understanding of probabilities (Piaget and Inhelder, 1975) and described in great detail the steps that children take in order to understand the quantification of probabilities. In the most comprehensive of their studies, the children were shown pairs of decks of cards with different numbers of cards, some marked with a cross and others unmarked. The children were asked to judge which deck they would choose to draw from if they wanted to have a better chance of drawing a card marked with a cross.

Piaget and Inhelder observed that many of the young children treated the number of marked and unmarked cards as if they were independent: sometimes they chose one deck because it had more marked cards than the other and sometimes they chose a deck because it had fewer blank cards than the other. This approach can lead to correct responses when either the number of marked cards or the number of unmarked cards is the same in both decks, and children aged about seven years were able to make correct choices in such problems. This is rather similar to the observations of children's successes and difficulties in comparing fractions reported earlier on: they can reach the correct answer when the denominator is constant or when the numerator is constant, as this allow them to focus on the other value. When they must think of different denominators and numerators, the questions become more difficult.

Around the age of nine, children started making correspondences between marked and unmarked cards within each deck and were able to identify equivalences using this type of procedure. For example, if asked to compare a deck with one marked and two unmarked cards (1/3 probability) with another deck with two marked and four unmarked cards (2/6 probability), the children would re-organise the second deck in two lots, setting one marked card in correspondence with two unmarked, and conclude that it did not make any difference which deck they picked a card from. Piaget and Inhelder saw these as empirical proportional solutions, which were a step towards the abstraction that characterises proportional reasoning.

Noelting (1980 *a* and *b*) replicated these results with another intensive quantity, the taste of orange juice made from a mixture of concentrate and water. In broad terms, he described children's thinking and its development in the same way as Piaget and Inhelder had done. This is an important replication of Piaget's results results considering that the content of the problems differed marked across the studies, probability and concentration of juice.

Nunes, Desli and Bell (2003) compared students' ability to solve problems about extensive and intensive quantities that involved the same type of reasoning. Extensive quantities can be represented by a single whole number (e.g. 5 kilos, 7 cows, 4 days) whereas intensive quantities are represented by a ratio between two numbers. In spite of these differences, it is possible to create problems which are comparable in other aspects but differ with respect to whether the quantities are extensive or intensive. Intensive quantities problems always involve three variables. For example, three variables might be amount of orange concentrate, amount of water, and the taste of the orange juice, which is the intensive quantity. The amount of orange concentrate is directly related to how orangey the juice tastes whereas the amount of water is inversely related to how orangey the juice tastes. A comparable extensive quantities problem would involve three extensive quantities, with the one under scrutiny being inversely proportional to one of the variables and directly proportional to the other. For example, the number of days that the food bought by a farmer lasts is directly proportional to the amount of food purchased and inversely proportional to the number of animals she has to feed. In our study, we analysed students' performance in comparison problems where they had to consider either intensive quantities (e.g. how orangey a juice would taste) or extensive quantities (e.g. the number of days the farmer's food supply would last). Students performed significantly better in the extensive quantities problems even though both types of problem involved proportional reasoning and the same number of variables. So, although the difficulties shown by children across the two types of problem are similar, their level of success was higher with extensive than intensive quantities. This indicates that students find it difficult to form a concept where two variables must be coordinated into a single construct,[8] and therefore it may be important for schools and teachers to consider how they might promote this development in the classroom.

We shall not review the large literature on intensive quantities here (see, for example, Erickson, 1979; Kaput, 1985; Schwartz, 1988; Stavy, Strauss, Orpaz and Carmi, 1982; Stavy and Tirosh, 2000), but there is little doubt that students' difficulties in understanding intensive quantities are very similar to those that they have when thinking about fractions which represent quantities smaller than the unit. They treat the values independently, they find it difficult to think about inverse relations, and they might think of the relations between the numbers as additive instead of multiplicative.

There is presently little information to indicate whether students can transfer what they have learned about fractions in the context of representing quantities smaller than the unit to the representation and understanding of intensive quantities. Brousseau, Brousseau, and Warfield (2004) suggest both that teachers believe that students will easily go from one use of fractions to another, and that nonetheless the differences between these two types of situation could actually result in interference rather than in easy transfer of insights across situations. In contrast, Lachance and Confrey (2002) developed a curriculum for teaching third grade students (estimated age about 8 years) about ratios in a variety of problems, including intensive quantities problems, and then taught the same students in fourth grade (estimated age about 9 years) about decimals. Their hypothesis is that students would show positive transfer from learning about ratios to learning about decimals. They claimed that their students learned significantly more about decimals than students who had not participated in a similar curriculum and whose performance in the same questions had been described in other studies.

We believe that it is not possible at the moment to form clear conclusions on whether knowledge of fractions developed in one type of situation transfers easily to the other, shows no transfer, or actually interferes with learning about the other type of situation. In order to settle this issue, we must carry out the appropriate teaching studies and comparisons.

However, there is good reason to conclude that the use of rational numbers to represent intensive quantities should be explicitly included in the curriculum. This is an important concept in everyday life and science, and causes difficulties for students.

Learning to use mathematical procedures to determine the equivalence and order of rational numbers

Piaget's (1952) research on children's understanding of natural numbers shows that young children, aged about four, might be able to count two sets of objects, establish that they have the same number, and still not conclude that they are equivalent if the sets are displayed in very different perceptual arrangements. Conversely, they might establish the equivalence between two sets by placing their elements in correspondence and, after counting the elements in one set, be unable to infer what the number in the other set is (Piaget, 1952; Frydman and Bryant, 1988). As we noted in Paper 2, Understanding whole numbers, counting is a procedure for creating equivalent sets and placing sets in order but many young children who know how to count do not use counting when asked to compare or create equivalent sets (see, for example, Michie, 1984, Cowan and Daniels, 1989; Cowan, 1987; Cowan, Foster and Al-Zubaidi, 1993; Saxe, Guberman and Gearhart, 1987).

Procedures to establish the equivalence and order of fractional quantities are much more complex than counting, particularly when both the denominator and the numerator differ. Students are taught different procedures in different countries. The procedure that seems most commonly taught in England is to check the equivalence by analysing the multiplicative relation between or within the fractions. For example, when comparing 1/3 with 4/12, students are taught to find the factor that connects the numerators (1 and 4) and then apply the same factor to the denominators. If the numerator and the denominator of the second fraction are the product of the numerator and the denominator of the first fraction by the same number, 4 in this case, they are equivalent. An alternative approach is to find whether the multiplicative relation between the numerator and the denominator of each fraction is the same (3 in this case): if it is, the fractions are equivalent.

If students learned this procedure and applied it consistently, it should not matter whether the factor is, for example, 2, 3 or 5, because these are well-known multiplication associations. It should also not matter whether the fraction with larger numerator and denominator is the first or the second. However, research shows that these variations affect students' performance. Hart *et al.* (1985) presented students with the task of identifying the missing values in equivalent fractions. The children were presented with the item below and asked which numbers should replace the square and the triangle:

2/7 = □/14 = 10/△

The rate of correct responses by 11 to 12 and 12- to 13-year-olds for the second question was about half that for the first one: about 56% for the first question and 24% for the second. The within-fraction method cannot be easily applied in these cases but the factors are 2 and 5, and these multiplication tables should be quite easy for students at this age level.

We recently replicated these different levels of difficulty in a study with 8- to 10-year-olds. The easiest questions were those where the common factor was 2; the rates of correct responses for 1/3 = 2/□ and 6/8 = 3/□ were 52% and 45%, respectively. The most difficult question was 4/12 = 1/□ this was only answered correctly by 16% of the students. It is unlikely that the difficulty of computation could explain the differences in performance: even weak students in this age range should be able to identify 3 as the factor connecting 4 and 12, if they had been taught the within-fraction method, or 4 as the factor connecting 1 and 4, if they were taught the between-fractions method.

A noteworthy aspect of our results was the low correlations between the different items: although most were significant (due to the large sample size; *N* = 188), only two of the nine correlations were above .4. This suggests that the students were not able to use the procedure that they learned consistently to solve five items that had the same format and could be solved by the same procedure.

Our assessment, like the one by Hart *et al.* (1985), also included an equivalence question set in the context of a story: two boys have identical chocolate bars, one cuts his into 8 equal parts and eats 4 and the other cuts his into 4 equal parts and eats 2; the children are asked to indicate whether the boys eat the same amount of chocolate and, if not, who eats more. This item is usually seen as assessing children's understanding of quantities as it is not expressed in fraction terms. In our sample, no student wrote the fractions 4/8 and 2/4 and compared them by means of a procedure. We analysed the correlations between this item and the five items described in the previous paragraph. If the students used the same reasoning or the same procedure to solve the items, there should be a high correlation between them. This was not so: the highest of the correlations between this item and each of the five previous ones was 0.32, which is low. This result exemplifies the separation between understanding fractional quantities and knowledge of procedures in the domain of rational numbers. This is much the same as observed in the domain of natural numbers.

It is possible that understanding the relations between quantities gives students an advantage in learning the procedures to establish the equivalence of fractions, but it may not guarantee that they will actually learn it if teachers do connect their understanding with the procedure. When we separated the students into two groups, one that answered the question about the boys and the chocolates correctly and the other that did not, there was a highly significant difference between the two groups in the rate of correct responses in the procedural items: the group who succeeded in the chocolate question showed 38% correct responses to the procedural items whereas the group who failed only answered 18% of the procedural questions correctly.

A combination of longitudinal and intervention studies is required to clarify whether students who understand fractional quantities benefit more when taught how to represent and compare fractions. There are presently no studies to clarify this matter.

Research that analyses students' knowledge of procedures used to find equivalent fractions and its connection with conceptual knowledge of fractions has shown that there can be discrepancies between these two forms of knowledge. Rittle-Johnson, Siegler and Alibali (2001) argued that procedural and conceptual knowledge develop *in tandem* but Kerslake (1986) and Byrnes and colleagues (Byrnes, 1992; Byrnes and Wasik, 1991), among others, identified clear discrepancies between students' conceptual and procedural knowledge of fractions. Recently we (Hallett, Nunes and Bryant, 2007) analysed a large data set (N = 318 children in Years 4 and 5) and observed different profiles of relative performance in items that assess knowledge of procedures to compare fractions and understanding of fractional quantities. Some children show greater success in procedural questions than would be expected from their performance in conceptual items, others show better performance in conceptual items than expected from their performance in the procedural items, and still others do not show any discrepancy between the two. Thus, some students seem to learn procedures for finding equivalent fractions without an understanding of why the procedures work, others base their approach to fractions on their understanding of quantities without mastering the relevant procedures, and yet others seem able to co-ordinate the two forms of knowledge. Our results show that the third group is more successful not only in a test about fractions but also in a test about intensive quantities, which did not require the use of fractions in the representation of the quantities.

Finally, we ask whether students are better at using procedures to compare decimals than to compare ordinary fractions. The students in some of the grade levels studied by Resnick and colleagues (1989) would have been taught how to add and subtract decimals: they were in grades 5 and 6 (the estimated age for U.S. students is about 10 and 11 years) and one of the early uses of decimals in the curriculum in the three participating countries is addition and subtraction of decimals. When students are taught to align the decimal numbers by placing the decimal points one under the other before adding – for example, when adding 0.8, 0.26 and 0.361 you need to align the decimal points before carrying out the addition – they may not realise that they are using a procedure that automatically converts the values to the same denominator: in this case, $x/1000$. It is possible that students may use this procedure of aligning the decimal point without fully understanding that this is a conversion to the same denominator and thus that it should help them to compare the value of the fractions: after learning how to add and subtract with decimals, they may still think that 0.8 is less than 0.36 but probably would not have said that 0.80 is less than 0.36.

To conclude, we find in the domain of rational numbers a similar separation between understanding quantities and learning to operate with representations when judging the equivalence and order of magnitude of quantities. Students are taught procedures to test whether fractions are equivalent but their knowledge of these procedures is limited, and they do not apply it across items consistently. Similarly, students who solve equivalence problems in context are not necessarily experts in solving problems when the fractions are presented without context.

The significance of children's difficulties in understanding equivalence of fractions cannot be overstressed: in the domain of rational numbers, students cannot learn to add and subtract with understanding if they do not realise that fractions must be equivalent in order to be added. Adding 1/3 and 2/5 without transforming one of these into an equivalent fraction with the same denominator as the other is like adding bananas and tins of soup: it makes no sense. Above and beyond the fact that one cannot be said to understand numbers without understanding their equivalence and order, in the domain of rational numbers equivalence is a core concept for computing addition and subtraction. Kerslake (1986) has shown that students learn to implement the procedures for adding and subtracting fractions without having a glimpse at why they convert the fractions into common denominators first. This separation between the meaning of fractions and the procedures cannot bode well for the future of these learners.

Conclusions and educational implications

- Rational numbers are essential for the representation of quantities that cannot be represented by a single natural number. For this reason, they are needed in everyday life as well as science, and should be part of the curriculum in the age range 5 to 16.

- Children learn mathematical concepts by applying schemes of action to problem solving and reflecting about them. Two types of action schemes are available in division situations: partitioning, which involves dividing a whole into equal parts, and correspondence situations, where two quantities (or measures) are involved, a quantity to be shared and a number of recipients of the shares.

- Children as young as five or six years in age are quite good at establishing correspondences to produce equal shares, whereas they experience much difficulty in partitioning continuous quantities. Reflecting about these schemes and drawing insights from them places children in different paths for understanding rational number. When they use the correspondence scheme, they can achieve some insight into the equivalence of fractions by thinking that, if there are twice as many things to be shared and twice as many recipients, then each one's share is the same. This involves thinking about a direct relation between the quantities. The partitioning scheme leads to understanding equivalence in a different way: if a whole is cut into twice as many parts, the size of each part will be halved. This involves thinking about an inverse relation between the quantities in the problem. Research consistently shows that children understand direct relations better than inverse relations.

- There are no systematic and controlled comparisons to allow for unambiguous conclusions about the outcomes of instruction based on correspondences or partitioning. The available evidence suggests that testing this hypothesis appropriately could result in more successful teaching and learning of rational numbers.

- Children's understanding of quantities is often ahead of their knowledge of fractional representations when they solve problems using the correspondence scheme. Schools could make use of children's informal knowledge of fractional quantities and work with problems about situations, without requiring them to use formal representations, to help them consolidate this reasoning and prepare them for formalization.

- Research has identified the arguments that children use when comparing fractions and trying to see whether they are equivalent or to order them by magnitude. It would be important to investigate next whether increasing teachers' awareness of children's own arguments would help teachers guide children's learning more effectively.

- In some countries, greater attention is given to decimal representation than to ordinary fractions in primary school whereas in others ordinary fractions continue to play an important role. The argument that decimals are easier to understand than ordinary fractions does not find support in surveys of students' performance: students find it difficult to make judgements of equivalence and order both with decimals and with ordinary fractions.

- Some researchers (e.g. Nunes, 1997; Tall, 1992; Vergnaud, 1997) argue that different representations shed light onto the same concepts from different perspectives. This would suggest that a way to strengthen students' learning of rational numbers is to help them connect both representations. Moss and Case (1999) analysed this possibility in the context of a curriculum based on measurements, where ordinary fractions and percentages were used to represent the same information. Their results are encouraging, but the study does not include the appropriate controls that would allow for establishing firmer conclusions.

- Students can learn procedures for comparing, adding and subtracting fractions without connecting these procedures with their understanding of equivalence and order of fractional quantities, independently of whether they are taught with ordinary or decimal representation. This is not a desired outcome of instruction, but seems to be a quite common one. Research that focuses on the use of children's informal knowledge suggests that it is possible to help students make connections (e.g. Mack, 1990), but the evidence is limited. There is now considerably more information regarding children's informal strategies to allow for new teaching programmes to be designed and assessed.

- Finally, this review opens the way for a fresh research agenda in the teaching and learning of fractions. The source for the new research questions is the finding that children achieve insights into relations between fractional quantities before knowing how to represent them. It is possible to envisage a research agenda that would not focus on children's misconceptions about fractions, but on children's possibilities of success when teaching starts from thinking about quantities rather than from learning fractional representations.

Endnotes

1 Rational numbers can also be used to represent relations that cannot be described by a single whole number but the representation of relations will not be discussed here.

2 The authors report a successful programme of instruction where they taught the students to establish connections between their understanding of ratios and decimals. The students had received two years of instruction on ratios. A full discussion of this very interesting work is not possible here as the information provided in the paper is insufficient.

3 There are different hypotheses regarding what types of subconstructs or meanings for rational numbers should be distinguished (see, for example, Behr, Harel, Post and Lesh, 1992; Kieren, 1988) and how many distinctions are justifiable. Mathematicians and psychologists may well use different criteria and consequently reach different conclusions. Mathematicians might be looking for conceptual issues in mathematics and psychologists for distinctions that have an impact on children's learning (i.e. show different levels of difficulty or no transfer of learning across situations). We have decided not to pursue this in detail but will consider this question in the final section of the paper.

4 Steffe and his colleagues have used a different type of problem, where the size of the part is fixed and the children have to identify how many times it fits into the whole.

5 This classification should not be confused with the classification of division problems in the mathematics education literature. Fischbein, Deri, Nello and Marino (1985) define partitive division (which they also term sharing division) as a model for situations in which 'an object or collection of objects is divided into a number of equal fragments or sub-collections. The dividend must be larger than the divisor; the divisor (operator) must be a whole number; the quotient must be smaller than the dividend (operand)… In quotative division or measurement division, one seeks to determine how many times a given quantity is contained in a larger quantity. In this case, the only constraint is that the dividend must be larger than the divisor. If the quotient is a whole number, the model can be seen as repeated subtraction.' (Fischbein, Deri, Nello and Marino, 1985, p.7). In both types of problems discussed by Fishbein et al., the scheme used in division is the same, partitioning, and the situations are of the same type, part-whole.

6 Empson, Junk, Dominguez and Turner (2005) have stressed that 'the depiction of equal shares of, for example, sevenths in a part–whole representation is not a necessary step to understanding the fraction 1/7 (for contrasting views, see Charles and Nason, 2000; Lamon, 1996; Pothier and Sawada, 1983). What is necessary, however, is understanding that 1/7 is the amount one gets when 1 is divided into 7 same-sized parts.'

7 Not all intensive quantities are represented by fractions; speed, for example, is represented by a ratio, such as in 70 miles per hour.

8 Vergnaud (1983) proposed this hypothesis in his comparison between isomorphism of measures and product of measures problems. This issue is discussed in greater detail in another paper 4 of this review.

References

Behr, M., Harel, G., Post, T., and Lesh, R. (1992). Rational number, ratio, proportion. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 296-333). New York: Macmillan.

Behr, M. J., Harrel, G., Post, R., & Lesh, R. (1993). Toward a unified discipline of scientific enquiry. In T. Carpenter, P. E. Fennema & T. A. Romberg (Eds.), *Rational Numbers: an integration of research* (pp. 13-48). Hillsdale, NJ: Lawrence Erlbaum Ass.

Behr, M., Lesh, R., Post, T., and Silver, E. A. (1983). Rational number concepts. In R. Lesh and M. Landau (Eds.), *Acquisition of Mathematical Concepts and Processes* (pp. 91-126). New York: Academic Press.

Behr, M. J., Wachsmuth, I., Post, T. R., and Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*(5), 323-341.

Brousseau, G., Brousseau, N., and Warfield, V. (2004). Rationals and decimals as required in the school curriculum Part 1: Rationals as measurement. *Journal of Mathematical Behavior, 23*, 1–20.

Brousseau, G., Brousseau, N., and Warfield, V. (2007). Rationals and decimals as required in the school curriculum Part 2: From rationals to decimals. *Journal of Mathematical Behavior 26* 281–300.

Booth, L. R. (1981). Child-methods in secondary mathematics. *Educational Studies in Mathematics, 12* 29-41.

Byrnes, J. P. (1992). The conceptual basis of procedural learning. *Cognitive Development, 7*, 235-237.

Byrnes, J. P. and Wasik, B. A. (1991). Role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology, 27(5)*, 777-786.

Carpenter, T. P., and Moser, J. M. (1982). The development of addition and subtraction problem solving. In T. P. Carpenter, J. M. Moser and T. A. Romberg (Eds.), *Addition and subtraction: A cognitive perspective* (pp. 10-24). Hillsdale (NJ): Lawrence Erlbaum.

Charles, K. and Nason, R. (2000). Young children's partitioning strategies. *Educational Studies in Mathematics*, 43, 191–221.

Cooney, T. J., Grouws, D. A., and Jones, D. (1988). An agenda for research on teaching mathematics. In D. A. Grouws and T. J. Cooney (Eds.), *Effective mathematics teaching* (pp. 253-261). Reston, VA: National Council of Teachers of Mathematics and Hillsdale, NJ: Erlbaum.

Correa, J. (1994). *Young children's understanding of the division concept.* University of Oxford, Oxford.

Correa, J., Nunes, T., and Bryant, P. (1998). Young children's understanding of division: The relationship between division terms in a noncomputational task. *Journal of Educational Psychology, 90*, 321-329.

Cowan, R., and Daniels, H. (1989). Children's use of counting and guidelines in judging relative number. *British Journal of Educational Psychology, 59*, 200-210.

Cowan, R. (1987). When do children trust counting as a basis for relative number judgements? *Journal of Experimental Child Psychology, 43*, 328-345.

Cowan, R., Foster, C. M., and Al-Zubaidi, A. S. (1993). Encouraging children to count. *British Journal of Developmental Psychology, 11*, 411-420.

Davis, G. E., and Hunting, R. P. (1990). Spontaneous partitioning: Preschoolers and discrete items. *Educational Studies in Mathematics, 21*, 367-374.

Davis, G. E., and Pepper, K. L. (1992). Mathematical problem solving by pre-school children. *Educational Studies in Mathematics, 23*, 397-415.

Davis, G. E., and Pitkethly, A. (1990). Cognitive aspects of sharing. *Journal for Research in Mathematics Education, 21*, 145-153.

Empson, S. B. (1999). Equal sharing and shared meaning: The development of fraction concepts in a First-Grade classroom. *Cognition and Instruction, 17*(3), 283-342.

Empson, S. B., Junk, D., Dominguez, H., and Turner, E. (2005). Fractions as the coordination of multiplicatively related quantities: a cross-sectional study of children's thinking. *Educational Studies in Mathematics, 63*, 1-28.

Erickson, G. (1979). Children's conceptions of heat and temperature. *Science Education, 63*, 221-230.

Fennell, F. S., Faulkner, L. R., Ma, L., Schmid, W., Stotsky, S., Wu, H.-H., et al. (2008). *Report of the Task Group on Conceptual Knowledge and Skills* Washington DC: U.S. Department of Education, The Mathematics Advisory Panel.

Fischbein, E., Deri, M., Nello, M., and Marino, M. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 16*, 3-17.

Frydman, O., and Bryant, P. E. (1988). Sharing and the understanding of number equivalence by young children. *Cognitive Development, 3*, 323-339.

Frydman, O., and Bryant, P. (1994). Children's understanding of multiplicative relationships in the construction of quantitative equivalence. *Journal of Experimental Child Psychology, 58*, 489-509.

Ginsburg, H. (1977). *Children's Arithmetic: The Learning Process*. New York: Van Nostrand.

Gréco, P. (1962). Quantité et quotité: nouvelles recherches sur la correspondance terme-a-terme et la conservation des ensembles. In P. Gréco and A. Morf (Eds.), *Structures numeriques elementaires: Etudes d'Epistemologie Genetique Vol 13* (pp. 35-52). Paris: Presses Universitaires de France.

Hart, K. (1986). *The step to formalisation*. Insititute of Education, University of London: Proceedings of the 10th International Conference of the Psychology of Mathematics Education, pp 159-164.

Hart, K., Brown, M., Kerslake, D., Kuchermann, D., & Ruddock, G. (1985). *Chelsea Diagnostic Mathematics Tests. Fractions 1*. Windsor (UK): NFER-Nelson.

Hallett, D., Nunes, T., and Bryant, P. (2007). Individual differences in conceptual and procedural knowledge when learning fractions. *under review*.

Hiebert, J., and Tonnessen, L. H. (1978). Development of the fraction concept in two physical contexts: An exploratory investigation. Journal for Research in Mathematics Education, 9(5), 374-378.

Hunting, R. P. and Sharpley, C. F. (1988 *a*). Preschoolers' cognition of fractional units. British Journal of Educational Psychology, 58, 172-183.

Hunting, R. P. and Sharpley, C. F. (1988 *b*). Fractional knowledge in preschool children. Journal for Research in Mathematics Education, 19(2), 175-180.

Kamii, C., and Clark, F. B. (1995). Equivalent Fractions: Their Difficulty and Educational Implications. *Journal of Mathematical Behavior, 14*, 365-378.

Kaput, J. (1985). *Multiplicative word problems and intensive quantities: An integrated software response* (Technical Report 85-19). Cambridge (MA): Harvard University, Educational Technology Center.

Kerslake, D. (1986). *Fractions: Children's Strategies and Errors: A Report of the Strategies and Errors in Secondary Mathematics Project*. Windsor: NFER-Nelson.

Kieren, T. (1988). Personal knowledge of rational numbers: Its intuitive and formal development. In J. Hiebert and M. Behr (Eds.), *Number concepts and operations in the middle-grades* (pp. 53-92). Reston (VA): National Council of Teachers of Mathematics.

Kieren, T. E. (1993). Rational and Fractional Numbers: From Quotient Fields to Recursive Understanding. In T. Carpenter, E. Fennema and T. A. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 49-84). Hillsdale, N.J.: Lawrence Erlbaum Associates.

Kieren, T. (1994). Multiple Views of Multiplicative Structures. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 389-400). Albany, New York: State University of New York Press.

Kornilaki, K., and Nunes, T. (2005). Generalising Principles in spite of Procedural Differences: Children's Understanding of Division. *Cognitive Development, 20*, 388-406.

Kouba, V. L., Brown, C. A., Carpenter, T. P., Lindquist, M. M., Silver, E. A., and Swafford, J. (1988). Results of the Fourth NAEP Assessment of Mathematics: Number, operations, and word problems. *Arithmetic Teacher, 35* (4), 14-19.

Lachance, A., and Confrey, J. (2002). Helping students build a path of understanding from ratio and proportion to decimal notation. *Journal of Mathematical Behavior, 20* 503–526.

Lamon, S. J. (1996). The Development of Unitizing: Its Role in Children's Partitioning Strategies. *Journal for Research in Mathematics Education, 27*, 170-193.

Larson, C. N. (1980). Locating proper fractions on number lines: Effect of length and equivalence. *School Science and Mathematics, 53*, 423-428.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education, 21*, 16-32.

Mamede, E., Nunes, T., & Bryant, P. (2005). *The equivalence and ordering of fractions in part-whole and quotient situations.* Paper presented at the 29th PME Conference, Melbourne; Volume 3, 281-288.

Mamede, E. P. B. d. C. (2007). *The effects of situations on children's understanding of fractions.* Unpublished PhD Thesis, Oxford Brookes University, Oxford.

Michie, S. (1984). Why preschoolers are reluctant to count spontaneously. *British Journal of Developmental Psychology, 2*, 347-358.

Miller, K. (1984). Measurement procedures and the development of quantitative concepts. In C. Sophian (Ed.), Origins of cognitive skills. The Eighteen Annual Carnegie Symposium on Cognition (pp. 193-228). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Moss, J., and Case, R. (1999). Developing children's understanding of rational numbers: A new model and an experimental curriculum. *Journal for Research in Mathematics Education, 30*, 122-147.

Noelting, G. (1980 a). The development of proportional reasoning and the ratio concept Part I - Differentiation of stages. *Educational Studies in Mathematics, 11*, 217-253.

Noelting, G. (1980 b). The development of proportional reasoning and the ratio concept Part II - Problem-structure at sucessive stages: Problem-solving strategies and the mechanism of adaptative restructuring. *Educational Studies in Mathematics, 11*, 331-363.

Novillis, C. F. (1976). An Analysis of the Fraction Concept into a Hierarchy of Selected Subconcepts and the Testing of the Hierarchical Dependencies. *Journal for Research in Mathematics Education, 7*, 131-144.

Nunes, T. (1997). Systems of signs and mathematical reasoning. In T. Nunes and P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 29-44). Hove (UK): Psychology Press.

Nunes, T., and Bryant, P. (2004). *Children's insights and strategies in solving fractions problems.* Paper presented at the First International Conference of the Greek Association of Cognitive Psychology, Alexandropolis, Greece.

Nunes, T., Bryant, P., Pretzlik, U., and Hurry, J. (2006). *Fractions: difficult but crucial in mathematics learning*. London Institute of Education, London: ESRC-Teaching and Learning Research Programme.

Nunes, T., Bryant, P., Pretzlik, U., Bell, D., Evans, D., & Wade, J. (2007). La compréhension des fractions chez les enfants. In M. Merri (Ed.), *Activité humaine et conceptualisation. Questions à Gérard Vergnaud* (pp. 255-262). Toulouse, France: Presses Universitaires du Mirail.

Nunes, T., Bryant, P., Pretzlik, U., Evans, D., Wade, J., and Bell, D. (2008). Children's understanding of the equivalence of fractions. An analysis of arguments in quotient situations. *submitted*.

Nunes, T., Desli, D., and Bell, D. (2003). The development of children's understanding of intensive quantities. *International Journal of Educational Research, 39*, 652-675.

Piaget, J. (1952). *The Child's Conception of Number*. London: Routledge and Kegan Paul.

Piaget, J., and Inhelder, B. (1975). *The Origin of the Idea of Chance in Children*. London: Routledge and Kegan Paul.

Piaget, J., Inhelder, B., and Szeminska, A. (1960). *The Child's Conception of Geometry*. New York: Harper and Row.

Ohlsson, S. (1988). Mathematical meaning and applicational meaning in the semantics of fractions and related concepts. In J. Hiebert and M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 53-92). Reston (VA): National Council of Mathematics Teachers.

Olive, J., and Steffe, L. P. (2002). The construction of an iterative fractional scheme: the case of Joe. *Journal of Mathematical Behavior, 20* 413-437.

Olive, J., and Vomvoridi, E. (2006). Making sense of instruction on fractions when a student lacks necessary fractional schemes: The case of Tim. *Journal of Mathematical Behavior, 25* 18-45.

Piaget, J. (1952). *The Child's Conception of Number*. London: Routledge and Kegan Paul.

Piaget, J., Inhelder, B., and Szeminska, A. (1960). *The Child's Conception of Geometry*. New York: Harper and Row.

Pitkethly, A., and Hunting, R. (1996). A Review of Recent Research in the Area of Initial Fraction Concepts. *Educational Studies in Mathematics, 30*, 5-38.

Pothier, Y., and Sawada, D. (1983). Partitioning: The Emergence of Rational Number Ideas in Young Children. *Journal for Research in Mathematics Education, 14*, 307-317.

Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., and Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education, 20*, 8-27.

Riley, M., Greeno, J. G., and Heller, J. I. (1983). Development of children's problem solving ability in arithmetic. In H. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 153-196). New York: Academic Press.

Rittle-Johnson, B., Siegler, R. S., and Alibali, M. W. (2001). Developing Conceptual Understanding and Procedural Skill in Mathematics: An Iterative Process. *Journal of Educational Psychology, 93*(2), 46-36.

Saenz-Ludlow, A. (1994). Michael's Fraction Schemes. *Journal for Research in Mathematics Education, 25*, 50-85.

Saxe, G., Guberman, S. R., and Gearhart, M. (1987). Social and developmental processes in children's understanding of number. *Monographs of the Society for Research in Child Development, 52*, 100-200.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert and M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Hillsdale,NJ: Erlbaum.

Stafylidou, S., and Vosniadou, S. (2004). The development of students' understanding of the numerical value of fractions. *Learning and Instruction 14*, 503-518.

Stavy, R., Strauss, S., Orpaz, N., and Carmi, G. (1982). U-shaped behavioral growth in ration comparisons, or that's funny I would not have thought you were U-ish. In S. Strauss and R. Stavy (Eds.), *U-shaped Behavioral Growth* (pp. 11-36). New York: Academic Press.

Stavy, R., and Tirosh, D. (2000). *How Students (Mis-)Understand Science and Mathematics. Intuitive Rules*. New York: Teachers College Press.

Steffe, L. P. (2002). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior, 102*, 1-41.

Steffe, L. P., and Tzur, R. (1994). Interaction and children's mathematics. *Journal of Research in Childhood Education, 8*, 99- 91 16.

Streefland, L. (1987). *How lo Teach Fractions So As to Be Useful*. Utrecht, The Netherlands: The State University of Utrecht.

Streefland, L. (1990). Free productions in Teaching and Learning Mathematics. In K. Gravemeijer, M. van den Heuvel & L. Streefland (Eds.), *Contexts Free Production Tests and Geometry in Realistic Mathematics Education* (pp. 33-52). Utrecht Netherlands: Research group for Mathematical Education and Educational Computer Centre - State University of Utrecht.

Streefland, L. (1993). Fractions: A Realistic Approach. In T. P. Carpenter, E. Fennema and T. A. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 289-326). Hillsdale, NJ: Lawrence Erlbaum Associates.

Streefland, L. (1997). Charming fractions or fractions being charmed? In T. Nunes and P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 347-372). Hove (UK): Psychology Press.

Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495-511). New York: Macmillan.

Tzur, R. (1999). An Integrated Study of Children's Construction of Improper Fractions and the Teacher's Role in Promoting That Learning. *Journal for Research in Mathematics Education, 30*, 390-416.

Vamvakoussi, X., and Vosniadou, S. (2004). Understanding the structure of the set of rational numbers: a conceptual change approach. *Learning and Instruction, 14* 453-467.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes and P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 1-28). Hove (UK): Psychology Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 128-175). London: Academic Press.

Yackel, E., Cobb, P., Wood, T., Wheatley, G., and Merkel, G. (1990). The importance of social interaction in children's construction of mathematical knowledge. In T. J. Cooney and C. R. Hirsch (Eds.), *Teaching and learning mathematics in the 1990s. 1990 Yearbook of the National Council of Teachers of Mathematics* (pp. 12-21). Reston, VA: National Council of Teachers of Mathematics.

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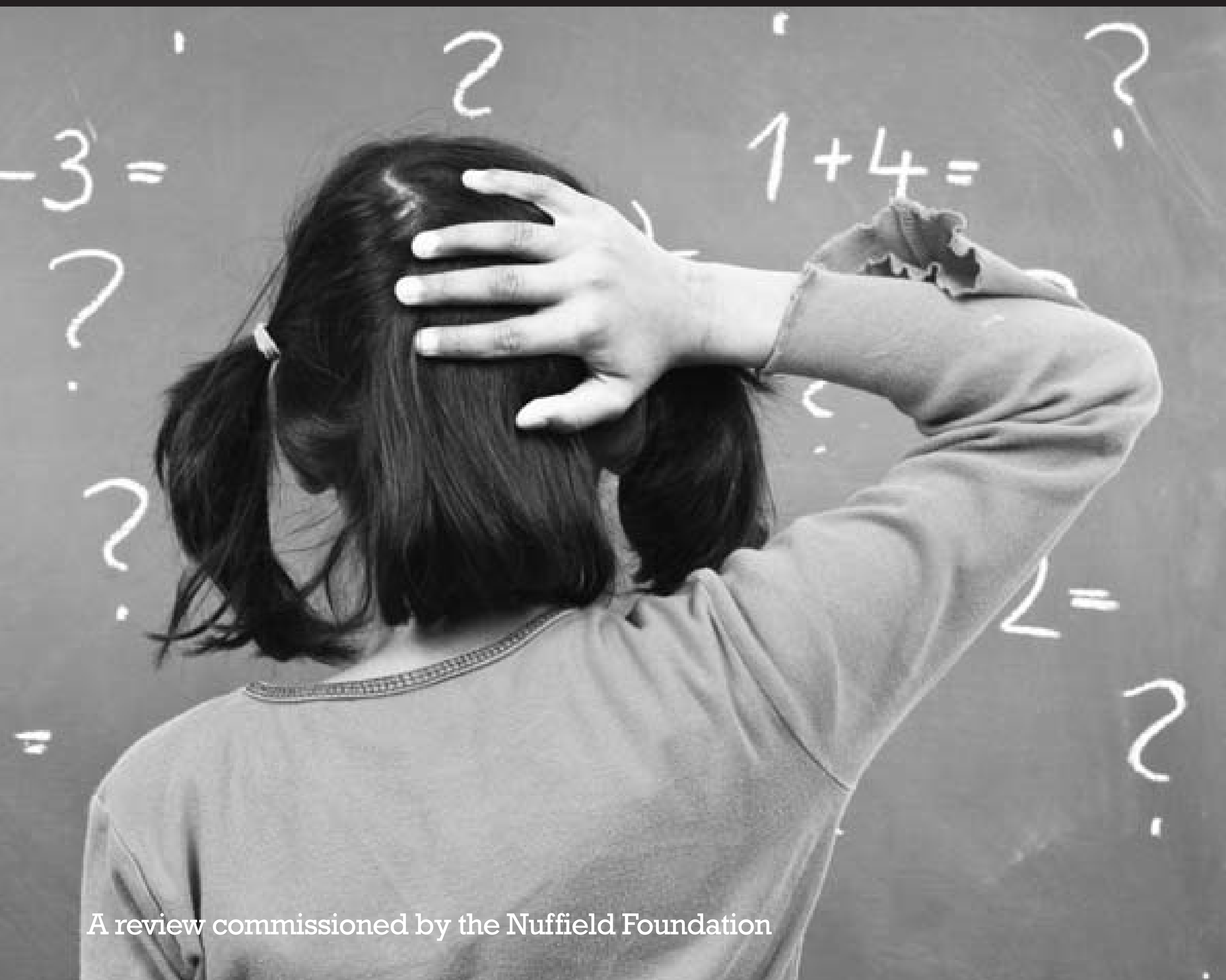
4

Key understandings in mathematics learning

Paper 4: Understanding relations and their graphical representation

By Terezinha Nunes and Peter Bryant, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 4: Understanding relations and their graphical representation

Headlines

- Children have greater difficulty in understanding relations than in understanding quantities. This is true in the context of both additive and multiplicative reasoning problems.
- Primary and secondary school students often apply additive procedures to solve multiplicative reasoning problems and also apply multiplicative procedures to solve additive reasoning problems.
- Explicit instruction to help students become aware of relations in the context of additive reasoning problems can lead to significant improvement in children's performance.
- The use of diagrams, tables and graphs to represent relations facilitates children's thinking about and discussing the nature of the relations between quantities in problems.
- Excellent curriculum development work has been carried out to design instruction to help students develop awareness of their implicit knowledge of multiplicative relations. This programme has not been systematically assessed so far.
- An alternative view is that students' implicit knowledge should not be the starting point for students to learn about proportional relations; teaching should focus on formalisations rather than informal knowledge and seek to connect mathematical formalisations with applied situations only later.
- There is no research comparing the results of these diametrically opposed ideas.

Children need to learn to co-ordinate their knowledge of numbers with their understanding of quantities. This is critical for mathematics learning in primary school so that they can use their understanding of quantities to support their knowledge of numbers and *vice versa*. But this is not all that students need to learn to be able to use mathematics sensibly. Using mathematics also involves thinking about relations between quantities. Research shows quite unambiguously that it is more difficult for children to solve problems that involve relations than to solve problems that involve only quantities.

A simple problem about quantities is: Paul had 5 marbles. He played two games with his friend. In the first game, he won 6 marbles. In the second game he lost 4 marbles. How many marbles does he have now? The same numerical information can be used differently, making the problem into one which is all about relations: Paul played three games of marbles. In the first game, he won 5 marbles. In the second game, he won 6. In the third game, he lost 4. Did he end up winning or losing marbles? How many?

The arithmetic that children need to use to solve is the same in both problems: add 5 and 6 and subtract 4. But the second problem is significantly more difficult for children because it is all about relations. They don't know how many marbles Paul actually had at any time, they only know that he had 5 more after the first game than before, and 6 more after the second game, and 4 fewer after the third game. Some children say that this problem cannot be solved because we don't know how many marbles Paul had to begin with: they recognise that it is possible to operate on quantities, but do not recognise that it is possible to operate on relations. Why should this be so?

One possible explanation is the way in which we express relations. When we speak about quantities, we say that Paul won marbles or lost marbles; these are two opposite statements. When we speak about relations, statements that use opposite words may mean the same thing: after winning 5 marbles, we can say that Paul now has 5 more marbles or that before he had 5 fewer. In order to grasp the concept of relations fully, students must be able to view these two different statements as meaning the same thing. Research shows that some students are able to treat these different statements as having the same meaning but others find this difficult. Students who realise that the two statements mean the same thing are more successful in solving problems about relations.

A second plausible explanation is that many children do not distinguish clearly between quantities and relations when they use numbers. When they are given a problem about relations, they interpret the relations as quantities. If they are given a problem like 'Tom, Fred, and Rhoda put their apples into a bag. Tom and Fred together had 17 more apples than Rhoda. Tom had 7 apples. Rhoda had 5 apples. How many apples did Fred have?', they write down that Tom and Fred had 17 apples together (instead of 17 more than Rhoda). When they make this interpretation error, the problem seems very easy: if Tom had 7, Fred had 10. The information about Rhoda seems irrelevant. But of course this is not the solution. It is possible to teach children to represent quantities and relations differently, and thus to distinguish the two: for example, they can be taught to write 'plus 17' to show that this is not a quantity but a relation. Children aged seven to nine years can adopt this notation and at the same time improve their ability to solve relational problems. However, even after this teaching, they still seem to be tempted to interpret relations as quantities. So, learning to represent relations helps children take a step towards distinguishing relations and quantities but they need plenty of opportunity to think about this distinction.

A third difficulty is that relational thinking involves building a model of a problem situation in order to treat the relations in the problem mathematically. In primary school, children have little opportunity to explore situations in their mathematics lessons before solving a problem. If they make a mistake in solving a problem when their computation was correct, the error is explained as 'choice of the wrong operation', but the wrong choice of operation is a symptom, not an explanation for what went wrong during problem solving.

Models of situations are ways of thinking about them, and more than one way may be appropriate. It all depends on the question that we want to answer. Suppose there are 12 girls and 18 boys in a class and they are assigned to single-sex groups during French lessons. If there were not enough books for all of them and the Head Teacher decided to give 4 books to the girls and 6 books to the boys, would this be fair? If you give one book to each girl, there are 8 girls left without books; if you give one book to each boy, there are 12 boys left without books. This seems unfair. If you ask all the children to share, 3 girls will share one book and 3 boys will share one book. This seems fair. The first model is additive: the questions it answers are 'How many more girls than books?' and 'How many more boys than books?' The second model is multiplicative: it examines the ratio between girls and books and the ratio between boys and books. If the Head Teacher is planning to buy more books, she needs an additive model. If the Head Teacher is not planning to buy more books, the ratio is more informative. A model of a situation is constructed by the problem solver for a purpose; additive and multiplicative relations answer different questions about the same situation.

Children, but also adults, often make mistakes in the choice of operation when solving problems: they sometimes use additive reasoning when they should have used multiplicative reasoning but they can also make the converse mistake, and use multiplicative reasoning when additive reasoning would be appropriate. So, we need to examine research that explains how children can become more successful in choosing the appropriate model to answer a question.

Experts often use diagrams, tables and graphs to help them analyse situations. These resources could support children's thinking about situations. But children seem to have difficulty in using these resources and have to learn how to use them. They have to become literate in the use of these mathematical tools in order to interpret them correctly. A question that has not been addressed in the literature is whether children can learn about using these tools and about analysing situations mathematically at the same time. Research about interpreting tables and graphs has been carried out either to assess students' previous knowledge (or misconceptions) before they are taught or to test ways of making them literate in the use of these tools.

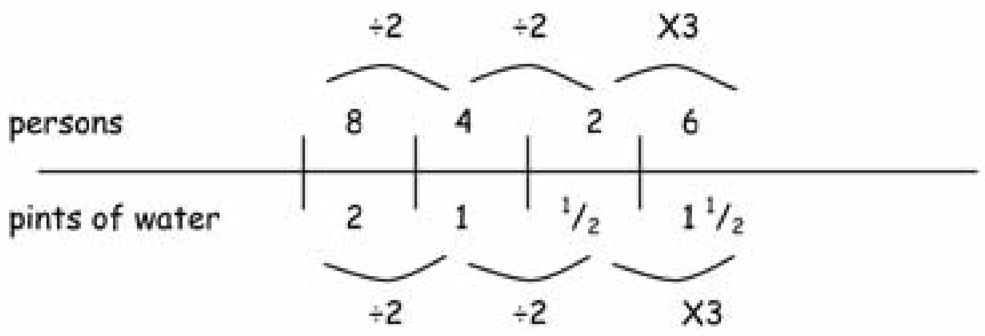
A remarkable exception is found in the work of researchers in the Freudenthal Institute. One of their explicit aims for instruction in mathematics is to help students mathematise situations: i.e. to help them build a model of a situation and later transform it into a model for other situations through their awareness of the relations in the model. They argue that we need to use diagrams, tables and graphs during the process of mathematising situations. These are built by students (with teacher guidance) as they explore the situations rather than presented to the students ready made for interpretation. Students are encouraged to use their implicit knowledge of relations; by building these representations, they can become aware of which models they are using. The process of solution is thus not to choose an operation and calculate but to analyse the relations in the problem and work towards solution. This process allows the students to become aware of the relations that are conserved throughout the different steps.

Streefland worked out in detail how this process would work if students were asked to solve Hart's famous onion soup recipe problem. In this problem, students are presented with a recipe of onion soup for 8 people and asked how much of each ingredient they would need if they were preparing the soup for 6 people. Many students use their everyday knowledge of relations in searching for a solution: they think that you need half of the original recipe (which would serve 4) plus half of this (which would serve 2 people) in order to have a recipe for 6 people. This perfectly sound reasoning is actually a mixture of additive and multiplicative thinking: half of a recipe for 8 serves 4 people (multiplicative reasoning) and half of the latter serves 2 (multiplicative reasoning); 6 people is 2 more than 4 (additive); a recipe for 6 is the same as the recipe for 4 plus the recipe for 2 (additive).

Streefland and his colleagues suggested that diagrams and tables provide the sort of representation that helps students think about the relations in the problem. It is illustrated here by the ratio table showing how much water should be used in the soup. The table can be used to help students become aware that the first two steps in their reasoning are multiplicative: they divide the number of persons in half and also the amount of water in half. Additive reasoning does not work: the transformation from 8 to 4 people would mean subtracting 4 whereas the parallel transformation in the amount of water would be to subtract 1. So the relation is not the same. If they can discover that multiplicative reasoning preserves the relation, whereas additive reasoning does not, they could be encouraged to test whether there is a multiplicative relation that they can use to find the recipe for 6; they could come up with x3, trebling the recipe for 2. Streefland's ratio table can be used as a model for testing if other situations fit this sort of multiplicative reasoning. The table can be expanded to calculate the amounts of the other ingredients.

An alternative approach in curriculum development is to start from formalisations and not to base teaching on students' informal knowledge. The aim of this approach is to establish links between different formal representations of the same relations. A programme proposed by Adjiage and Pluvinage starts with lines divided into segments: students learn how to represent segments with the same fraction even though the lengths of the lines differ (e.g. 3/5 of lines of different lengths). Next they move to using these formal representations in other types of problems: for example, mixtures of chocolate syrup and milk where the number of cups of each ingredient differs but the ratio of chocolate to total number of cups is the same. Finally, students are asked to write abstractions that they learned in these situations and formulate rules for solving the problems that they solved during the lessons. An example of generalisation expected is 'seven divided by four is equal to seven fourths' or '7 ÷ 4 = 7/4'. An example of a rule used in problem solving would be 'Given an enlargement in which a 4 cm length becomes a 7 cm length, then any length to be enlarged has to be multiplied by 7/4.'

There is no systematic research that compares these two very different approaches. Such research would provide valuable insight into how children come to understand relations.



Recommendations

Research about mathematical learning	Recommendations for teaching and research
Numbers are used to represent quantities and relations. Primary school children often interpret statements about relations as if they were about quantities and thus make mistakes in solving problems.	**Teaching** Teachers should be aware of children's difficulties in distinguishing between quantities and relations during problem solving.
Many problem situations involve both additive and multiplicative relations; which one is used to solve a problem depends on the question being asked. Both children and adults can make mistakes in selecting additive or multiplicative reasoning to answer a question.	**Teaching** The primary school curriculum should include the study of relations in situations in a more explicit way. **Research** Evidence from experimental studies is needed on which approaches to making students aware of relations in problem situations improve problem solving.
Experts use diagrams, tables and graphs to explore the relations in a problem situation before solving a problem.	**Teaching** The use of tables and graphs in the classroom may have been hampered by the assumption that students must first be literate in interpreting these representations before they can be used as tools. Teachers should consider using these tools as part of the learning process during problem solving. **Research** Systematic research on how students use diagrams, tables and graphs to represent relations during problem solving and how this impacts their later learning is urgently needed. Experimental and longitudinal methods should be combined.
Some researchers propose that informal knowledge interferes with students' learning. They propose that teaching should start from formalisations which are only later applied to problem situations.	**Teaching** Teachers who start from formalisations should try to promote links across different types of mathematical representations through teaching. **Research** There is a need for experimental and longitudinal studies designed to investigate the progress that students make when teaching starts from formalisations rather than from students' informal knowledge and the long-term consequences of this approach to teaching students about relations.

Understanding relations and their graphical representation

Relations and their importance in mathematics

In our analysis of how children come to understand natural and rational numbers, we examined the connections that children need to make between quantities and numbers in order to understand what numbers mean. Numbers are certainly used to represent quantities, but they are also used to represent relations. The focus of this section is on the use of numbers to represent relations. Relations do not have to be quantified: we can simply say, for example, that two quantities are equivalent or different. This is a qualitative statement about the relation between two quantities. But relations can be quantified also: if there are 20 children in the class and 17 books, we can say that there are 3 more children than books. The number 3 quantifies the additive relation between 17 and 20 and so we can say that 3 quantifies a relation.

When we use numbers to represent quantities, the numbers are the result of a measurement operation. Measures usually rely on culturally developed systems of representation. In order to measure discrete quantities, we count their units, and in order to measure continuous quantities, we use systems that have been set up to allow us to represent them by a number of conventional units. Measures are usually described by a number followed by a noun, which indicates the unit of quantity the number refers to: 5 children, 3 centimetres, 200 grams. And we can't replace the noun with another noun without changing what we are talking about. When we quantify a relation, the number does not refer to a quantity. We can say '3 more children than books' or '3 books fewer than children': it makes no difference which noun comes after the number because the number refers to the relation between the two quantities, how many more or fewer.

When we use qualitative statements about the relations between two quantities, the quantities may or may not have been expressed numerically. For example, we can look at the children and the books in the class and know that there are more children without counting them, especially if the difference is quite large. So we can say that there are more children than books without knowing how many children or how many books. But in order to quantify a relation between two quantities, the quantities need to be measurable, even if, in the case of differences, we can evaluate the relationship without actually measuring them. The ability to express the relationship quantitatively, without knowing the actual measures, is one of the roots of algebra (see Paper 5). For this reason, we will often use the term 'measures' in this section, instead of 'quantities', to refer to quantities that are represented numerically.

It is perfectly possible that when children first appear to succeed in quantifying relations, they are actually still thinking about quantities: when they say '3 children more than books', they might be thinking of the poor little things who won't have a book when the teacher shares the books out, not of the relation between the number of books and the number of children. This hypothesis is consistent with results of studies by Hudson (1983), described in Paper 2: young children are quite able to answer the question 'how many birds won't get worms' but they can't tell 'how many more birds than worms'. We, as adults, may think that they understand something about relations when they answer the first question, but they may be talking about quantities, i.e. the number of birds that won't get worms.

There is no doubt to us that children must grasp how numbers and quantities are connected in order to understand what numbers mean. But mathematics is not only about representing quantities with numbers. A major use of mathematics is to manipulate numbers that represent relations and arrive at conclusions without having to operate directly on the quantities. Attributing a number to a quantity is measuring; quantifying relations and manipulating them is quantitative reasoning. To quote Thompson (1993): 'Quantitative reasoning is the analysis of a situation into a quantitative structure – a network of quantities and quantitative relationships… A prominent characteristic of reasoning quantitatively is that numbers and numeric relationships are of secondary importance, and do not enter into the primary analysis of a situation. What is important is relationships among quantities' (p. 165). Elsewhere, Thompson (1994) emphasised that 'a quantitative operation is nonnumerical; it has to do with the *comprehension* [italics in the original] of a situation. Numerical operations (which we have termed measurement operations) are used to evaluate a quantity' (p. 187–188).

In order to reach the right conclusions in quantitative reasoning, one must use an appropriate representation of the relations between the quantities, and the representation depends on what we want to know about the relation between the quantities. Suppose you want to know whether you are paying more for your favourite chocolates at one shop than another, but the boxes of chocolates in the two shops are of different sizes. Of course the bigger box costs more money, but are you paying more for each chocolate? You don't know unless you quantify the relation between price and chocolates. This relation, price per chocolate, is not quantified in the same way as the relation 'more children than books'. When you want to know how many children won't have books, you subtract the number of books from the number of children (or vice versa). When you want to know the price per chocolate, you shouldn't subtract the number of chocolates from the price (or vice versa); you should divide the price by the number of chocolates. Quantifying relations depends on the nature of the question you are asking about the quantities. If you are asking how many more, you use subtraction; if you are asking a rate question, such as price per chocolate, you use division. So quantifying relations can be done by additive or multiplicative reasoning. Additive reasoning tells us about the difference between quantities; multiplicative reasoning tells us about the ratio between quantities. The focus of this section is on multiplicative reasoning but a brief discussion of additive relations will be included at the outset to illustrate the difficulties that children face when they need to quantify and operate on relations. However, before we turn to the issue of quantification of relations, we want to say why we use the terms additive and multiplicative reasoning, instead of speaking about the four arithmetic operations.

Mathematics educators (e.g. Behr, Harel, Post and Lesh, 1994; Steffe, 1994; Vergnaud, 1983) include under the term 'additive reasoning' those problems that are solved by addition and subtraction and under the term 'multiplicative reasoning' those that are solved by multiplication and division. This way of thinking, focusing on the problem structure rather than on the arithmetic operations used to solve problems, has become dominant in mathematics education research in the last three decades or so. It is based on some assumptions about how children learn mathematics, three of which are made explicit here. First, it is assumed that in order to understand addition and subtraction properly, children must also understand the inverse relation between them; similarly, in order to understand multiplication and division, children must understand that they also are the inverse of each other. Thus a focus on specific and separate operations, which was more typical of mathematics education thinking in the past, is justified only when the focus of teaching is on computation skills. Second, it is assumed that the links between addition and subtraction, on one hand, and multiplication and division, on the other, are conceptual: they relate to the connections between quantities within each of these domains of reasoning. The connections between addition and multiplication and those between subtraction and division are procedural: you can multiply by carrying out repeated additions and divide by using repeated subtractions. Finally, it is assumed that, in spite of the procedural links between addition and multiplication, these two forms of reasoning are distinct enough to be considered as separate conceptual domains. So we will use the terms additive and multiplicative reasoning and relations rather than refer to the arithmetic operations.

Quantifying additive relations

The literature about additive reasoning consistently shows that compare problems, which involve relations between quantities, are more difficult than those that involve combining sets or transformations. This literature was reviewed in Paper 1. Our aim in taking up this theme again here is to show that there are three sources of difficulties for students in quantifying additive relations:

- to interpret relational statements as such, rather than to interpret them as statements about quantities
- to transform relational statements into equivalent statements which help them think about the problem in a different way
- to combine two relational statements into a third relational statement without falling prey to the temptation of treating the result as a statement about a quantity.

This discussion in the context of additive reasoning illustrates the role of relations in quantitative reasoning. The review is brief and selective, because the main focus of this section is on multiplicative reasoning.

Interpreting relational statements as quantitative statements

Compare problems involve two quantities and a relation between them. Their general format is: A had x; B has y; the relation between A and B is z. This allows for creating a number of different compare problems. For example, the simplest compare problems are of the form: Paul has 8 marbles; Alex had 5 marbles; how many more does Paul have than Alex? or How many fewer does Alex have than Paul? In these problems, the quantities are known and the relation is the unknown.

Carpenter, Hiebert and Moser (1981) observed that 53% of the first grade (estimated age about 6 years) children that they assessed in compare problems answered the question 'how many more does A have than B' by saying the number that A has. This is the most common mistake reported in the literature: the relational question is answered as a quantity mentioned in the problem. The explanation for this error cannot be children's lack of knowledge of addition and subtraction, because about 85% of the same children used correct addition and subtraction strategies when solving problems that involved joining quantities or a transformation of an initial quantity. Carpenter and Moser report that many of the children did not seem to know what to do when asked to solve a compare problem.

Transforming relational statements into equivalent relational statements

Compare problems can also state how many items A has, then the value of the relation between A's and B's quantities, and then ask how much B has. Two problems used by Vershaffel (1994) will be used to illustrate this problem type. In the problem 'Chris has 32 books. Ralph has 13 more books than Chris. How many books does Ralph have?', the relation is stated as '13 more books' and the answer is obtained by addition; this problem type is referred to by Lewis and Mayer (1987) as involving consistent language. In the problem 'Pete has 29 nuts. Pete has 14 more nuts than Rita. How many nuts does Rita have?', the relation is stated as '14 more nuts' but the answer is obtained by subtraction; this problem types is referred to as involving inconsistent language. Verschaffel found that Belgian students in sixth grade (aged about 12) gave 82% correct responses to problems with consistent language and 71% correct responses to problems with inconsistent language. The operation itself, whether it was addition or subtraction, did not affect the rate of correct responses.

Lewis and Mayer (1987) have argued that the rate of correct responses to relational statements with consistent or inconsistent language varies because there is a higher cognitive load in processing inconsistent sentences. This higher cognitive load is due to the fact that the subject of the sentence in the question 'how many nuts does Rita have?' is the object of the relational sentence 'Pete has 14 more nuts than Rita'. It takes more effort to process these two sentences than other two, in which the subject of the question is also the subject of the relational statement. They provided some evidence for this hypothesis, later confirmed by Verschaffel (1994), who also asked the students in his study to retell the problem after the students had already answered the question.

In the problems where the language was consistent, almost all the students who gave the right answer simply repeated what the researcher had said: there was no need to rephrase the problem. In the problems where the language was inconsistent, about half of the students (54%) who gave correct answers retold the problem by rephrasing it appropriately. Instead of saying that 'Pete has 14 nuts more than Rita', they said that 'Rita has 14 nuts less than Pete', and thus made Rita into the subject of both sentences. Verschaffel interviewed some of the students who had used this correct rephrasing by showing them the written problem that he had read and asking them whether they had said the same thing. Some said that they changed the phrase intentionally because it was easier to think about the question in this way; they stressed that the meaning of the two sentences was the same. Other students became confused, as if they had said something wrong, and were no longer certain of their answers. In conclusion, there is evidence that at least some students do reinterpret the sentences as hypothesised by Lewis and Mayer; some do this explicitly and others implicitly. However, almost as many students reached correct answers without seeming to rephrase the problem, and may not experience the extra cognitive load predicted.

It is likely that, under many conditions, we rephrase relational statements when solving problems. So two significant findings arise from these studies:

- rephrasing relational statements seems to be a strategy used by some people, which may place extra cognitive demands on the problem solver but nevertheless helps in the search for a solution

- rephrasing may be done intentionally and explicitly, as a strategy, but may also be carried out implicitly and apparently unintentionally, producing uncertainty in the problem solvers' minds if they are asked about the rephrasing.

Combining relational statements into a third relational statement

Compare problems typically involve two quantities and a relation between them but it is possible to have problems that require children to work with more quantities and relations than these simpler problems. In these more complex problems, it may be necessary to combine two relational statements to identify a third one.

Thompson (1993) analysed students' reasoning in complex comparison problems which involved at least three quantities and three relations. His aim was to see how children interpreted complex relational problems and how their reasoning changed as they tackled more problems of the same type. To exemplify his problems, we quote the first one: 'Tom, Fred, and Rhoda combined their apples for a fruit stand. Fred and Rhoda together had 97 more apples than Tom. Rhoda had 17 apples. Tom had 25 apples. How many apples did Fred have?' (p. 167). This problem includes three quantities (Tom's, Fred's and Rhoda's apples) and three relations (how many more Fred and Rhoda have than Tom; how many fewer Rhoda has than Tom; a combination of these two relations). He asked six children who had achieved different scores in a pre-test (three with higher and three with middle level scores) sampled from two grade levels, second (aged about seven) and fifth (aged about nine) to discuss six problems presented over four different days. The children were asked to think about the problems, represent them and discuss them.

On the first day the children went directly to trying out calculations and represented the relations as quantities: the statement '97 more apples than Tom' was interpreted as '97 apples'. They did not know how to represent '97 more'. This leads to the conclusion that Fred has 80 apples because Rhoda has 17. On the second day, working with problems about marbles won or lost during the games, the researcher taught the children to use representations by writing, for example, 'plus 12' to indicate that someone had won 12 marbles and 'minus 1' to indicate that someone had lost 1 marble. The children were able to work with these representations with the researcher's support, but when they combined two statements, for example minus 8 and plus 14, they thought that the answer was 6 marbles (a quantity), instead of plus six (a relation). So at first they represented relational statements as statements about quantities, apparently because they did not know how to represent relations. However, after having learned how to represent relational statements, they continued to have difficulties in thinking only relationally, and unwittingly converted the result of operations on relations into statements about quantities. Yet, when asked whether it would always be true that someone who had won 2 marbles in a game would have 2 marbles, the children recognised that this would not necessarily be true. They did understand that relations and quantities are different but they interpreted the result of combining two relations as a quantity.

Thompson describes this tension between interpreting numbers as quantities or relations as the major difficulty that the students faced throughout his study. When they seemed to understand 'difference' as a relation between two quantities arrived at by subtraction, they found it difficult to interpret the idea of 'difference' as a relation between two relations. The children could correctly answer, when asked, that if someone has 2 marbles more than another person, this does not mean that he has two marbles; however, after combining two relations (minus 8 and plus 14), instead of saying that this person ended up with plus 6 marbles, they said that he now had 6 marbles.

Summary

1 At first, children have difficulties in using additive reasoning to quantify relations; when asked about a relation, they answer about a quantity.

2 Once they seem to conquer this, they continue to find it difficult to combine relations and stay within relational reasoning: the combination of two relations is often converted into a statement about a quantity.

3 So children's difficulties with relations are not confined to multiplicative reasoning: they are also observed in the domain of additive reasoning.

Quantifying multiplicative relations

Research on how children quantify multiplicative relations has a long tradition. Piaget and his colleagues (Inhelder and Piaget, 1958; Piaget and Inhelder, 1975) originally assumed that children first think of quantifying relations additively and can only think of relations multiplicatively at a later age. This hypothesis led to the prediction of an 'additive phase' in children's solution to multiplicative reasoning problems, before they would be able to conceive of two variables as linked by a multiplicative relation. This hypothesis led to much research on the development of proportional reasoning, which largely supported the claim that many younger students offer additive solutions to proportions problems (e.g. Hart, 1981 b; 1984; Karplus and Peterson, 1970; Karplus, Pulos and Stage, 1983; Noelting, 1980 *a* and *b*). These results are not disputed but their interpretation will be examined in the next sections of this paper because current studies suggest an alternative interpretation.

Work carried out mostly by Lieven Verschaffel and his colleagues (e.g. De Bock, Verschaffel *et al.*, 2002; 2003) shows that students also make the converse mistake, and multiply when they should be adding in order to solve some relational problems. This type of error is not confined to young students: pre-service elementary school teachers in the United States (Cramer, Post and Currier, 1993) made the same sort of mistake when asked to solve the problem: *Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run?* The relation between Sue's and Julie's numbers of laps should be quantified additively: because they were running at the same speed, this difference would (in principle) be constant. However, 32 of 33 pre-service teachers answered 45 (15 × 3), apparently using the ratio between the first two measures (9 and 3 laps) to calculate Sue's laps. This latter type of mistake would not be predicted by Piaget's theory.

The hypothesis that we will pursue in this chapter, following authors such as Thompson 1994) and Vergnaud (1983), is that additive and multiplicative reasoning have different origins. Additive reasoning stems from the actions of joining, separating, and placing sets in one-to-one correspondence. Multiplicative reasoning stems from the action of putting two variables in one-to-many correspondence (one-to-one is just a particular case), an action that keeps the ratio between the variables constant. Thompson (1994) made this point forcefully in his discussion of quantitative operations: 'Quantitative operations originate in actions: The quantitative operation of combining two quantities additively originates in the actions of putting together to make a whole and separating a whole to make parts; the quantitative operation of comparing two quantities additively originates in the action of matching two quantities with the goal of determining excess or deficits; the quantitative operation of comparing two quantities multiplicatively originates in matching and subdividing with the goal of sharing. As one interiorizes actions, making mental operations, these operations in the making imbue one with the ability to comprehend situations representationally and enable one to draw inferences about numerical relationships that are not present in the situation itself' (pp. 185–186).

We suggest that, if students solve additive and multiplicative reasoning problems successfully but they are guided by implicit models, they will find it difficult to distinguish between the two models. According to Fischbein (1987), implicit models and informal reasoning provide a starting point for learning, but one of the aims of mathematics teaching in primary school is to help students formalize their informal knowledge (Treffers, 1987). In this process, the models will change and become more explicitly connected to the systems of representations used in mathematics.

In this section, we analyse how students establish and quantify relations between quantities in multiplicative reasoning problems. We first discuss the nature of multiplicative reasoning and present research results that describe how children's informal knowledge of multiplicative relations develops. In the subsequent section, we discuss the representation of multiplicative relations in tables and graphs. Next we analyse how children establish other relations between measures, besides linear relations. The final section sets out some hypotheses about the nature of the difficulty in dealing with relations in mathematics and a research agenda for testing current hypotheses systematically.

The development of multiplicative reasoning

Multiplicative reasoning is important in many ways in mathematics learning. Its role in understanding numeration systems with a base and place value was already discussed in Paper 2. In this section, we focus on a different role of multiplicative reasoning in mathematics learning, its role in understanding relations between measures or quantities, which has already been recognised by different researchers (e.g. Confrey, 1994; Thompson, 1994; Vergnaud, 1983; 1994).

Additive and multiplicative reasoning problems are essentially different: additive reasoning is used in one-variable problems, when quantities of the same kind are put together, separated or compared, whereas multiplicative reasoning involves two variables in a fixed-ratio to each other. Even the simplest multiplicative reasoning problems involve two variables in a fixed ratio. For example, in the problem 'Hannah bought 6 sweets; each sweet costs 5 pence; how much did she spend?' there are two variables, number of sweets and price per sweet. The problem would be solved by a multiplication if, as in this example, the total cost is unknown. The same problem situation could be presented with a different unknown quantity, and would then be solved by means of a division: 'Hannah bought some sweets; each sweet costs 5p; she spent 30p; how many sweets did she buy?'

Even before being taught about multiplication and division in school, children can solve multiplication and division problems such as the one about Hannah. They use the schema of one-to-many correspondence.

Different researchers have investigated the use of one-to-many correspondences by children to solve multiplication and division problems before they are taught about these operations in school. Piaget's work (1952), described in Paper 3, showed that children can understand multiplicative equivalences: they can construct a set A equivalent to a set B by putting the elements in A in the same ratio that B has to a comparison set.

Frydman and Bryant (1988; 1994) also showed that young children can use one-to-many correspondences to create equivalent sets. They used sharing in their study because young children seem to have much experience with correspondence when sharing. In a sharing situation, children typically use a one-for-you one-for-me procedure, setting the shared elements (sweets) into one-to-one correspondence with the recipients (dolls). Frydman and Bryant observed that children in the age range five to seven years became progressively more competent in dealing with one-to-many correspondences and equivalences in this situation. In their task, the children were asked to construct equivalent sets but the units in the sets were of a different value. For example, one doll only liked her sweets in double units and the second doll liked his sweets in single units. The children were able to use one-to-many correspondence to share fairly in this situation: when they gave a double to the first doll, they gave two singles to the second. This flexible use of correspondence to construct equivalent sets was interpreted by Frydman and Bryant as an indication that the children's use of the procedure was not merely a copy of previously observed and rehearsed actions: it reflected an understanding of how one-to-many correspondences can result in equivalent sets. They also replicated one of Piaget's previous findings: some children who succeed with the 2:1 ratio found the 3:1 ratio difficult. So the development of the one-to-many correspondence schema does not happen in an all-or-nothing fashion.

Kouba (1989) presented young children in the United States, in first, second and third grade (aged about six to eight years), with multiplicative reasoning problems that are more typical of those used in school; for example: in a party, there were 6 cups and 5 marshmallows in each cup; how many marshmallows were there?

Kouba analysed the children's strategies in great detail, and classified them in terms of the types of actions used and the level of abstraction. The level of abstraction varied from direct representation (i.e. all the information was represented by the children with concrete materials), through partial representation (i.e. numbers replaced concrete representations for the elements in a group and the child counted in groups) up to the most abstract form of representation available to these children, i.e. multiplication facts.

For the children in first and second grade, who had not received instruction on multiplication and division, the most important factor in predicting the children's solutions was which quantity was unknown. For example, in the problem above, about the 6 cups with 5 marshmallows in each cup, when the size of the groups was known (i.e. the number of marshmallows in each cup), the children used correspondence strategies: they paired objects (or tallies to represent the objects) and counted or added, creating one-to-many correspondences between the cups and the marshmallows. For example, if they needed to find the total number of marshmallows, they pointed 5 times to a cup (or its representation) and counted to 5, paused, and then counted from 6 to 10 as they pointed to the second 'cup', until they reached the solution. Alternatively, the may have added as they pointed to the 'cup'.

In contrast, when the number of elements in each group was not known, the children used dealing strategies: they shared out one marshmallow (or its representation) to each cup, and then another, until they reached the end, and then counted the number in each cup. Here they sometimes used trial-and-error: they shared more than one at a time and then might have needed to adjust the number per cup to get to the correct distribution.

Although the actions look quite different, their aims are the same: to establish one-to-many correspondences between the marshmallows and the cups.

Kouba observed that 43% of the strategies used by the children, including first, second, and third graders, were appropriate. Among the first and second grade children, the overwhelming majority of the appropriate strategies was based on correspondences, either using direct representation or partial representation (i.e. tallies for one variable and counting or adding for the other); few used recall of multiplication facts. The recall of number facts was significantly higher after the children had received instruction, when they were in third grade.

The level of success observed by Kouba among children who had not yet received instruction is modest, compared to that observed in two subsequent studies, where the ratios were easier. Becker (1993) asked kindergarten children in the United States, aged four to five years, to solve problems in which the correspondences were 2:1 or 3:1. As reported by Piaget and by Frydman and Bryant, the children were more successful with 2:1 than 3:1 correspondences, and the level of success improved with age. The overall level of correct responses by the five-year-olds was 81%.

Carpenter, Ansell, Franke, Fennema and Weisbeck (1993) also gave multiplicative reasoning problems to U.S. kindergarten children involving correspondences of 2:1, 3:1 and 4:1. They observed 71% correct responses to these problems.

The success rates leave no doubt that many young children start school with some understanding of one-to-many correspondence, which they can use to learn to solve multiplicative reasoning problems in school. These results do not imply that children who use one-to-many correspondence to solve multiplicative reasoning problems consciously recognise that in a multiplicative situation there is a fixed ratio linking the two variables. Their actions maintain the ratio fixed but it is most likely that this invariance remains, in Vergnaud's (1997) terminology, as a 'theorem in action'.

The importance of informal knowledge

Both Fischbein (1987) and Treffers (1987) assumed that children's informal knowledge is a starting point for learning mathematics in school but it is important to consider this assumption further. If children start school with some informal knowledge that can be used for learning mathematics in school, it is necessary to consider whether this knowledge facilitates their learning or, quite the opposite, is an obstacle to learning. The action of establishing one-to-many correspondences is not the same as the concept of ratio or as multiplicative reasoning: ratio may be implicit in their actions but it is possible that the children are more aware of the methods that they used to figure out the numerical values of the quantities, i.e. they are aware of counting or adding.

Children's methods for solving multiplication problems can be seen as a starting point, if they form a basis for further learning, but also an obstacle to learning, if children stick to their counting and addition procedures instead of learning about ratio and multiplicative reasoning in school. Resnick (1983) and Kaput and West (1994) argue that an important lesson from psychological and mathematics education research is that, even after people have been taught new concepts and ideas, they still resort to their prior methods to solve problems that differ from the textbook examples on which they have applied their new knowledge. The implementation of the one-to-many correspondence schema to solve problems requires adding and counting, and students have been reported to resort to counting and adding even in secondary school, when they should be multiplying (Booth, 1981). So is this informal knowledge an obstacle to better understanding or does it provide a basis for learning?

It is possible that a precise answer to this question cannot be found: whether informal knowledge helps or hinders children's learning might depend on the pedagogy used in their classroom. However, it is possible to consider this question in principle by examining the results of longitudinal and intervention studies. If it is found in a longitudinal study that children who start school with more informal mathematical knowledge achieve better mathematics learning in school, then it can be concluded that, at least in a general manner, informal knowledge does provide a basis for learning. Similarly, if intervention studies show that increasing children's informal knowledge when they are in their first year school has a positive impact on their school learning of mathematics, there is further support for the idea that informal knowledge can offer a foundation for learning. In the case of the correspondence schema, there is clear evidence from a longitudinal study but intervention studies with the appropriate controls are still needed.

Nunes, Bryant, Evans, Bell, Gardner, Gardner and Carraher (2007) carried out the longitudinal study. In this study, British children were tested on their understanding of four aspects of logical-mathematical reasoning at the start of school; one of these was multiplicative reasoning. There were five items which were multiplicative reasoning problems that could be solved by one-to-many correspondence. The children were also given the British Abilities Scale (BAS-II; Elliott, Smith and McCulloch, 1997) as an assessment of their general cognitive ability and a Working Memory Test, Counting Recall (Pickering and Gathercole, 2001), at school entry. At the beginning of the study, the children's age ranged from five years and one month to six years and six months. About 14 months later, the children were given a state-designed and teacher-administered mathematics achievement test, which is entirely independent of the researchers and an ecologically valid measure of how much they have learned in school. The children's performance in the five items on correspondence at school entry was a significant predictor of their mathematics achievement, after controlling for: (1) age at the time of the achievement test; (2) performance on the BAS-II excluding the subtest of their knowledge of numbers at school start; (3) knowledge of number at school entry (a subtest of the BAS-II); (4) performance on the working memory measure; and (5) performance on the multiplicative reasoning, one-to-many correspondence items. Nunes *et al.* (2007) did not report the analysis of longitudinal prediction based separately on the items that assess multiplicative reasoning; so these results are reported here. The results are presented visually in Figure 4.1 and described in words subsequently.

The total variance explained in the mathematics achievement by these predictors was 66%; age explained 2% (non significant), the BAS general score (excluding the Number Skills subtest) explained a further 49% ($p < 0.001$); the sub-test on number skills explained a further 6% ($p < 0.05$); working memory explained a further 4% ($p < 0.05$), and the children's understanding of multiplicative reasoning at school entry explained a further 6% ($p = 0.005$). This result shows that children's understanding of multiplicative reasoning at school entry is a specific predictor of mathematics achievement in the first two years of school. It supports the hypothesis that, in a general way, this informal knowledge forms a basis for their school learning of mathematics: after 14 months and after controlling for general cognitive factors at school entry, performance on an assessment of multiplicative reasoning still explained a significant amount of variance in the children's mathematics achievement in school.

It is therefore quite likely that instruction will be an important factor in influencing whether students continue to use the one-to-many schema of action to solve such problems, even if replacing objects with numbers but still counting or adding instead of multiplying, or whether they go on to adopt the use of the operations of multiplication and division. Treffers (1987) and Gravemeijer (1997) argue that students do and should use their informal knowledge in the classroom when learning about multiplication and division, but that it should be one of the aims of teaching to help them formalise this knowledge, and in the process develop a better understanding of the arithmetic operations themselves. We do not review this work here but recognise the importance of their argument, particularly in view of the strength of this informal knowledge and students' likelihood of using it even after having been taught other forms of knowledge in school. However, it must be pointed out that there is no evidence that teaching students about arithmetic operations makes them more aware of the invariance of the ratio when they use one-to-many correspondences to solve problems. Kaput and West (1994) also designed a teaching programme which aimed at using students' informal knowledge of correspondences to promote their understanding of multiplicative reasoning. In contrast to the programme designed by Treffers for the operation of multiplication and by Gravemeijer for the operation of division, Kaput and West's programme used simple calculations and tried to focus the students' attention on the invariance of ratio in the correspondence situations. They used different sorts of diagrams which treated the quantities in correspondence as composite units: for example, a plate and six pieces of tableware formed a single unit, a set-place for one person. The ideas proposed in these approaches to instruction are very ingenious and merit further research with the appropriate controls and measures. The lack of control groups and appropriate pre- and post-test assessments in these intervention studies makes it difficult to reach conclusions regarding the impact of the programmes.

Park and Nunes (2001) carried out a brief intervention study where they compared children's success in multiplicative reasoning problems after the children had participated in one of two types of intervention. In the first, they were taught about multiplication as repeated addition, which is the traditional approach used in British schools and is based on the procedural connection between multiplication and addition. In the second intervention group, the children were taught about multiplication by considering one-to-many correspondence situations, where these

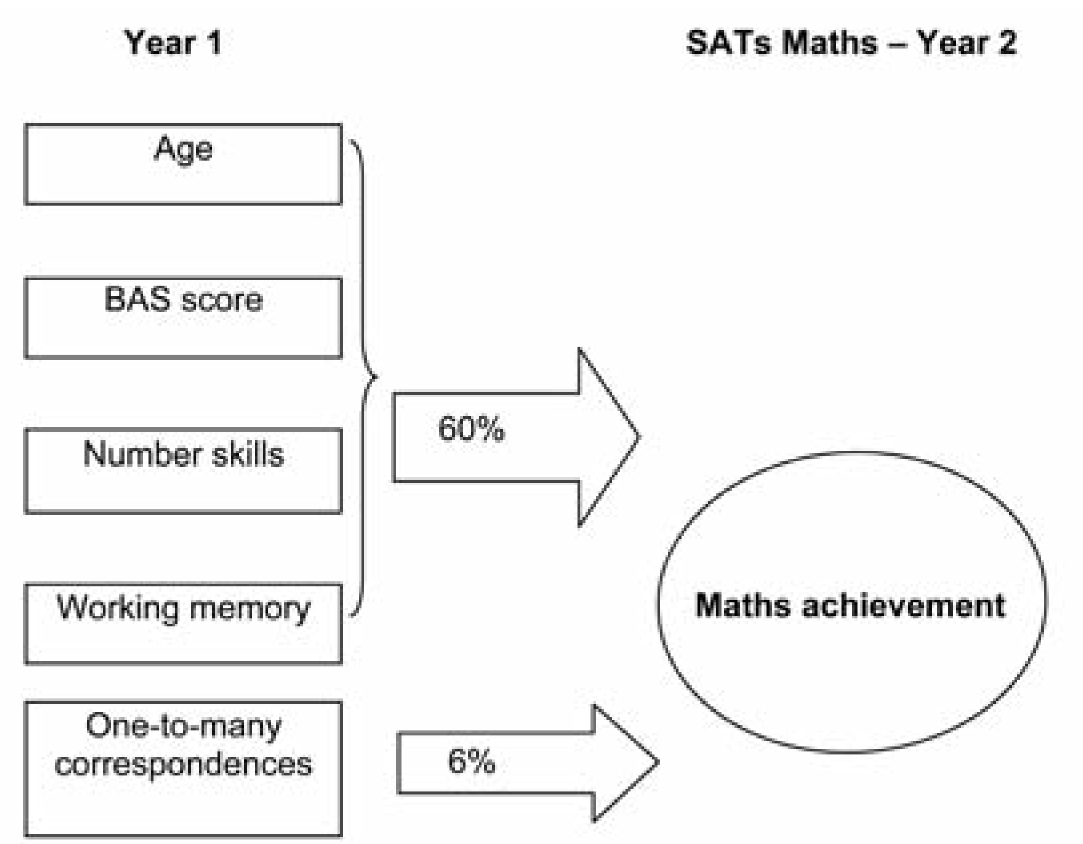


Figure 4.1: A schematic representation of the degree to which individual differences in mathematics achievement are explained by the first four factors and the additional amount of variance explained by children's informal knowledge at school entry.

correspondences were represented explicitly. A third group of children, the control group, solved addition and subtraction problems, working with the same experimenter for a similar period of time. The children in the one-to-many correspondence group made significantly more progress in solving multiplicative reasoning problems than those in the repeated addition and in the control group. This study does include the appropriate controls and provides clear evidence for more successful learning of multiplicative reasoning when instruction draws on the children's appropriate schema of action. However, this was a very brief intervention with a small sample and in one-to-one teaching sessions. It would be necessary to replicate it with larger numbers of children and to compare its level of success with other interventions, such as those used by Treffers and Gravemeijer, where the children's understanding of the arithmetic operations of multiplication and division was strengthened by working with larger numbers.

Summary

1 Additive and multiplicative reasoning have their origins in different schemas of action. There does not seem to be an order of acquisition, with young children understanding at first only additive reasoning and only later multiplicative reasoning. Children can use schemas of action appropriately both in additive and multiplicative reasoning situations from an early age.

2 The schemas of one-to-many correspondence and sharing (or dealing) allow young children to succeed in solving multiplicative reasoning problems before they are taught about multiplication and division in school.

3 There is evidence that children's knowledge of correspondences is a specific predictor of their mathematics achievement and, therefore, that their informal knowledge can provide a basis for further learning. However, this does not mean that they understand the concept of ratio: the invariance of ratio in these situations is likely to be known only as a theorem in action.

4 Two types of programmes have been proposed with the aim of bridging students' informal and formal knowledge. One type (Treffers, 1987; Gravemeijer, 1997) focuses on teaching the children more about the operations of multiplication and division, making a transition from small to large numbers easier for the students. The second type (Kaput and West, 1994; Park and Nunes, 2001) focused on making the students more aware of the schema of one-to-many correspondences and the theorems in action that it represents implicitly. There is evidence that, with younger children solving small number problems, an intervention that focuses on the schema of correspondences facilitates the development of multiplicative reasoning.

Finally, it is pointed out that all the examples presented so far dealt with problems in which the children were asked questions about quantities. None of the problems focused on the relation between quantities. In the subsequent section, we present a classification of multiplicative reasoning problems in order to aid the discussion of how quantities and relations are handled in the context of multiplicative reasoning problems.

Different types of multiplicative reasoning problems

We argued previously that many children solve problems that involve additive relations, such as compare problems, by thinking only about quantities. In this section, we examine different types of multiplicative reasoning problems and analyse students' problem solving methods with a view to understanding whether they are considering only quantities or relations in their reasoning. In order to achieve this, it is necessary to think about the different types of multiplicative reasoning problems.

Classifications of multiplicative reasoning situations vary across authors (Brown, 1981; Schwartz, 1988; Tourniaire and Pulos, 1985; Vergnaud, 1983), but there is undoubtedly agreement on what characterises multiplicative situations: in these situations there are always two (or more) variables with a fixed ratio between them. Thus, it is argued that multiplicative reasoning forms the foundation for children's understanding of proportional relations and linear functions (Kaput and West, 1994; Vergnaud, 1983).

The first classifications of problem situations considered distinct possibilities: for example, rate and ratio problem situations were distinguished initially. However, there seemed to be little agreement amongst researchers regarding which situations should be classified as rate and which as ratio. Lesh, Post and Behr (1988) wrote some time ago: 'there is disagreement about the essential characteristics that distinguish, for example rates from ratios… In fact, it is common to find a given author changing terminology from one publication to another' (p. 108). Thompson (1994) and Kaput and West (1994) consider this distinction to apply not to situations, but to the mental operations that the problem solver uses. These different mental operations could be used when thinking about the same situation: ratio refers to understanding a situation in terms of the particular values presented in the problem (e.g. travelling 150 miles over 3 hours) and rate refers to understanding the constant relation that applies to any of the pairs of values (in theory, in any of the 3 hours one would have travelled 50 miles). 'Rate is a reflectively abstracted constant ratio' (Thompson, 1994, p. 192).

In this research synthesis, we will work with the classification offered by Vergnaud (1983), who distinguished three types of problems.

- In isomorphism of measures problems, there is a simple proportional relation between two measures (i.e. quantities represented by numbers): for example, number of cakes and price paid for the cakes, or amount of corn and amount of corn flour produced.

- In product of measures problems, there is a Cartesian composition between two measures to form a third measure: for example, the number of T-shirts and number of shorts a girl has can be composed in a Cartesian product to give the number of different outfits that she can wear; the number of different coloured cloths and the number of emblems determines the number of different flags that you can produce.

- In multiple proportions problems, a measure is in simple proportion to (at least) two other measures: for example, the consumption of cereal in a Scout camp is proportional to number of persons and the number of days.

Because this classification is based on measures, it offers the opportunity to explore the difference between a quantity and its measure. Although this may seem like a digression, exploring the difference between quantities and measures is helpful in this chapter, which focuses on the quantification of relations between measures. A quantity, as defined by Thompson (1993) is constituted when we think of a quality of an object in such a way that we understand the possibility of measuring it. 'Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them' (p. 166). Two quantities, area and volume, can be used here to illustrate the difference between quantities and measures.

Hart (1981 a) pointed out that the square unit can be used to measure area by different measurement operations. We can attribute a number to the area of a rectangle, for example, by covering it with square units and counting them: this is a simple measurement operation, based on iteration of the units. If we don't have enough bricks to do this (see Nunes, Light and Mason, 1993), we can count the number of square units that make a row along the base, and establish a one-to-many correspondence between the number of rows that fit along the height and the number of square units in each row. We can calculate the area of the rectangle by conceiving of it as an isomorphism of measures problem: 1 row corresponds to x units. If we attribute a number to the area of the rectangle by multiplying its base by its height, both measured with units of length, we are conceiving this situation as a product of measures: two measures, the length of the base and that of its height, multiplied produce a third measure, the area in square units. Thus a quantity in itself is not the same as its measure, and the way it is measured can change the complexity (i.e. the number of relations to be considered) of the situation.

Nunes, Light and Mason (1993) showed that children aged 9 to 10 years were much more successful when they compared the area of two figures if they chose to use bricks to measure the areas than if they chose to use a ruler. Because the children did not have sufficient bricks to cover the areas, most used calculations. They had three quantities to consider – the number of rows that covered the height, the number of bricks in each row along the base, and the area, and the relation between number of rows and number of bricks in the row. These children worked within an isomorphism of measures situation.

Children who used a ruler worked within a product of measures situation and had to consider three quantities – the value of the base, the value of the height, and the area; and three relations to consider – the relation between the base and the height, the relation between base and area, and the relation between the height and the area (the area is proportional to the base if the height is constant and proportional to the height if the base is constant).

The students who developed an isomorphism of measures conception of area were able to use their conception to compare a rectangular with a triangular area, and thus expanded their understanding of how area is measured. The students who worked within a product of measures situation did not succeed in expanding their knowledge to think about the area of triangles. Nunes, Light and Mason speculated that, after this initial move, students who worked with an isomorphism of measures model might subsequently be able to re-conceptualise area once again and move on to a product of measures approach, but they did not test this hypothesis.

Hart (1981 a) and Vergnaud (1983) make a similar point with respect to the measurement of volume: it can be measured by iteration of a unit (how many litres can fit into a container) or can be conceived as a problem situation involving the relations between base, height and width, and described as product of measures. Volume as a quantity is itself neither a uni-dimensional nor a three-dimensional measure and one measure might be useful for some purposes (add 2 cups of milk to make the pancake batter) whereas a different one might be useful for other purposes (the volume of a trailer in a lorry can be easily calculated by multiplying the base, the height and the width). Different systems of representation and different measurement operations allow us to attribute different numbers to the same quantity, and to do so consistently within each system.

Vergnaud's classification of multiplicative reasoning situations is used here to simplify the discussion in this chapter. We will focus primarily on isomorphism of measures situations, because the analysis of how this type of problem is solved by students of different ages and by schooled and unschooled groups will help us understand the operations of thought used in solving them.

A diagram of isomorphism of measures situations, presented in Figure 4.2 and adapted from Vergnaud (1983), will be used to facilitate the discussion.

This simple schema shows that there are two sets of relations that can be quantified in this situation:

- the relation between *a* and *c* is the same as that between *b* and *d*; this is the scalar relation, which links two values in the same measure space
- the relation between *a* and *b* is the same as that between *c* and *d*; this is the functional relation, or the ratio, which links the two measure spaces.

The psychological difference (i.e. the difference that it makes for the students) between scalar and functional relations is very important, and it has been discussed in the literature by many authors (e.g. Kaput and West, 1994; Nunes, Schliemann and Carraher, 1993; Vergnaud, 1983). It had also been discussed previously by Noelting (1980 *a* and *b*) and Tourniaire and Pulos (1985), who used the terms within and between quantities relations. This paper will use only the terms scalar and functional relations or reasoning.

'For a mathematician, a proportion is a statement of equality of two ratios, i.e., $a/b = c/d$' (Tourniaire and Pulos, 1985, p. 181). Given this definition, there is no reason to distinguish between what has been traditionally termed multiplication and division problems and proportions problems. We think that the distinction has been based, perhaps only implicitly, on the use of a ratio with reference to the unit in multiplication and division problems. So one

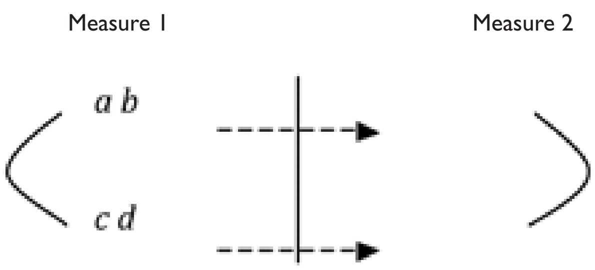


Figure 4.2: Schema of an isomorphism of measures situation. Measures 1 and 2 are connected by a proportional relation.

should not be surprised to see that one-to-many correspondences reasoning is used in the beginning of primary school by children to solve simple multiplicative reasoning problems and continues to be used by older students to solve proportions problems in which the unit ratio is not given in the problem description. Many researchers (e.g. Hart, 1981 b; Kaput and West, 1994; Lamon, 1994; Nunes and Bryant, 1996; Nunes, Schliemann and Carraher, 1993; Piaget, Grize, Szeminska and Bangh, 1977; Inhelder and Piaget, 1958; 1975; Ricco, 1982; Steffe, 1994) have described students' solutions based on correspondence procedures and many different terms have been used to refer to these, such as building up strategies, empirical strategies, halving or doubling, and replications of a composite unit. In essence, these strategies consist of using the initial values provided in the problem and changing them in one or more steps to arrive at the desired value. Hart's (1981 b) well known example of the onion soup recipe for 4 people, which has to be converted into a recipe for 6 people, illustrates this strategy well. Four people plus half of 4 makes 6 people, so the children take each of the ingredients in the recipe in turn, half the amount, and add this to the amount required for 4 people.

Students used yet another method in solving proportions problems, still related to the idea of correspondences: they first find the unit ratio and then use it to calculate the desired value. Although this method is taught in some countries (see Lave, 1988; Nunes, Schliemann and Carraher, 1993; Ricco, 1982), it is not necessarily used by all students after they have been taught; many students rely on building up strategies which change across different problems in terms of the calculations that are used, instead of using a single algorithm that aims at finding the unit ratio. Hart (1981 b) presented the following problem to a large sample of students (2257) aged 11 to 16 years in 1976: 14 metres of calico cost 63p; find the price of 24 metres. She reported that no child actually quoted the unitary method in their explanation, even though some children did essentially seek a unitary ratio. Ricco (1982), in contrast, found that some students explicitly searched for the unit ratio (e.g. 'First I need to know how much one notebook will cost and then we will see', p. 299, our translation) but others seem to search for the unit ratio without making explicit the necessity of this step in their procedure.

Building up methods and finding the unit ratio may be essentially an extension of the use of the one-to-many correspondence schema, which maintains the ratio invariant without necessarily bringing with it an awareness of the fixed relation between the variables. Unit ratio is a mathematical term but it is not clear whether the children who were explicitly searching for the price of one notebook in Ricco's study were thinking of ratio as the quantification of the relation between notebooks and money. When the child says '1 notebook costs 4 cents', the child is speaking about two quantities, not necessarily about the relation between them. A statement about the relation between the quantities would be 'the number of notebooks times 4 tells me the total cost'.

The use of these informal strategies by students in the solution of proportions problems is consistent with the hypothesis that multiplicative reasoning develops from the schema of one-to-many correspondences: students may be simply using numbers instead of objects when reasoning about the quantities in these problems. In the same way that they build up the quantities with objects, they can build up the quantities with numbers. It is unlikely that students are thinking of the scalar relation and quantifying it when they solve problems by means of building up strategies. We think that it can be concluded with some certainty that students realise that whatever transformation they make, for example, to the number of people in Hart's onion soup problem, they must also make to the quantities of ingredients. It is even less likely that they have an awareness of the ratio between the two domains of measures and have reached an understanding of a reflectively abstracted constant ratio, in Thompson's terms.

These results provoke the question of the role of teaching in developing students' understanding of functional relations. Studies of high-school students and adults with limited schooling in Brazil throw some light on this issue. They show that instruction about multiplication and division or about proportions *per se* is neither necessary for people to be able to solve proportions problems nor sufficient to promote students' thinking about functional relations. Nunes, Schliemann and Carraher (1993) have shown that fishermen and foremen in the construction industry, who have little formal school instruction, can solve proportions problems that are novel to them in three ways: (a) the problems use values that depart from the values they normally work with; (b) they are asked to calculate in a direction which they normally do not have to think about; or (c) the content of the problem is different from the problems with which they work in their everyday lives.

Foremen in the construction industry have to work with blue-prints as representations of distances in the buildings under construction. They have experience with a certain number of conventionally used scales (e.g. 1:50, 1:100 and 1:1000). When they were provided with a scale drawing that did not fit these specifications (e.g. 1:40) and did not indicate the ratio used (e.g. they were shown a distance on the blue-print and its value in the building), most foremen were able to use correspondences to figure out what the scale would be and then calculate the measure of a wall from its measure on the blueprint. They were able to do so even when fractions were involved in the calculations and the scale had an unexpected format (e.g. 3 cm:1 m uses different units whereas scales typically use the same unit) because they have extensive experience in moving across units (metres, centimetres and millimetres). Completely illiterate foremen (N = 4), who had never set foot in a school due to their life circumstances, showed 75% correct responses to these problems. In contrast, students who had been taught the formal method known as the rule of three, which involves writing an equation of the form $a/b = c/d$ and solving for the unknown value, performed significantly worse (60% correct). Thus schooling is not necessary for multiplicative reasoning to develop and proportions problems to be solved correctly, and teaching students a general formula to solve the problem is not a guarantee that they will use it when the opportunity arises.

These studies also showed that both secondary school students and adults with relatively little schooling were more successful when they could use building up strategies easily, as in problems of the type A in Figure 4.3. Problem B uses the same numbers but arranged in a way that building up strategies are not so easily implemented; the relation that is easy to quantify in problem B is the functional relation.

The difference in students' rate of success across the two types of problem was significant: they solved about 80% of type A problems correctly and only 35% of type B problems. For the adults (fishermen), there was a difference between the rate of correct responses (80% correct in type A and 75% correct in type B) but this was not statistically significant. Their success, however, was typically a result of prowess with calculations when building up a quantity, and very few answers might have resulted from a quantification of the functional relation.

These results suggest three conclusions.

- Reasoning about quantities when solving proportional problems seems to be an extension of correspondence reasoning; schooling is not necessary for this development.
- Most secondary school students seem to use the same schema of reasoning as younger students; there is little evidence of an impact of instruction on their approach to proportional problems.
- Functional reasoning is more challenging and is not guaranteed by schooling; teaching students a

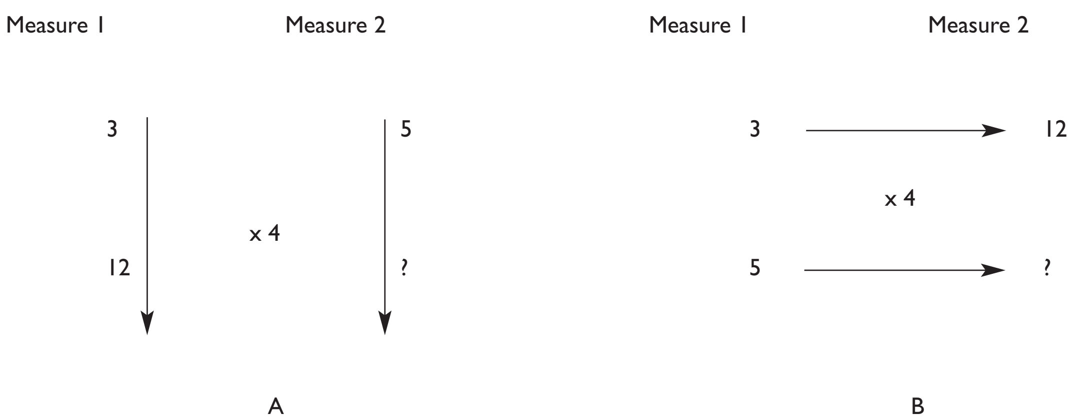


Figure 4.3: For someone who can easily think about scalar or functional relations, there should be no difference in the level of difficulty of the two problems. For those who use building up strategies and can only work with quantities, problem A is significantly easier.

formal method, which can be used as easily for both problem types, does not make functional reasoning easier (see Paper 6 for further discussion).

The results observed with Brazilian students do not differ from those observed by Vergnaud (1983) in France, and Hart (1981 b; 1984) in the United Kingdom. The novelty of these studies is the demonstration that the informal knowledge of multiplicative reasoning and the ability to solve multiplicative reasoning problems through correspondences develop into more abstract schemas that allow for calculating in the absence of concrete forms of representation, such as maniputatives and tallies. Both the students and the adults with low levels of schooling were able to calculate, for example, what should the actual distances in a building be from their size in blueprint drawings. Relatively unschooled adults who have to think about proportions in the course of their occupations and secondary school students seem to rely on these more abstract schemas to solve proportions problems. The similarity between these two groups, rather than the differences, in the forms of reasoning and rates of success is striking. These results suggest that informal knowledge of correspondences is a powerful thinking schema and that schooling does not easily transform it into a more powerful one by incorporating functional understanding into the schema.

Different hypotheses have been considered in the explanation of why this informal knowledge seems so resistant to change. Hart (1981 b) considered the possibility that this may rest on the difficulty of the calculations but the comparisons made by Nunes, Schliemann and Carraher (1993) rule out this hypothesis: the difficulty of the calculations was held constant across problems of type A and type B, and quantitative reasoning on the basis of the functional relation remained elusive.

An alternative explanation, explored by Vergnaud (1983) and Hart (personal communication), is that informal strategies are resistant to change because they are connected to reasoning about quantities, and not about relations. It makes sense to say that if I buy half as much fish, I pay half as much money: these are manipulations of quantities and their representations. But what sense does it make to divide kilos of fish by money?

There has been some discussion of the difference between reasoning about quantities and relations in the literature. However, we have not been able to find studies that establish whether the difficulty of thinking about relations might be at the root of students' difficulties in transforming their informal into formal mathematics knowledge. The educational implications of these hypotheses are considerable but there is, to our knowledge, no research that examines the issue systematically enough to provide a firm ground for pedagogical developments. The importance of the issue must not be underestimated, particularly in the United Kingdom, where students seem do to well enough in the international comparisons in additive reasoning but not in multiplicative reasoning problems (Beaton, Mullis, Martin, Gonzalez, Kelly AND Smith, 1996, p. 94–95).

Summary

We draw some educational implications from these studies, which must be seen as hypotheses about what is important for successful teaching of multiplicative reasoning about relations.

1 Before children are taught about multiplication and division in school, they already have schemas of action that they use to solve multiplicative reasoning problems. These schemas of action involve setting up correspondences between two variables and do not appear to develop from the idea of repeated addition. This informal knowledge is a predictor of their success in learning mathematics and should be drawn upon explicitly in school.

2 Students' schemas of multiplicative reasoning develop sufficiently for them to apply these schemas to numbers, without the need to use objects or tallies to represent quantities. But they seem to be connected to quantities, and it appears that students do not focus on the relations between quantities in multiplicative reasoning problems. This informal knowledge seems to be resistant to change under current conditions of instruction.

3 In many previous studies, researchers drew the conclusion that students' problems in understanding proportional relations were explained by their difficulties in thinking multiplicatively. Today, it seems more likely that students' problems are based on their difficulty in thinking about relations, and not about quantities, since even young children succeed in multiplicative reasoning problems.

4 Teaching approaches might be more successful in promoting the formalisation of students' informal knowledge if: (a) they draw on the students' informal knowledge rather than ignore it; (b) they offer the students a way of representing the relations between quantities and promote an awareness of these relations; and (c) they use a variety of situational contexts to help students extend their knowledge to new domains of multiplicative reasoning.

We examine now the conceptual underpinnings of two rather different teaching approaches to the development of multiplicative reasoning in search for more specific hypotheses regarding how greater levels of success can be achieved by U.K. students.

The challenge in attempting a synthesis of results is that there are many ways of classifying teaching approaches and there is little systematic research that can provide unambiguous evidence. The difficulty is increased by the fact that by the time students are taught about proportions, some time between their third and their sixth year in school, they have participated in a diversity of pedagogical approaches to mathematics and might already have distinct attitudes to mathematics learning. However, we consider it plausible that systematic investigation of different teaching approaches would prove invaluable in the analysis of pathways to help children understand functional relations. In the subsequent section, we explore two different pathways by considering the types of representations that are offered to students in order to help them become aware of functional relations.

Representing functional relations

The working hypothesis we will use in this section is that in order to become explicitly aware of something, we need to represent it. This hypothesis is commonplace in psychological theories: it is part of general developmental theories, such as Piaget's theory on reflective abstraction (Piaget, 1978; 2001; 2008) and Karmiloff-Smith's theory of representational re-descriptions in development (Karmiloff-Smith, 1992; Karmiloff-Smith and Inhelder, 1977). It is also used to describe development in specific domains such as language and literacy (Gombert, 1992; Karmiloff-Smith, 1992), memory (Flavell, 1971) and the understanding of others (Flavell, Green and Flavell, 1990). It is beyond the scope of this work to review the literature on whether representing something does help us become more aware of the represented meaning; we will treat this as an assumption.

The hypothesis concerning the importance of representations will be used in a different form here. Duval (2006) pointed out that 'the part played by signs in mathematics, or more exactly by semiotic systems of representation, is not only to designate mathematical objects or to communicate but also to work on mathematical objects and with them.' (p. 107). We have so far discussed the quantification of relations, and in particular of functional relations, as if the representation of functional relations could only be attained through the use of numbers. Now we wish to make explicit that this is not so. Relations, including functional relations, can be represented by numbers but there are many other ways in which relations can be represented before a number is attributed to them; to put it more forcefully, one could say that relations can be represented in different ways in order to facilitate the attribution of a number to them.

When students are taught to write an equation of the form $a/b = c/x$, for example, to represent a proportions problem in order to solve for x, this formula can be used to help them quantify the relations in the problem. Hart (1981 b) reports that this formula was taught to 100 students in one school where she carried out her investigations of proportions problems but that it was only used by 20 students, 15 of whom were amongst the high achievers in the school. This formula can be used to explore both scalar and functional relations in a proportions problem but it can also be taught as a rule to solve the problem without any exploration of the scalar or functional relations that it symbolises. In some sense, students can learn to use the formula without developing an awareness of the nature of the relations between quantities that are assumed when the formula is applied.

Researchers in mathematics education have been aware for at least two decades that one needs to explore different forms of representation in order to seek the best ways to promote students' awareness of reasoning about the relations in a proportions problem. It is likely that the large amount of research on proportional reasoning, which exposed students' difficulties as well as their reliance on their own methods even after teaching, played a crucial role in this process. It did undoubtedly raise teachers' and researchers' awareness that the representation through formulae ($a/b = c/x$) or algorithms did not work all that well. In this section, we will seek to examine the underlying assumptions of two very different approaches to teaching students about proportions.

Two approaches to the representation of functional relations

Kieren (1994) suggested that there are two approaches to research about, and to the teaching of, multiplicative reasoning in school. The first is *analytic-functional*: it is human in focus, and investigates actions, action schemes and operations used in giving meaning to multiplicative situations. The second is *algebraic*: this focuses on mathematical structures, and investigates structures used in this domain of mathematics. Although the investigation of mathematics structures is not incompatible with the analytic-functional approach, these are alternatives in the choice of starting point for instruction. They delineate radically distinct pathways for guiding students' learning trajectories.

Most of the research carried out in the past about students' difficulties did not describe what sort of teaching students had participated in; one of the exceptions is Hart's (1981 b) description of the teaching in one school, where students were taught the $a/b = c/d$, algebraic approach: the vast majority of the students did not use this formula when they were interviewed about proportions in her study, and its use was confined to the higher achievers in their tests. It is most urgent that a research programme that systematically compares these two approaches should be carried out, so that U.K. students can benefit from better understanding of the consequences of how these different pathways contribute to learning of multiplicative reasoning. In the two subsequent sections, we present one well developed programme of teaching within each approach.

The analytic-functional approach: from schematic representations of quantities in correspondence to quantifying relations

Streefland and his colleagues (Streefland, 1984; 1985 a and b; van den Brink and Streefland, 1979) highlighted the role that drawing and visualisation can play in making children aware of relations. In an initial paper, van den Brink and Streefland (1979) analysed a boy's reactions to proportions in drawings and also primary school children's reactions in the classroom when visual proportions were playfully manipulated by their teacher.

The boy's reactions were taken from a discussion between the boy and his father. They saw a poster for a film, where a man is bravely standing on a whale and trying to harpoon it. The whale's size is exaggerated for the sake of sensation. The father asked what was wrong with the picture and the boy eventually said: 'I know what you mean. That whale should be smaller. When we were in England we saw an orca and it was only as tall as three men' (van den Brink and Streefland, 1979, p. 405). In line with Bryant (1974), van den Brink and Streefland argued that visual proportions are part of the basic mechanisms of perception, which can be used in learning in a variety of situations, and suggested that this might be an excellent start for making children aware of relations between quantities.

Van den Brink and Streefland then developed classroom activities where six- to eight-year-old children explored proportional relations in drawings. Finally, the teacher showed the children a picture of a house and asked them to mark their own height on the door of the house. The children engaged in measurements of themselves and the door of the classroom in order to transpose this size relation to the drawing and mark their heights on the door. This activity generated discussions relevant to the question of proportions but it is not possible to assess the effect of this activity on their understanding of proportions, as no assessments were used. The lesson ended with the teacher showing another part of the same picture: a girl standing next to the house. The girl was much taller than the house and the children concluded that this was actually a doll house. Surprise and playfulness were considered by Streefland an important factor in children's engagement in mathematics lessons.

Van den Brink and Streefland suggested that children can use perceptual mechanisms to reflect about proportions when they judge something to be out of proportion in a picture. They argued that it is not only of psychological interest but also of mathematical-didactical interest to discover why children can reason in ratio and proportion terms in such situations, abstracting from perceptual mechanisms.

Streefland (1984) later developed further activities in a lesson series with the theme 'with a giant's regard', which started with activities that explored the children's informal sense of proportions and progressively included mathematical representations in the lesson. The children were asked, for example, to imagine how many steps would a normal man take to catch up with one of the giant's steps; later, they were asked to represent the man's and the giant's steps on a number line and subsequently by means of a table. Figure 4.4 presents one example of the type of diagram used for a visual comparison.

In a later paper, Streefland (1985 a and b) pursued this theme further and illustrated how the diagrams used to represent visual meanings could be used in a progressively more abstract way, to represent correspondences between values in other problems that did not have a visual basis. This was illustrated using, among others, Hart's (1981 b) onion soup problem, where a recipe for onion soup for 8 people is to be adapted for 4 or 6 people. The diagram proposed by Streefland, which the teacher should encourage the pupils to construct, shows both (a) the correspondences between the values, which the children can find using their own, informal building up strategies, and (b) the value of the scalar transformation. See Figure 4.5 for an example.

Streefland suggested that these schematic representations could be used later in Hart's onion soup problem in a vertical orientation, more common for tables than the above diagram, and with all ingredients listed on the same table in different rows. The top row would list the number of people, and the subsequent rows would list each ingredient. This would help students realise that the same scalar transformation is applied to all the ingredients for the taste to be preserved when the amounts are adjusted. Streefland argued that 'the ratio table is a permanent record of proportion as an equivalence relation, and in this way contributes to acquiring the correct concept. Applying the ratio table contributes to the detachment from the context… In this quality the ratio table is, as it were, a unifying model for a variety of ratio contexts, as well as for the various manifestations of ratio… The ratio table can contribute to discovering, making conscious and applying all properties that characterise ratio-preserving mappings and to their use in numerical problems' (Streefland, 1985, a, p. 91). Ratio tables are then related to graphs, where the relation between two variables can be discussed in a new way.

Streefland emphasises that 'mathematizing reality involves model building' (Streefland, 1985a, p. 86); so students must use their intuitions to develop a model and then learn how to represent it in order to assess its appropriateness. He (Streefland, 1985 *b*; in van den Heuvel-Panhuizen, 2003) argued that children's use of such schematic *models of* situations that they understand well can become a *model for* new situations that they would encounter in the future. The representation of their knowledge in such schematic form helps them understand what is implied in the model, and make explicit a relation that they had used only implicitly before.

This hypothesis is in agreement with psychological theories that propose that reflection and representation help make implicit knowledge explicit (e.g. Karmiloff-Smith, 1992; Piaget, 2001). However, the concept proposes a pedagogical strategy in Streefland's work: the model is chosen by the teacher, who guides the student to use it and adds

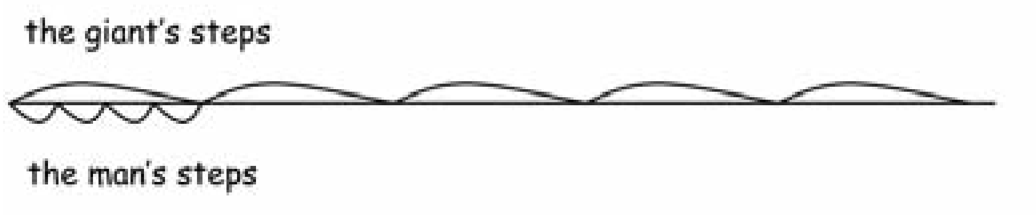


Figure 4.4: The giant's steps and the man's steps on a line; this drawing can be converted into a number line and a table which displays the numerical correspondences between the giant's and the man's steps.

elements, such as the explicit representation of the scalar factor. The model is chosen because it can be easily stripped of the specifics in the situation and because it can help the students move from thinking about the context to discussing the mathematical structures (van den Heuvel-Panhuizen, 2003). So children's informal knowledge is to be transformed into formal knowledge through changes in representation that highlight the mathematical relations that remain implicit when students focus on quantities.

Finally, Streefland also suggested that teaching children about ratio and proportions could start much earlier in primary school and should be seen as a longer project than prescribed by current practice. Starting from children's informal knowledge is a crucial aspect of his proposal, which is based on Freudenthal's (1983) and Vergnaud's (1979) argument that we need to know about children's implicit mathematical models for problem situations, not just their arithmetic skills, when we want to develop their problem solving ability. Streefland suggests that, besides the visual and spatial relations that he worked with, there are other concepts which children aged eight to ten years can grasp in primary school, such as comparisons between the density (or crowdedness) of objects in space and probabilities. Other concepts, such as percentages and fractions, were seen by him as related to proportions, and he argued that connections should be made across these concepts. However, Streefland considered that they merited their own analyses in the mathematics classroom. He argued, citing Vergnaud (1979) that 'different properties, almost equivalent to the mathematician, are not all equivalent for the child (Vergnaud, 1979, p. 264). So he also developed programmes for the teaching of percentages (Streefland and van den Heuvel-Panhuizen, 1992; van den Heuvel-Panhuizen, 2003) and fractions (Streefland, 1993; 1997). Marja van den Heuvel-Panhuizen and her colleagues (Middleton and van den Heuvel-Panhuizen, 1995; Middleton, van den Heuvel-Panhuizen and Shew, 1998) detailed the use of the ratio table in teaching students in their 3rd year in school about percentages and connecting percentages, fractions and proportions.

In all these studies, the use of the ratio table is seen as a tool for computation and also for discussion of the different relations that can be quantified in the problem situations. Their advice is that teachers should allow students to use the table at their own level of understanding but always encourage students to make their reasoning explicit. In this way, students can compare their own reasoning with their peers' approach, and seek to improve their understanding through such comparisons.

Streefland's proposal is consistent with many of the educational implications that we drew from previous research. It starts from the representation of the correspondences between quantities and moves to the representation of relations. It uses schematic drawings and tables that bring to the fore of each student's activity the explicit representation of the two (or more) measures that are involved in the problem. It is grounded on students' informal knowledge because students use their building up solutions in order to construct tables and schematic drawings. It systematizes the students' solutions in tables and re-represents them by means of graphs. After exploring students' work on quantities, students' attention is focused on scalar relations, which they are asked to represent explicitly using the same visual records. It draws on a variety of contexts

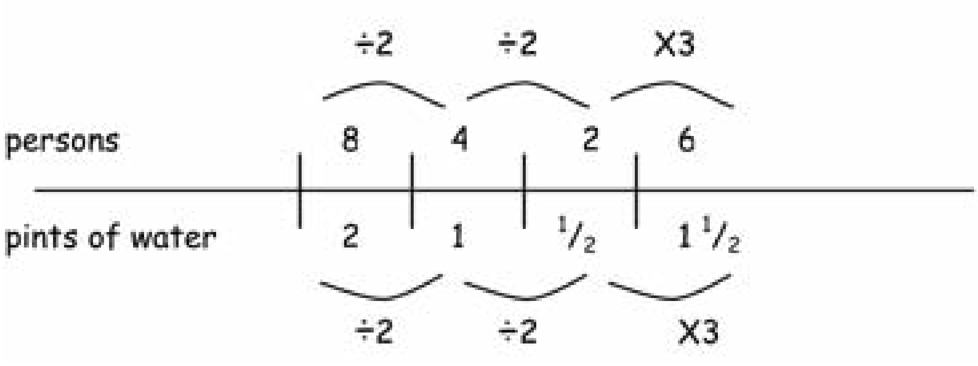


Figure 4.5: A table showing the answers that the children can build up and the representation of the scalar transformations.

that have been previously investigated and which students have been able to handle successfully. Finally, it uses graphs to explore the linear relations that are implied in proportional reasoning.

To our knowledge, there is no systematic investigation of how this proposal actually works when implemented either experimentally or in the classroom. The work by Treffers (1987) and Gravemeijer (1997) on the formalisation of students' understanding of multiplication and division focused on the transition from computation with small to large numbers. The work by van den Heuvel-Panhuizen and colleagues focused on the use of ratio tables in the teaching of percentages and equivalence of fractions. In these papers, the authors offer a clear description of how teachers can guide students' transition from their own intuitions to a more formal mathematical representation of the situations. However, there is no assessment of how the programmes work and limited systematic description of how students' reasoning changes as the programmes develop.

The approach by researchers at the Freudenthal Institute is described as developmental research and aims at constructing a curriculum that is designed and improved on the basis of students' responses (Gravemeijer, 1994). This work is crucial to the development of mathematics education. However, it does not allow for the assessment of the effects of specific teaching approaches, as more experimental intervention research does. It leaves us with the sense that the key to formalising students' multiplicative reasoning may be already to hand but we do not know this yet. Systematic research at this stage would offer an invaluable contribution to the understanding of how students learn and to education.

Streefland was not the only researcher to propose that teaching students about multiplicative relations should start from their informal understanding of the relations between quantities and measures. Kaput and West (1994) developed an experimental programme that took into account students' building up methods and sought to formalise them through connecting them with tables. Their aim was to help students create composite units of quantity, where the correspondences between the measures were represented iconically on a computer screen. For example, if in a problem the quantities are 3 umbrellas for 2 animals, the computer screen would display cells with images for 3 umbrellas and 2 animals in each cell, so that the group of umbrellas and animals became a higher-level unit. The cells in the computer screen were linked up with tables, which showed the values corresponding to the cells that had been filled with these composite units: for example, if 9 cells had been filled in with the iconic representations, the table displayed the values for 1 through 9 of the composite units in columns headed by the icons for umbrellas and animals. Subsequently, students worked with non-integer values for the ratios between the quantities: for example, they could be asked to enlarge a shape and the corresponding sides of the two figures had a non-integer ratio between them (e.g. one figure had a side 21 cm long and the other had the corresponding side 35 cm long).

Kaput and West's programme was delivered over 11 lessons in two experimental classes, which included 31 students. Two comparison classes, with a total of 29 students, followed the instruction previously used by their teachers and adopted from textbooks. One comparison class had 13 lessons: the first five lessons were based on a textbook and covered exercises involving ratio and proportion; the last eight consisted of computer-based activities using function machines with problems about rate and profit. The second comparison class had only three lessons; the content of these is not described by the authors. The classes were not assigned randomly to these treatments and it is not clear how the teachers were recruited to participate in the study.

At pre-test, the students in the experimental and comparison classes did not differ in the percentage of correct solutions in a multiplicative reasoning test. At post-test, the students in the experimental group significantly out-performed those in both comparison classes. They also showed a larger increase in the use of multiplicative strategies than students in the comparison classes. It is not possible from Kaput and West's report to know whether these were building up, scalar or functional solutions, as they are considered together as multiplicative solutions.

In spite of the limitations pointed out, the study does provide evidence that students benefit from teaching that develops their building up strategies into more formalised approaches to solution, by linking the quantities represented by icons of objects to tables that represent the same quantities. This result goes against the view that informal methods are an obstacle to students' learning in and of themselves; it is more likely that they are an obstacle *if* the teaching they are exposed to does not build on the students' informal strategies and does not help students connect what they know with the new forms of mathematical representation that the teacher wants them to learn.

The algebraic approach: representing ratios and equivalences

In contrast to the functional approach to the teaching of proportions that was described in the previous section, some researchers have proposed that teaching should not start from students' understanding of multiplicative reasoning, but from a formal mathematical definition of proportions as the equality of two ratios. We found the most explicit justification for this approach in a recent paper by Adjiage and Pluvinage (2007). Adjiage and Pluvinage, citing several authors (Hart, 1981 b); Karplus, Pulos and Stage, 1983; Lesh, Post, and Behr, 1988), argue that building up strategies are a weak indicator of proportionality reasoning and that the link between 'interwoven physical and mathematical considerations, present in the build-up strategy' (2007, p. 151) should be the representation of problems through rational numbers. For example, a mixture that contains 3 parts concentrate and 2 parts water should be represented as 3/5, using numbers or marks on a number line. The level 1, which corresponds to an iconic representation of the parts used in the mixture, should be transformed into a level 2, numerical representation, and students should spend time working on such transformations. Similarly, a scale drawing of a figure where one side is reduced from a length of 5 cm to 3 cm should be represented as 3/5, also allowing for the move from an iconic to a numerical representation. Finally, the representation by means of an equivalence of ratios, as in 3/5 = 6/10, should be introduced, to transform the level 2 into a level 3 representation. The same results could be obtained by using decimals rather than ordinary fractions representations.

In brief, level 1 allows for an articulation between physical quantities: the students may realise that a mixture with 3 parts concentrate and 2 parts water tastes the same as another with 6 parts concentrate and 4 parts water. Level 2 allows for articulations between the physical quantities and a mathematical representation: students may realise that two different situations are represented by the same number. Level 3 allows for articulations within the mathematical domain as well as conversions from one system of representation to another: 3/5 = 0.6 or 6/10.

Adjiage and Pluvinage (2007) argue that it is important to separate the physical from the mathematical initially in order to articulate them later, and propose that three rational registers should be used to facilitate students' attainment of level 3: linear scale (a number line with resources such as subdividing, sliding along the line, zooming), fractional writing, and decimal writing should be used in the teaching of ratio and proportions.

It seems quite clear to us that this proposal does not start from students' intuitions or strategies for solving multiplicative reasoning problems, but rather aims to formalise the representation of physical situations from the start and to teach students how to work with these formalisations. The authors indicate that their programme is inspired by Duval's (1995) theory of the role of representations in mathematical thinking but we believe that there is no necessary link between the theory and this particular approach to teaching students about ratio and proportions.

In order to convey a sense for the programme, Adjiage and Pluvinage (2007) describe five moments experienced by students. The researchers worked with two conditions of implementation, which they termed the full experiment and the partial experiment. Students in the partial experiment did not participate in the first moment using a computer; they worked with pencil and paper tasks in moments 1 and 2.

- *Moment 1* The students are presented with three lines, divided into equal spaces. They are told that the lines are drawn in different scales. The lines have different numbers of subdivisions – 5, 3 and 4, respectively. Points equivalent to 3/5, 2/3 and 1/4 are marked on the line. The students are asked to compare the segments from the origin to the point on the line. This is seen as a purely mathematical question, executed in the computer by students in the full experiment condition. The computer has resources such as dividing the lines into equal segments, which the students can use to execute the task.

- *Moment 2* A similar task is presented with paper and pencil.

- *Moment 3* The students are shown two pictures that represent two mixtures: one is made with 3 cups of chocolate and 2 of milk (the cups are shown in the pictures in different shades) and the other with 2 cups of chocolate and 1 of milk. The students are asked which mixture tastes more chocolaty. This problem aims to link the physical and the mathematical elements.

- *Moment 4* The students are asked in what way are the problems in moments 2 and 3 similar. Students are expected to show on the segmented line which portion corresponds to the cups of chocolate and which to the cups of milk.

- *Moment 5* This is described as institutionalization in Douady's (1984) sense: the students are asked to make abstractions and express rules. For instance, expressions such as these are expected: '7 divided by 4 is equal to seven fourths ($7 \div 4 = 7/4$)'; 'Given an enlargement in which a 4 cm length becomes a 7 cm length, then any length to be enlarged has to be multiplied by 7/4.' (Adjiage and Pluvinage, 2007, pp. 160–161).

The teaching programme was implemented over two school years, starting when the students were in their 6th year (estimated age about 11 years) in school. A pre-test was given to them before they started the programme; the post-test was carried out at the end of the students' 7th year (estimated age about 12 years) in school.

Adjiage and Pluvinage (2007) worked with an experienced French mathematics teacher, who taught two classes using their experimental programme. In both classes, the students solved the same problems but in one class, referred to as a partial experimental, the students did not use the computer-based set of activities whereas in the other one, referred to as full experimental, they had access to the computer activities. The teacher modified only his approach to teaching ratio and proportions; other topics in the year were taught as previously, before his engagement in the experiment.

The performance of students in these two experimental classes was compared to results obtained by French students in the same region (the baseline group) in a national assessment and also to the performance of non-specialist, prospective school-teachers on a ratio and proportions task. The tasks given to the three groups were not the same but the researchers considered them comparable.

Adjiage and Pluvinage reported positive results from their teaching programme. When the pupils in the experimental classes were in grade 6 they had a low rate of success in ratio and proportions problems: about 13%. At the end of grade 7, they attained 39% correct answers whereas the students in the sample from the same region (baseline group) attained 15% correct responses in the national assessment. The students in the full experimental classes obtained significantly better results than those in the partial experimental classes but the researchers did not provide separate percentages for the two groups. Prospective teachers attained 83% on similar problems. The researchers were not satisfied with these results because, as they point out, the students performed significantly worse than the prospective teachers, who were taken to represent educated adults.

Although there are limitations to this study, it documents some progress among the students in the experimental classes. However, it is difficult to know from their report how much time was devoted to the teaching programme over the two years and how this compares to the instruction received by the baseline group.

In brief, this approach assumes that students' main difficulties in solving proportions problems result from their inability to co-ordinate different forms of mathematical representations and to manipulate them. There is no discussion of the question of quantities and relations and there is no attempt to make students aware of the relations between quantities in the problems. The aims of teaching are to:
- develop students' understanding of how to use number line and numerical representations together in order to compare rational numbers
- promote students' reflection on how the numerical and linear representations relate to problem situations that involve physical elements (3 cups of chocolate and 2 of milk)
- promote students' understanding of the relations between the different mathematical representations and their use in solving problems.

A comparison between this example of the algebraic approach and the functional approach as exemplified by Streefland's work suggests that this algebraic approach does not offer students the opportunity to distinguish between quantities and relations. The three forms of representation offered in the Adjiage and Pluvinage programme focus on quantities; the relations between quantities are left implicit. Students are expected to recognise that mixtures of concentrate that are numerically represented as 3/5 and 6/10 are equivalent. In the number line, they are expected to manipulate the representations of quantities in order to compare them. We found no evidence in the description of their teaching programme that students were asked to think about their implicit models of the situations and explicitly discuss the transformations that would maintain the equivalences.

Summary

1 It is possible to identify in the literature two rather different views of how students can best be taught about multiplicative reasoning. Kieren identified these as the functional and the algebraic approach.

2 The functional approach proposes that teaching should start from students' understanding of quantities and seek to make their implicit models of relations between quantities explicit.

3 The algebraic approach seeks to represent quantities with mathematical symbols and lead students to work with symbols as soon as possible, disentangling physical and mathematical knowledge.

4 There is no systematic comparison between these two approaches. Because their explicit description is relatively recent, this paper is the first detailed comparison of their characteristics and provides a basis for future research.

Graphs and functional relations

The previous sections focused on the visual and numerical representations of relations. This section will briefly consider the question of the representation of relations in the Cartesian plane. We believe that this is a form of representation that merits further discussion because of the additional power that it can add to students' reflections, if properly explored.

Much research on how students interpret graphs has shown that graph reading has to be learned, just as one must learn how to read words or numbers. Similarly to other aspects of mathematics learning, students have some ideas about reading graphs before they are taught, and researchers agree that these ideas should be considered when one designs instruction about graph reading. Several papers can be of interest in this context but this research is not reviewed here, as it does not contribute to the discussion of how graphs can be used to help students understand functional relations (for complementary reviews, see Friel, Curcio and Bright, 2001; Mevarech and Kramarsky, 1997). We focus here on the possibilities of using graphs to help students understand functional relations.

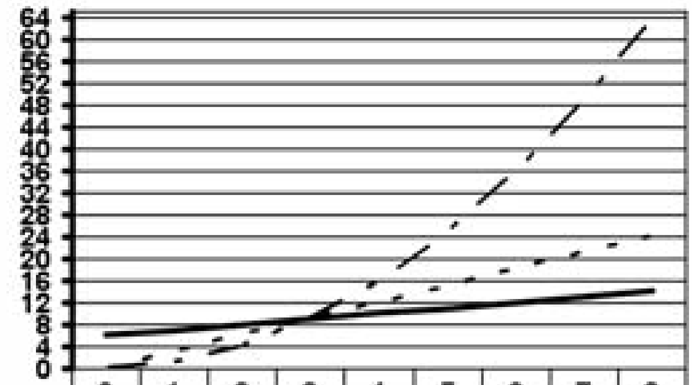
As reported earlier in this chapter, Lieven Verschaffel and his colleagues have shown that students make multiplicative reasoning errors in additive situations as well as additive errors in multiplicative situations, and so there is a need for students to be offered opportunities to reflect on the nature of the relation between quantities in problems. Van Dooren, Bock, Hessels, Janssens and Verschaffel (2004) go as far as suggesting that students fall prey to what they call an illusion of linearity, but we think that they have overstated their case in this respect. In fact, some of the examples that they use to illustrate the so-called illusion of linearity are indeed examples of linear functions, but perhaps not as simple as the typical linear functions used in school. In two examples of their 'illusion of linearity' discussed here, there is a linear function connecting the two variables but the problem situation is more complex than many of the problems used in schools when students are taught about ratio and proportions. In our view, these problems demonstrate the importance of working with students to help them reflect about the relations between the quantities in the situations.

In one example, taken from Cramer, Post and Currier (1993) and discussed earlier on in this paper, two girls, Sue and Julie, are supposed to be running on a track at the same speed. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie completed 15 laps, how many laps had Sue run? Although prospective teachers wrongly quantified the relation between the number of laps in a multiplicative way, we do not think that they fell for the 'illusion of linearity', as argued by De Bock, Verschaffel *et al.*, (2002; 2003). The function actually is linear, as illustrated in Figure 4.6. However, the intercept between Sue's and Julie's numbers of laps is not at zero, because Sue must have run 6 laps before Julie starts. So the prospective teachers' error is not an illusion of linearity but an inability to deal with intercepts different from zero.

Figure 4.6 shows that three different curves would be obtained: (1) if the girls were running at the same speed but one started before the other, as in the Cramer, Post and Currier problem; (2) if one were running faster than the other and this difference in speed were constant; and (3) if they started out running at the same speed but one girl became progressively more tired whereas the other was able to speed up as she warmed up. Students might hypothesise that this latter example is better described by a quadratic than a linear function, if the girl who was getting tired went from jogging to walking, but they could find that the quadratic function would exaggerate the difference between the girls: how could the strong girl run 25 laps while the weak one ran 5?

The aim of this illustration is to show that relations between quantities in the same context can vary and that students can best investigate the nature of the relation between quantities is if they have a tool to do so. Streefland suggested that tables and graphs can be seen as tools that allow students to explore relations between quantities; even though they could be used to help students' reasoning in this problem, we do not know of research where it has been used.

Van Dooren, *et al.* (2004) used graphs and tables in an intervention programme designed to help student overcome the 'illusion of linearity' in a second problem, which we argue also involve mislabelling of the phenomenon under study. In several studies, De Bock, Van Dooren and their colleagues (De Bock, Verschaffel and Janssens, 1998; De Bock, Van Dooren, Janssens and Verschaffel, 2002; De Bock, Verschaffel and Janssens, 2002; De Bock, Verschaffel, Janssens, Van Dooren and Claes, 2003) claim to have identified this illusion in questions exemplified in this problem: 'Farmer Carl needs approximately 8 hours to manure a square piece of land with a side of 200 m. How many hours would he need to manure a square piece of land with a side of 600 m?' De Bock, Van Dooren and colleagues worked with relatively large numbers of Belgian students across their many studies, in the age range 12 to 16 years. They summarise their findings by indicating that 'the vast majority of students (even 16-year-olds) failed on this type of problem because of their alarmingly strong tendency to apply linear methods' (Van Dooren, Bock, Hessels, Janssens and Verschaffel, 2004, p. 487) and that even with considerable support many students were not able to overcome this difficulty. Some students who did become more cautious about over-using a linear model, subsequently failed to use it when it was appropriate.



	0	1	2	3	4	5	6	7	8
Same speed, different starts	6	7	8	9	10	11	12	13	14
Multiplicative relation	0	3	6	9	12	15	18	21	24
Quadratic function	0	1	4	9	16	25	36	49	64

Same speed, different starts
Multiplicative relation
Quadratic function

Figure 4.6: Three graphs showing different relations between the number of laps run by two people over time

We emphasise here that in this problem, as in the previous one, students were not falling prey to an illusion of linearity. The area of a rectangular figure is indeed proportional to its side *when the other side is held constant*; this is a case of multiple proportions and thus the linear relation between the side and the area can only be appreciated if the other side does not change. Because the rectangle in their problem is the particular case of a square, if one side changes, so does the other; with both measures changing at the same time, the area is not a simple linear function of one of the measures.

Van Dooren *et al.* (2004) describe an intervention programme, in which students used graphs and tables to explore the relation between the measure of the side of a square, its area and its perimeter. The intervention contains interesting examples in which students have the opportunity to examine diagrams that display squares progressively larger by 1 cm, in which the square units (1 cm^2) are clearly marked. Students thus can see that when the side of a square increases, for example, from 1 cm to 2 cm, its area increases from 1 cm^2 to 4 cm^2, and when the increase is from 2 cm to 3 cm, the area increases to 9 cm^2. The graph associated with this table displays a quadratic function whereas the graph associate with the perimeter displays a linear function.

Their programme was not successful in promoting students' progress: the experimental group significantly decreased the rate of responses using simple proportional reasoning to the area problems but also decreased the rate of correct responses to perimeter problems, although the perimeter of a square is connected to its sides by a simple proportion.

We believe that the lack of success of their programme may be due not to a lack of effectiveness of the use of graphs and tables in promoting students' reasoning but from their use of an inadequate mathematical analysis of the problems. Because the graphs and tables used only two variables, measure of the side and measure of the area, the students had no opportunity to appreciate that in the area problem there is a proportional relation between area and each the two sides. The two sides vary at the same time in the particular case of the square but in other rectangular figures there isn't a quadratic relation between side and area. The relation between sides and perimeter is additive, not multiplicative: it happens to be multiplicative in the case of the square because all sides are equal; so to each increase by 1 cm in one side corresponds a 4 cm increase in the perimeter.

We think that it would be surprising if the students had made significant progress in understanding the relations between the quantities through the instruction that they received in these problems: they were not guided to an appropriate model of the situation, and worked with one measure, side, instead of two measures, base and height. One of the students remarked at the end of the intervention programme, after ten experimental lessons over a two week period: 'I really do understand now why the area of a square increases 9 times if the sides are tripled in length, since the enlargement of the area goes in two dimensions. But suddenly I start to wonder why this does not hold for the perimeter. The perimeter also increases in two directions, doesn't it?' (Van Dooren *et al.*, 2004, p. 496). This student seems to have understood that the increase in one dimension of the square implies a similar increase along the other dimension and that these are multiplicatively related to the area but apparently missed the opportunity to understand that sides are additively related to the perimeter.

In spite of the shortcomings of this study, the intervention illustrates that it is possible to relate problem situations to tables and graphs systematically to stimulate students' reflection about the implicit models. It is a current hypothesis by many researchers (e.g. Carlson, Jacobs, Coe, Larsen and Hsu, 2002; Hamilton, Lesh, Lester and Yoon, 2007; Lesh, Middleton, Caylor and Gupta, 2008) that modelling data, testing the adequacy of models through graphs, and comparing different model fits can make an important contribution to students' understanding of the relations between quantities. It is consistently acknowledged that this process must be carefully designed: powerful situations must be chosen, clear means of hypothesis testing must be available, and appropriate teacher guidance should be provided. Shortcomings in any of these aspects of teaching experiments could easily result in negative results.

The hypothesis that modelling data, testing the adequacy of models through graphs, and comparing different model fits can promote student's understanding of different types of relations between quantities seems entirely plausible but, to our knowledge, there is no research to provide clear support to it. We think that there are now many ideas in the literature that can be implemented to assess systematically how effective the use of graphs and tables is as tools to support students' understanding of the different types of relation that can exist between measures. This research has the potential to make a huge contribution to the improvement of mathematics education in the United Kingdom.

Conclusions and implications

This review has identified results in the domain of how children learn mathematics that have significant implications for education. The main points are highlighted here.

1 Children form concepts about quantities from their everyday experiences and can use their schemas of action with diverse representations of the quantities (iconic, numerical) to solve problems. They often develop sufficient awareness of quantities to discuss their equivalence and order as well as how they can be combined.

2 It is significantly more difficult for them to become aware of the relations between quantities and operate on relations. Even after being taught how to represent relations, they often interpret the results of operations on relations as if they were quantities. Children find both additive and multiplicative relations significantly more difficult than understanding quantities.

3 There is little evidence that the design of instruction has so far taken into account the importance of helping students become aware of the difference between quantities and relations. Some researchers have carried out experimental teaching studies that suggest that it is possible to promote students' awareness of relations. Further research must be carried out to analyse how this knowledge affects mathematics learning. If positive results are found, there will be strong policy implications.

4 Previous research had led to the conclusion that students' problems with proportional reasoning stemmed from their difficulties with multiplicative reasoning. However, there is presently much evidence to show that, from a relatively early age (about five to six years in the United Kingdom), children already have informal knowledge that allows them to solve multiplicative reasoning problems. We suggest that students' problems with proportional reasoning stems from their difficulties in becoming explicitly aware of relations between quantities. This awareness would help them distinguish between situations that involve different types of relations: additive, proportional or quadratic, for example.

5 Multiplicative reasoning problems are defined by the fact that they involve two (or more) measures linked by a fixed ratio. Students' informal knowledge of multiplicative reasoning stems from the schema of one-to-many correspondence, which they use both in multiplication and division problems. When the product is unknown, children set the elements in the two measures in correspondence (e.g. 1 sweet costs 4p) and figure out the product (how much 5 sweets will cost). When the correspondence is unknown (e.g. if you pay 20p for 5 sweets, how much does each sweet cost), the children share out the elements (20p shared in 5 groups) to find what the correspondence is.

6 This informal knowledge is currently ignored in U.K. schools, probably due to the theory that multiplication is essentially repeated addition and division is repeated subtraction. However, the connections between addition and multiplication, on one hand, and subtraction and division, on the other hand, are procedural and not conceptual. So students' informal knowledge of multiplicative reasoning could be developed in school from an earlier age.

7 A considerable amount of research carried out independently in different countries has shown that students sometimes use additive reasoning about relations when the appropriate model is a multiplicative one. Some recent research has shown that students also use multiplicative reasoning in situations where the appropriate model is additive. These results suggest that children use additive and multiplicative models implicitly and do not make conscious decisions regarding which model is appropriate in a specific situation. The educational implication from these findings is that schools should take up the task of helping students become more aware of the models that they use implicitly and of ways of testing their appropriateness to particular situations.

8 Proportional reasoning stems from children's use of the schema of one-to-many correspondences, which is expressed in calculations as building-up strategies. Evidence suggests that many students who use these strategies are not aware of functional relations that characterises a linear function. This result reinforces the importance of the role that schools could play in helping students become aware of functional relations in proportions problems.

9 Two radically different approaches to teaching proportions and linear functions in schools can be identified in the literature. One, identified as functional and human in focus, is based on the notion that students' schemas of action should be the starting point for this teaching. Through instruction, they should become progressively more aware of the scalar and functional relations that can be identified in such problems. Diagrams, tables and graphs are seen as tools that could help students understand the models that they are using of situations and make them into models for other situations later. The second approach, identified as algebraic, proposes that there should be a sharp separation between students' intuitive knowledge, in which physical and mathematical knowledge are intertwined, and mathematical knowledge. Students should be led to formalisations early on in instruction and re-establish the connections between mathematical structures and physical knowledge at a later point. Representations using fractions, ordinary and decimal, and the number line are seen as the tools that can allow students to abstract early on from the physical situations. There is no unambiguous evidence to show how either of these approaches to teaching succeeds in promoting students' progress, nor that either of them is more successful than the less clearly articulated ideas that are implicit in current teaching in the classroom. Research that can clarify this issue is urgently needed and could have a major impact by promoting better learning in U.K. students.

10 Students need to learn to read graphs in order to be able to use them as tools for thinking about functions. Research has shown that students have ideas about how to read graphs before instruction and that these ideas should be taken into account when graphs are used in the classroom. It is possible to teach students to read graphs and to use them in order to think about relations but much more research is needed to show how students' thinking changes if they do learn to use graphs in order to analyse the type of relation that is most relevant in specific situations.

References

Adjiage, R., and Pluvinage, F. (2007). An experiment in teaching ratio and proportion. *Educational Studies in Mathematics, 65*, 149-175.

Beaton, A. E., Mullis, I. V. S., Martin, M. O., Gonzalez, E. J., Kelly, D. L., & Smith, T. A. (1996). *Mathematics achievement in the middle school years: IEA's Third International Mathematics and Science Study (TIMSS)*. Chestnut Hill, MA, USA: TIMSS International Study Center, Boston College.

Becker, J. (1993). Young children's numerical use of number words: counting in many-to-one situations. *Developmental Psychology, 19*, 458-465.

Behr, M. J., Harel, G., Post, T., and Lesh, R. (1994). Units of quantity: A conceptual basis common to additive and multiplicative structures. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 121-176). Albany (NY): SUNY Press.

Booth, L. R. (1981). Child-Methods in Secondary Mathematics. *Educational Studies in Mathematics, 12*, 29-41.

Brown, M. (1981). Number operations. In K. Hart (Ed.), *Children's Understanding of Mathematics: 11-16* (pp. 23-47). Windsor, UK: NFER-Nelson.

Bryant, P. (1974). *Perception and Understanding in Young Children*. London and Co. Ltd.: Methuen.

Carlson, M., Jacobs, S., Coe, E., Larsen, S., and Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events. *Journal for Research in Mathematics Education, 33*, 352–378.

Carpenter, T. P., Ansell, E., Franke, M. L., Fennema, E., and Weisbeck, L. (1993). Models of Problem Solving: A Study of Kindergarten Children's Problem-Solving Processes. *Journal for Research in Mathematics Education, 24*, 428-441.

Carpenter, T. P., Hiebert, J., and Moser, J. M. (1981). Problem structure and first grade children's initial solution processes for simple addition and subtraction problems. *Journal for Research in Mathematics Education, 12*, 27-39.

Confrey, J. (1994). Splitting, Similarity and Rate of Change: A New Approach to Multiplication and Exponential Functions. In G. Harel & J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 293-332). Albany, New York: State University of New York Press.

Cramer, K., Post, T., and Currier, S. (1993). Learning and teaching ratio and proportion: research implications. In D. T. Owens (Ed.), *Research ideas for the classroom: Middle grades mathematics* (pp. 159-178). New York: Macmillan.

De Bock, D., Verschaffel, L., and Janssens, D. (1998). The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics, 35*, 65-83.

De Bock, D., Verschaffel, L., and Janssens, D. (2002). The effects of different problem presentations and formulations on the illusion of linearity in secondary school students. *Mathematical Thinking and Learning, 4*, 65-89.

De Bock, D., Van Dooren, W., Janssens, D., and Verschaffel, L. (2002). Improper use of linear reasoning: an in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*, 311-334.

De Bock, D., Verschaffel, L., Janssens, D., Dooren, W., and Claes, K. (2003). Do realistic contexts and graphical representations always have a beneficial impact on students' performance? Negative evidence from a study on modelling non-linear geometry problems. *Learning and Instruction, 13*, 441-463.

Douady, R. (1984). *Jeux de cadres et dialectique outil-objet dans l'enseignement des mathématiques, une réalisation de tout le cursus primaire.* Unpublished, Doctorat d'état, Paris.

Duval, R. (1995). *Sémiosis et pensée humaine*. Bern: Peter Lang.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics, 61*, 103-131.

Elliott, C. D., Smith, P., and McCulloch, K. (1997). *British Ability Scale-II Technical Manual*. London: nferNelson Publishing Company.

Flavell, J. H. (1971). What is memory development the development of? *Human Development, 14*, 272-278.

Flavell, J. H., Green, F. L., and Flavell, E. R. (1990). Developmental Changes in young children's knowledge about the mind. *Cognitive Development, 5*, 1-27.

Freudenthal, H. (1983). *Didactical Phenomenology of Mathematical Structures*. Dordrecht: D. Reidel.

Friel, S. N., Curcio, F. R., and Bright, G. W. (2001). Making Sense of Graphs: Critical Factors Influencing Comprehension and Instructional Implications. *Journal for Research in Mathematics Education, 32*, 124-158.

Frydman, O., & Bryant, P. E. (1988). Sharing and the understanding of number equivalence by young children. *Cognitive Development, 3*, 323-339.

Gombert, J. E. (1992). *Metalinguistic Development*. Hemel Hempstead: Harverster Wheatsheaf.

Gravemeijer, K. (1994). Educational Development and Developmental Research in Mathematics Education. *Journal for Research in Mathematics Education, 25*, 443-471.

Gravemeijer, K. (1997). Mediating between concrete and abstract. In T. Nunes and P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 315-346). Hove (UK): Psychology Press.

Hamilton, E., Lesh, R., Lester, F., and Yoon, C. (2007). The use of reflection tools in building personal models of problem-solving. In R. Lesh, E. Hamilton and J. Kaput (Eds.), *Foundations for the future in mathematics education* (pp. 349–366). Mahwah, NJ: Lawrence Erlbaum Associates.

Hart, K. (1981 *a*). Measurement. In K. Hart (Ed.), *Children's Understanding of Mathematics: 11-16* (pp. 9-22). London: John Murray.

Hart, K. (1981 *b*). Ratio and proportion. In K. Hart (Ed.), *Children's Understanding of Mathematics: 11-16* (pp. 88-101). London: John Murray.

Hart, K. (1984). *Ratio: children's strategies and errors. A report of the strategies and errors in secondary mathematics project*. Windsor: NFER-Nelson.

Hudson, T. (1983). Correspondences and numerical differences between sets. *Child Development, 54*, 84-90.

Inhelder, B. and. Piaget, J. (1958). *The Growth of Logical Thinking for Childhood to Adolescence*. London: Routledge. and Kegan Paul.

Kaput, J., and West, M. M. (1994). Missing-Value proportional reasoning problems: Factors affecting informal reasoning patterns. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 237-292). Albany, New York: State University of New York Press.

Karmiloff-Smith, A. (1992). *Beyond modularity: a developmental perspective on cognitive science*. Cambridge, Mass: MIT Press.

Karmiloff-Smith, A., and Inhelder, B. (1977). 'If you want to get ahead, get a theory'. In P. N. Johnson-Laird and P. C. Wason (Eds.), *Thinking. Readings in Cognitive Science* (pp. 293-306). Cambridge: Cambridge University Press.

Karplus, R., and Peterson, R. W. (1970). Intellectual development beyond elementary school II: ratio, a survey. *School-Science and Mathematics, 70*, 813-820.

Karplus, R., Pulos, S., and Stage, E. K. (1983). Proportional reasoning of early adolescents. In R. Lesh and M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 45-90). London: Academic Press.

Kieren, T. (1994). Multiple Views of Multiplicative Structures. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 389-400). Albany, New York: State University of New York press.

Kouba, V. (1989). Children's solution strategies for equivalent set multiplication and division word problems. *Journal for Research in Mathematics Education, 20*, 147-158.

Lamon, S. (1994). *Ratio and Proportion: Cognitive Foundations in Unitizing and Norming.* In G. Harel and J. Confrey (Eds.), The Development of Multiplicative Reasoning in the Learning of Mathematics (pp. 89-122). Albany, New York: State University of New York Press.

Lave, J. (1988). *Cognition in practice*. Cambridge: Cambridge University Press.

Lesh, R., Middleton, J. A., Caylor, E., and Gupta, S. (2008). A science need: Designing tasks to engage students in modeling complex data. *Educational Studies in Mathematics, in press.*

Lesh, R., Post, T., and Behr, M. (1988). Proportional reasoning. In J. Hiebert and M. Behr (Eds.), *Number Concepts and Operations in the Middle Grades* (pp. 93-118). Reston, VA: National Council of Teachers of Mathematics.

Lewis, A., and Mayer, R. (1987). Students' miscomprehension of relational statements in arithmetic word problems. *Journal of Educational Psychology, 79*, 363-371.

Mevarech, Z. R., and Kramarsky, B. (1997). From Verbal Descriptions to Graphic Representations: Stability and Change in Students' Alternative Conceptions. *Educational Studies in Mathematics, 32*, 229-263.

Middleton, J. A., and Van den Heuvel-Panhuizen, M. (1995). The ratio table. *Mathematics Teaching in the Middle School, 1*, 282-288.

Middleton, J. A., Van den Heuvel-Panhuizen, M., and Shew, J. A. (1998). Using bar representations as a model for connecting concepts of rational number. *Mathematics Teaching in the Middle School, 3*, 302-311.

Noelting, G. (1980 *a*). The development of proportional reasoning and the ratio concept Part I - Differentiation of stages. *Educational Studies in Mathematics, 11*, 217-253.

Noelting, G. (1980 *b*). The development of proportional reasoning and the ratio concept Part II - Problem-structure at successive stages: Problem-solving strategies and the mechanism of adaptative restructuring. *Educational Studies in Mathematics, 11*, 331-363.

Nunes, T., & Bryant, P. (1996). *Children Doing Mathematics*. Oxford: Blackwell.

Nunes, T., Light, P., and Mason, J. (1993). Tools for thought: the measurement of lenght and area. *Learning and Instruction, 3*, 39-54.

Nunes, T., Schliemann, A. D., and Carraher, D. W. (1993). *Street Mathematics and School Mathematics*. New York: Cambridge University Press.

Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. N. (2007). The Contribution of Logical Reasoning to the Learning of Mathematics in Primary School. *British Journal of Developmental Psychology, 25*, 147-166.

Park, J., & Nunes, T. (2001). The development of the concept of multiplication. *Cognitive Development, 16*, 1-11.

Piaget, J. (1952). *The Child's Conception of Number*. London: Routledge & Kegan Paul.

Piaget, J. (1978). *The Grasp of Consciousness*. London: Routledge Kegan Paul.

Piaget, J. (2001). *Studies in reflecting abstraction: translated by R. Campbell*. Hove: Psychology Press.

Piaget, J. (2008). Intellectual Evolution from Adolescence to Adulthood. *Human Development, 51* (Reprint of Human Development 1972; 15:1–12), 40-47.

Piaget, J., Grize, J., Szeminska, A., and Bangh, V. (1977). *Epistemology and the Psychology of Functions*. Dordrecth, Netherlands: D. Reidel.

Piaget, J., and Inhelder, B. (1975). *The Origin of the Idea of Chance in Children*. London: Routledge and Kegan Paul.

Pickering, S., and Gathercole, S. (2001). *Working memory test battery for children (WMTB-C) Manual*. London: The Psychological Corporation.

Resnick, L. B. (1983). Mathematics and Science Learning: A New Conception. *Science, 220*, 477-478.

Ricco, G. (1982). Les première acquisitions de la notion de fonction linèaire chez l'enfant de 7 à 11 ans. *Educational Studies in Mathematics, 13*, 289-327.

Schwartz, J. (1988). Intensive quantity and referent transforming arithmetic operations. In J. Hiebert and M. Behr (Eds.), *Number concepts and operations in the middle grades* (pp. 41-52). Hillsdale, NJ: Erlbaum.

Steffe, L. (1994). Children's multiplying schemes. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics (pp. 3-40)*. Albany, New York: State University of New York Press.

Streefland, L. (1984). Search for the Roots of Ratio: Some Thoughts on the Long Term Learning Process (Towards...A Theory): Part I: Reflections on a Teaching Experiment. *Educational Studies in Mathematics, 15*, 327-348.

Streefland, L. (1985 *a*). Search for the Roots of Ratio: Some Thoughts on the Long Term Learning Process (Towards...A Theory): Part II: The Outline of the Long Term Learning Process. *Educational Studies in Mathematics, 16*, 75-94.

Streefland, L. (1985 *b*). Wiskunde als activiteit en de realiteit als bron. *Nieuwe Wiskrant, 5*, 60-67.

Streefland, L. (1993). Fractions: A Realistic Approach. In T. P. Carpenter, E. Fennema and T. A. Romberg (Eds.), *Rational Numbers: An Integration of Research* (pp. 289-326). Hillsdale, NJ: Lawrence Erlbaum Associates.

Streefland, L. (1997). Charming fractions or fractions being charmed? In T. Nunes and P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 347-372). Hove (UK): Psychology Press.

Streefland, L., and Van den Heuvel-Panhuizen, M. (1992). *Evoking pupils' informal knowledge on percents.* Paper presented at the Proceedings of the Sixteenth PME Conference, University of New Hampshire, Durham, NH.

Thompson, P.W. (1993). Quantitative Reasoning, Complexity, and Additive Structures. *Educational Studies in Mathematics, 25*, 165-208.

Thompson, P. (1994). The Development of the Concept of Speed and Its Relationship to Concepts of Rate. In G. Harel and J. Confrey (Eds.), *The Development of Multiplicative Reasoning in the Learning of Mathematics* (pp. 181-236). Albany, New York: State University of New York Press.

Tourniaire, F., and Pulos, S. (1985). Proportional reasoning: A review of the literature. *Educational Studies in Mathematics, 16* 181-204.

Treffers, A. (1987). Integrated column arithmetic according to progressive schematisation. *Educational Studies in Mathematics, 18* 125-145.

Van den Brink, J. and Streefland, L. (1979). Young Children (6-8): Ratio and Proportion. *Educational Studies in Mathematics, 10*, 403-420.

Van de Heuvel-Panhuizen, M. (2003). The Didactical Use of Models in Realistic Mathematics Education: An Example from a Longitudinal Trajectory on Percentage. Realistic Mathematics Education Research: Leen Streefland's Work Continues. *Educational Studies in Mathematics, 54*, 9-35.

Van Dooren, W., Bock, D. D., Hessels, A., Janssens, D., and Verschaffel, L. (2004). Remedying secondary school students' illusion of linearity: a teaching experiment aiming at conceptual change. *Learning and Instruction, 14*, 485–501.

Vergnaud, G. (1979). The Acquisition of Arithmetical Concepts. *Educational Studies in Mathematics, 10*, 263-274.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh and M. Landau (Eds.), *Acquisition of Mathematics Concepts and Processes* (pp. 128-175). London: Academic Press.

Vergnaud, G. (1994). *Multiplicative Conceptual Field: What and Why? In G. Harel & J. Confrey (Eds.), The Development of Multiplicative Reasoning in the Learning of Mathematics (pp. 41-60)*. Albany, New York: State University of New York press.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes & P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 1-28). Hove (UK): Psychology Press.

Verschaffel, L. (1994). Using Retelling Data to Study Elementary School Children's Representations and Solutions of Compare Problems. *Journal for Research in Mathematics Education, 25*, 141-165.

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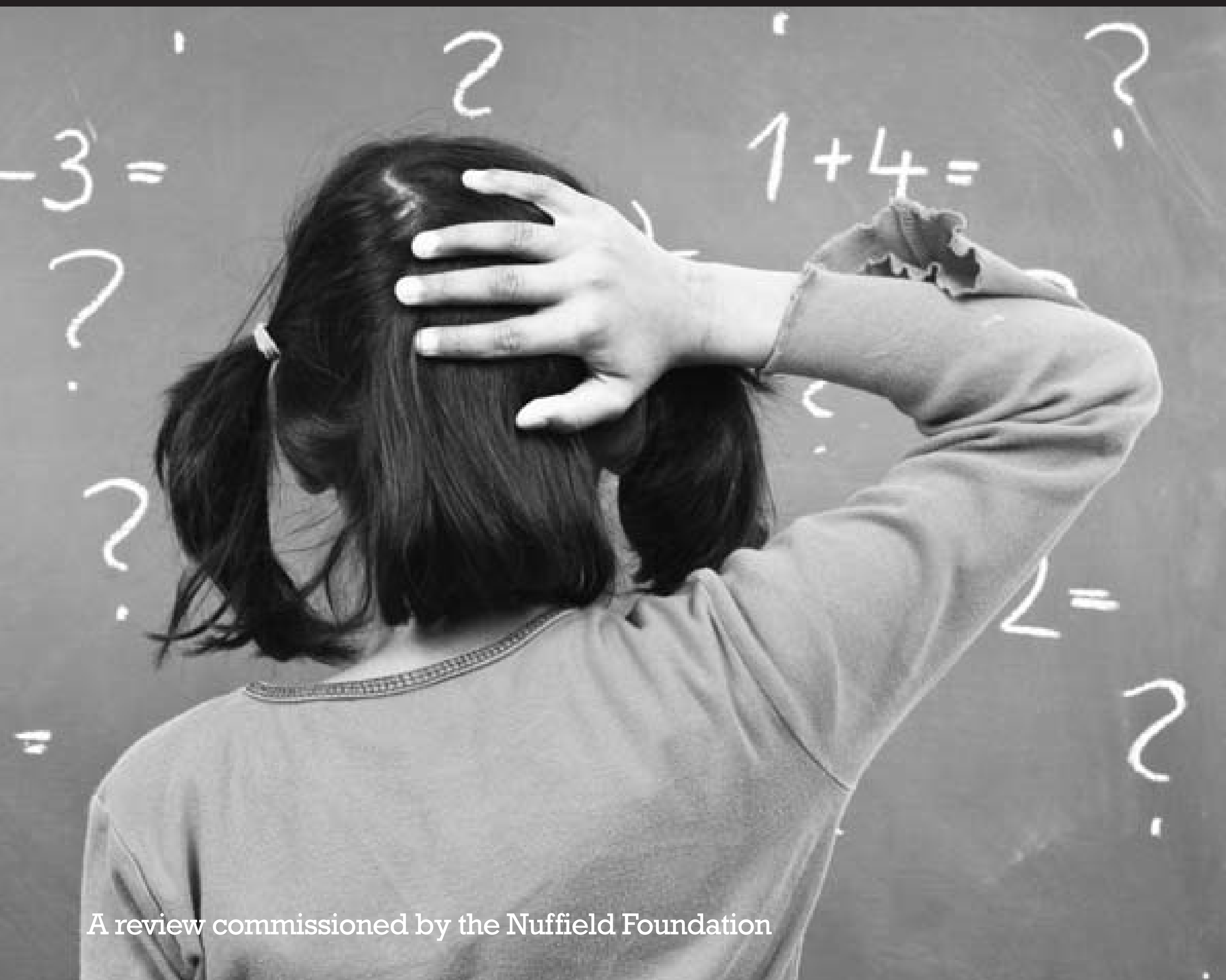
5

Key understandings in mathematics learning

Paper 5: Understanding space and its representation in mathematics

By Peter Bryant, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

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About the author

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 5: Understanding space and its representation in mathematics

Headlines

- Children come to school with a great deal of knowledge about spatial relations. One of the most important challenges in mathematical education is how best to harness this implicit knowledge in lessons about space.

- Children's pre-school implicit knowledge of space is mainly relational. Teachers should be aware of kinds of relations that young children recognise and are familiar with, such as their use of stable background to remember the position and orientation of objects and lines.

- Measuring of length and area poses particular problems for children, even though they are able to understand the underlying logic of measurement. Their difficulties concern iteration of standard units, which is a new idea for them, and also the need to apply multiplicative reasoning to the measurement of area.

- From an early age children are able to extrapolate imaginary straight lines, which allows them to learn how to use Cartesian co-ordinates to plot specific positions in space with no difficulty. However, they need instruction about how to use co-ordinates to work out the relation between different positions.

- Learning how to represent angle mathematically is a hard task for young children, even though angles are an important part of their everyday life. There is evidence that children are more aware of angle in the context of movement (turns) than in other contexts and learn about the mathematics of angle relatively easily in this context. However, children need a great deal of help from to teachers to understand how to relate angles across different contexts.

- An important aspect of learning about geometry is to recognise the relation between transformed shapes (rotation, reflection, enlargement). This also can be difficult, since children's pre-school experiences lead them to recognise the same shapes as equivalent across such transformations, rather than to be aware of the nature of the transformation. However, there is very little research on this important question.

- Another aspect of the understanding of shape is the fact that one shape can be transformed into another, by addition and subtraction of its subcomponents. For example, a parallelogram can be transformed into a rectangle of the same base and height by the addition and subtraction of equivalent triangles and adding two equivalent triangles to a rectangle creates a parallelogram. Research demonstrates that there is a danger that children might learn about these transformations only as procedures without understanding their conceptual basis.

- There is a severe dearth of psychological research on children's geometrical learning. In particular we need long-term studies of the effects of intervention and a great deal more research on children's understanding of transformations of shape.

At school, children often learn formally about matters that they already know a great deal about in an informal and often quite implicit way. Sometimes their existing, informal understanding, which for the most part is based on experiences that they start to have long before going to school, fits well with what they are expected to learn in the classroom. At other times, what they know already, or what they think they know, clashes with the formal systems that they are taught at school and can even prevent them from grasping the significance of these formal systems.

Geometry is a good and an obvious example. Geometry lessons at school deal with the use of mathematics and logic to analyse spatial relations and the properties of shapes. The spatial relations and the shapes in question are certainly a common part of any child's environment, and psychological research has established that from a very early age children are aware of them and quite familiar with them. It has been shown that even very young babies not only discriminate regular geometric shapes but can recognise them when they see them at a tilt, thus co-ordinating information about the orientation of an object with information about the pattern of its contours.

Babies are also able to extrapolate imaginary straight lines (a key geometric skill) at any rate in social situations because they can work out what someone else is looking at and can thus construct that person's line of sight. Another major early achievement by young children is to master the logic that underlies much of the formal analysis of spatial relations that goes on in geometry. By the time they first go to school young children can make logical transitive inferences (A > B, B > C, therefore A > C; A = B, B = C, therefore A = C), which are the logical basis of all measurement. In their first few years at school they also become adept at the logic of inversion (A + B – B), which is a logical move that is an essential part of studying the relation between shapes.

Finally, there is strong evidence that most of the information about space that children use and remember in their everyday lives is relational in nature. One good index of this is that children's memory of the orientation of lines is largely based on the relation between these lines and the orientation of stable features in the background. For this reason children find it much easier to remember the orientation of horizontal and vertical lines than of diagonal lines, because horizontal and vertical features are quite common in the child's stable spatial environment. For the same reason, young children remember and reproduce right angles (perpendicular lines) better than acute or obtuse angles. The relational nature of children's spatial perception and memory is potentially a powerful resource for learning about geometry, since spatial relations are the basic subject matter of geometry.

With so much relevant informal knowledge about space and shape to draw on, one might think that children would have little difficulty in translating this knowledge into formal geometrical understanding. Yet, it is not always that easy. It is an unfortunate and well-documented fact that many children have persistent difficulties with many aspects of geometry.

One evidently successful link between young children's early spatial knowledge and their more formal experiences in the classroom is their learning how to use Cartesian co-ordinates to plot positions in two-dimensional space. This causes schoolchildren little difficulty, although it takes some time for them to understand how to work out the relation between two positions plotted in this way.

Other links between informal and formal knowledge are harder for young children. The apparently simple act of measuring a straight line, for example, causes them problems even though they are usually perfectly able to make the appropriate logical moves and understand the importance of one-to-one correspondence, which is an essential part of relating the units on a ruler to the line being measured. One problem here is that they find it hard to understand the idea of iteration: iteration is about repeated measurements, so that a ruler consists of a set of iterated (repeated) units like centimetres. Iteration is necessary when a particular length being measured is longer than the measuring instrument. Another problem is that the one-to-one correspondence involved in measuring a line with a ruler is asymmetrical. The units (centimetres, inches) are visible and clear in the ruler but have to be imagined on the line itself. It is less of a surprise that it also takes children a great deal of time to come to terms with the fact that measurement of area usually needs some form of multiplication, e.g. height x width with rectangles, rather than addition.

The formal concept of angle is another serious stumbling block for children even though they are familiar enough with angles in their everyday spatial environments. The main problem is that they find it hard to grasp that two angles in very different contexts are the same, e.g. themselves turning 90^{O} and the corner of a page in a book. Abstraction is an essential part of geometry but it has very little to do with children's ordinary spatial perception and knowledge.

For much the same reason, decomposing a relatively complex shape into several simpler component shapes – again an essential activity in geometry – is something that many children find hard to do. In their ordinary lives it is usually more important for them to see shapes as unities, rather than to be able to break them up into other shapes. This difficulty makes it hard for them to work out relationships between shapes.

For example, children who easily grasp that $a + b - b = a$, nevertheless often fail to understand completely the demonstration that a rectangle and a parallelogram with the same base and height are equal in area because you can transform the parallelogram into the rectangle by subtracting a triangle from one end of the parallelogram and adding an exactly equivalent triangle to the other end.

We know little about children's understanding of transformations of shape or of any difficulties that they might have when they are taught about these transformations. This is a serious gap in research on children's mathematical learning. It is well recognised, however, that children and some adults confuse scale enlargements with enlargements of area. They think that doubling the length of the contour of a geometric shape such as a square or a rectangle also doubles its area, which is a serious misconception. Teachers should be aware of this potential difficulty when they teach children about scale enlargements.

Researchers have been more successful in identifying these obstacles than in showing us how to help children to surmount them. There are some studies of ways of preparing children for geometry in the pre-school period or in the early years at school. This research, however, concentrated on short-term gains in children's geometric understanding and did not answer the question whether these early teaching programmes would actually help children when they begin to learn about geometry in the classroom.

There has also been research on teaching children about angle, mostly in the context of computer-based teaching programmes. One of the most interesting points to come out of this research is that teaching children about angle in terms of movements (turning) is successful, and there is some evidence that children taught this way are quite likely to transfer their new knowledge about angle to other contexts that do not involve movement.

However, there has been no concerted research on how teachers could take advantage of children's considerable spatial knowledge when teaching them geometry. We badly need long-term studies of interventions that take account of children's relational approach to the spatial environment and encourage them to grasp other relations, such as the relation between shapes and the relation between shapes and their subcomponent parts, which go beyond their informal spatial knowledge.

Recommendations

Research about mathematical learning	Recommendations for teaching and research
Children's pre-school knowledge of space is relational. They are skilled at using stable features of the spatial framework to perceive and remember the relative orientation and position of objects in the environment. There is, however, no research on the relation between this informal knowledge and how well children learn about geometry.	**Teaching** Teachers should be aware of the research on children's considerable spatial knowledge and skills and should relate their teaching of geometrical concepts to this knowledge. **Research** There is a serious need for longitudinal research on the possible connections between children's pre-school spatial abilities and how well they learn about geometry at school.
Children already understand the logic of measurement in their early school years. They can make and understand transitive inferences, they understand the inverse relation between addition and subtraction, and they can recognise and use one-to-one correspondence. These are three essential aspects of measurement.	**Teaching** The conceptual basis of measurement and not just the procedures should be an important part of the teaching. Teachers should emphasise transitive inferences, inversion of addition and subtraction and also one-to-one correspondence and should show children their importance. **Research** Psychologists should extend their research on transitive inference, inversion and one-to-one correspondence to geometrical problems, such as measurement of length and area.

Recommendations (continued)

Research about mathematical learning	Recommendations for teaching and research
Many children have difficulties with the idea of iteration of standard units in measurement.	**Teaching** Teachers should recognise this difficulty and construct exercises which involve iteration, not just with standard units but with familiar objects like cups and hands. **Research** Psychologists should study the exact cause of children's difficulties with iteration.
Many children wrongly apply additive reasoning, instead of multiplicative reasoning, to the task of measuring area. Children understand this multiplicative reasoning better when they first think of it as the number of tiles in a row times the number of rows than when they try to use a base times height formula.	**Teaching** In lessons on area measurement, teachers can promote children's use of the reasoning 'number in a row times number of rows' by giving children a number of tiles that is insufficient to cover the area. They should also contrast measurements which do, and measurements which do not, rest on multiplication.
Even very young children can easily extrapolate straight lines and schoolchildren have no difficulty in learning how to plot positions using Cartesian co-ordinates, but it is difficult for them to work out the relation between different positions plotted in this way.	**Teaching** Teachers, using concrete material, should relate teaching about spatial co-ordinates to children's everyday experiences of extrapolating imaginary straight lines. **Research** There is a need for intervention studies on methods of teaching children to work out the relation between different positions, using co-ordinates.
Research on pre-school intervention suggests that it is possible to prepare children for learning about geometry by enhancing their understanding of space and shapes. However, this research has not included long-term testing and therefore the suggestion is still tentative.	**Research** There will have to be long-term predictive and long-term intervention studies on this crucial, but neglected, question
Children often learn about the relation between shapes (e.g. between a parallelogram and a rectangle) as a procedure without understanding the conceptual basis for these transformations.	**Teaching** Children should be taught the conceptual reasons for adding and subtracting shape components when studying the relation between shapes. **Research** Existing research on this topic was done a very long time ago and was not very systematic. We need well-designed longitudinal and intervention studies on children's ability to make and understand such transformations.
There is hardly any research on children's understanding of the transformation of shapes, but there is evidence of confusion in many children about the effects of enlargement: they consider that doubling the length of the perimeter of a square, for example, doubles its area.	**Teaching** Teachers should be aware of the risk that children might confuse scale enlargements with area enlargements. **Research** Psychologists could easily study how children understand transformations like reflection and rotation but they have not done so. We need this kind of research.

Understanding space and its representation in mathematics

From informal understanding to formal misunderstanding of space

This paper is about children's informal knowledge of space and spatial relations and about their formal learning of geometry. It also deals with the connection between these two kinds of knowledge. This connection is much the same as the one between knowledge about quantitative relations on the one hand and about number on the other hand, which we described in Papers 1 and 2. We shall show how young children build up a large and impressive, but often implicit, understanding of spatial relations before they go to school and how this knowledge sometimes matches the relations that they learn in geometry very well and sometimes does not.

There is a rich vein of research on children's spatial knowledge – knowledge which they acquire informally and, for the most part, long before they go to school – and this research is obviously relevant to the successes and the difficulties that they have when they are taught about geometry at school. Yet, with a few honourable exceptions, the most remarkable of which is a recent thorough review by Clements and Sarama (2007 *b*), there have been very few attempts indeed to link research on children's informal, and often implicit, knowledge about spatial relations to their ability to carry out the explicit analyses of space that are required in geometry classes.

The reason for this gap is probably the striking imbalance in the contribution made by psychologists and by maths educators to research on geometrical learning. Although psychologists have studied children's informal understanding of space in detail and with great success, they have virtually ignored children's learning about geometry, at any rate in recent years. Despite Wertheimer's (1945) and Piaget, Inhelder and Szeminska's (1960) impressive pioneering work on children's understanding of geometry, which we shall describe later, psychologists have virtually ignored this aspect of children's education since then. In contrast, mathematics educators have made steady progress in studying children's geometry with measures of what children find difficult and studies of the effects of different kinds of teaching and classroom experience.

One effect of this imbalance in the contribution of the two disciplines to research on learning about geometry is that the existing research tells us more about educational methods than about the underlying difficulties that children have in learning about geometry. Another result is that some excellent ideas about enhancing children's geometrical understanding have been proposed by educationalists but are still waiting for the kind of empirical test that psychologists are good at designing and carrying out.

The central problem for anyone trying to make the link between children's informal spatial knowledge and their understanding of geometry is easy to state. It is the stark contrast between children's impressive everyday understanding of their spatial environment and the difficulties that they have in learning how to analyse space mathematically. We shall start our review with an account of the basic spatial knowledge that children acquire informally long before they go to school.

Early spatial knowledge: perception

Shape, size, position and extrapolation of imagined straight lines

Spatial achievements begin early. Over the last 30 years, experimental work with young babies has clearly shown that they are born with, or acquire very early on in their life, many robust and effective perceptual abilities. They can discriminate objects by their shape, by their size and by their orientation and they can perceive depth and distinguish differences in distance (Slater, 1999; Slater and Lewis 2002; Slater, Field and Hernadez-Reif, 2002; Bremner, Bryant and Mareschal, 2006).

They can even co-ordinate information about size and distance (Slater, Mattock and Brown, 1990), and they can also co-ordinate information about an object's shape and its orientation (Slater and Morrison, 1985). The first co-ordination makes it possible for them to recognise a particular object, which they first see close up, as the same object when they see it again in the distance, even though the size of the visual impression that it now makes is much smaller than it was before. With the help of the second kind of co-ordination, babies can recognise particular shapes even when they see them from completely different angles: the shape of the impression that these objects make on the visual receptors varies, but babies can still recognise them as the same by taking the change in orientation into account.

We do not yet know how children so young are capable of these impressive feats, but it is quite likely that the answer lies in the relational nature of the way that they deal with size (and, as we shall see later, with orientation), as Rock (1970) suggested many years ago. A person nearby makes a larger visual impression on your visual system than a person in the distance but, if these two people are roughly the same size as each other, the relation between their size and that of familiar objects near each of them, such as cars and bus-stops and wheelie-bins, will be much the same.

The idea that children judge an object's size in terms of its relation to the size of other objects at the same distance receives some support from work on children's learning about relations. When four-year-old children are asked to discriminate and remember a particular object on the basis of its size, they do far better when it is possible to solve the problem on the basis of size relations (e.g. it is always the smaller one) than when they have to remember its absolute size (e.g. it is always exactly so large) (Lawrenson and Bryant, 1972).

Another remarkable early spatial achievement by infants, which is also relational and is highly relevant to much of what they later have to learn in geometry lessons, is their ability to extrapolate imaginary straight lines in three dimensional space (Butterworth, 2002). Extrapolation of imagined straight lines is, of course, essential for the use of Cartesian co-ordinates to plot positions in graphs and in maps, but it also is a basic ingredient of very young children's social communication (Butterworth and Cochrane, 1980; Butterworth and Grover, 1988). Butterworth and Jarrett (1991) showed this in a study in which they asked a mother to sit opposite her baby and then to stare at some predetermined object which was either in front and in full view of the child or was behind the child, so that he had to turn his head in order to see it. The question was whether the baby would then look at the same object, and to do this he would have to extrapolate a straight line that represented his mother's line of sight. Butterworth and Jarrett found that babies younger than 12 months manage to do this most of the time when the object in question was in front of them. They usually did not also turn their heads to look at objects behind them when these apparently caught their mothers' attention. But 15-month-old children did even that: they followed their mother's line of sight whether it led them to objects already in full view or to ones behind them. A slightly later development that also involves extrapolating imaginary straight lines is the ability to point and to look in the direction of an object that someone else is pointing at, which infants manage do with great proficiency (Butterworth and Morisette, 1996; Butterworth and Itakura, 2000).

Orientation and position

The orientation of objects and surfaces are a significant and highly regular and predictable part of our everyday spatial environments. Walls usually are, and usually have to be, vertical: objects stay on horizontal surfaces but tend to slide off sloping surfaces. The surface of still liquid is horizontal: the opposite edges of many familiar manufactured objects (doors, windows, television sets, pictures, book pages) are parallel: we ourselves are vertical when we walk, horizontal when we swim. Yet, children seem to have more difficulty distinguishing and remembering information about orientation than information about other familiar spatial variables.

Horizontals and verticals are not the problem. Five-year-old children take in and remember the orientation of horizontal and vertical lines extremely well (Bryant 1969, 1974; Bryant and Squire, 2001). In contrast, they have a lot of difficulty in remembering either the direction or slope of obliquely oriented lines. There is, however, an effective way of helping them over this difficulty with oblique lines. If there are other obliquely oriented lines in the background that are parallel to an oblique line that they are asked to remember, their memory of the slope and direction for this oblique line improves dramatically (see Figure 5.1). The children use the parallel relation between the line that they have to remember and stable features in the background framework to store and recognise information about the oblique line.

This result suggests a reason for the initial radical difference in how good their memory is for vertical and horizontal features and how poor it is for obliquely oriented ones. The reason, again, is about relations. It is that that there are usually ample stable horizontal and vertical features in the background to relate these lines to. Stable, background features that parallel particular lines which are not either vertical or horizontal are much less common. If this idea is right, young children are already relying on spatial relations that are at the heart of Euclidean geometry to store information about the spatial environment by the time that they begin to be taught formally about geometry.

However, children do not always adopt this excellent strategy of relating the orientation of lines to permanent features of the spatial environment. Piaget and Inhelder's (1963) deservedly famous and often-repeated experiment about children drawing the level of water in a tilted container is the best example. They showed the children tilted glass containers (glasses, bottles) with liquid in them (though the containers were tilted, the laws of nature dictated that the level of the liquid in them was horizontal). They also gave the children a picture of an empty, tilted container depicted as just above a table top which was an obvious horizontal background feature. The children's task was to draw in the level of the liquid in the drawing so that it was

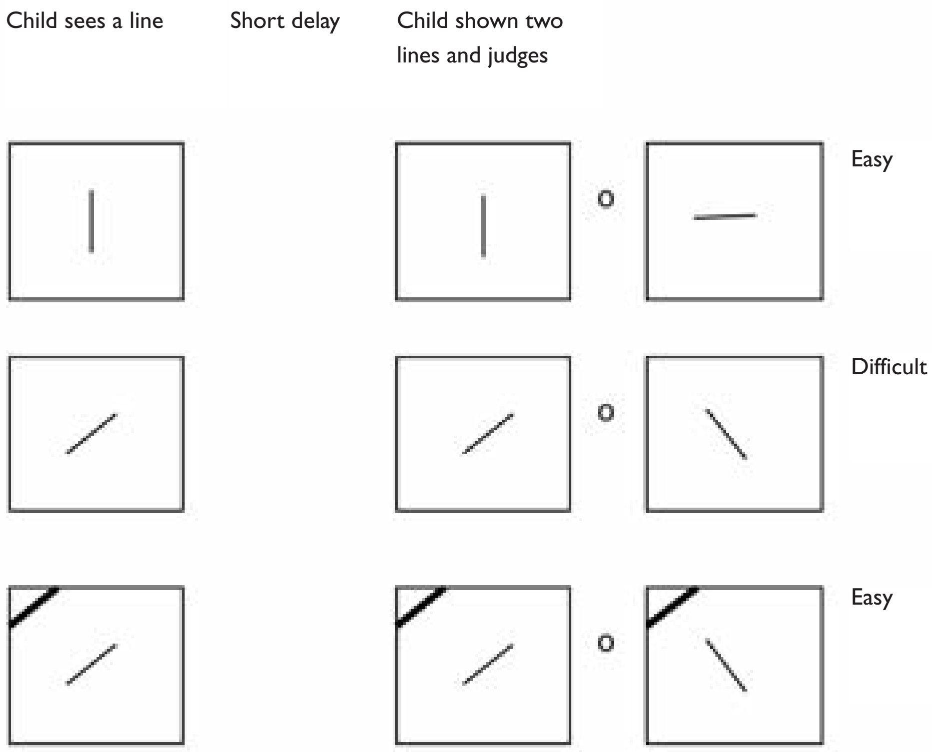

Figure 5.1: Children easily remember horizontal or vertical lines but not oblique lines unless they can relate oblique lies to a stable background feature.

exactly like the liquid in the experimenter's hand. The question that Piaget and Inhelder asked was whether they would draw the liquid as parallel to the table top or, in other words, as horizontal.

Children below the age of roughly eight years did not manage to do this. Many of them drew the liquid as perpendicular to the sides (when the sides were straight) and parallel to the bottom of the container. It seems that the children could not take advantage of the parallel relation between the liquid and the table top, probably because they were preoccupied with the glass itself and did not manage to shift their attention to an external feature.

Piaget and Inhelder treated the young child's difficulties in this drawing task as a failure on the child's part to notice and take advantage of a basic Euclidean relation, the parallel relation between two horizontal lines. They argued that a child who makes this mistake does not have any idea about horizontality: he or she is unaware that horizontal lines and surfaces are an important part of the environment and that some surfaces, such as still liquid, are constantly horizontal.

Piaget and Inhelder then extended their argument to verticality. They asked children to copy pictures of objects that are usually vertical, such as trees and chimneys. In the pictures that the children had to copy, these objects were positioned on obliquely oriented surfaces: the trees stood vertically on the side of a steeply sloping hill and vertical chimneys were placed on sloping roofs. In their copies of these pictures, children younger than about eight years usually drew the trees and chimneys as perpendicular to their baselines (the side of the hill or the sloping roof) and therefore with an oblique orientation. Piaget and Inhelder concluded that children of this age have not yet realised that the space around them is full of stable vertical and horizontal features.

There is something of a conflict between the two sets of results that we have just presented. One (Bryant 1969, 1974; Bryant and Squire, 2001) suggests that young children detect, and indeed rely on, parallel relations between objects in their immediate perception and stable background features. The other (Piaget and Inhelder, 1963) leads to the conclusion that children completely disregard these relations. However, this is not a serious problem. In the first set of experiments the use that children made of parallel relations was probably implicit. The second set of experiments involved drawing tasks, in which the children had to make explicit judgements about such

When children see 2-line figure A and are asked to copy in the missing line on B either by placing or drawing a straight wire, they represent the line as nearer to the perpendicular than it is

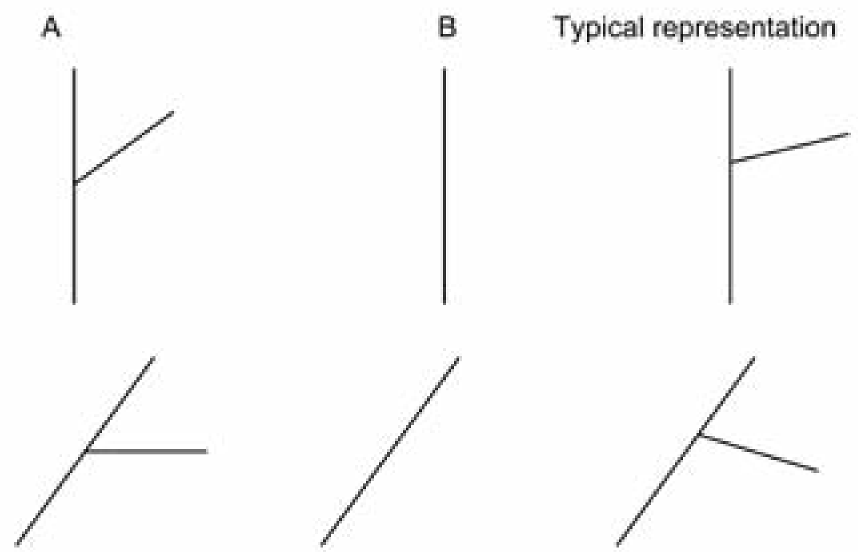

Figure 5.2: The perpendicular bias

relations. Children probably perceive and make use of parallel relations without being aware of doing so. The implication for teaching children is an interesting one. It is that one important task for the teacher of geometry is to transform their implicit knowledge into explicit knowledge.

There is another point to be made about the children's mistakes in Piaget and Inhelder's studies. One possible reason, or partial reason, for these mistakes might have been that in every case (the liquid in a tilted container, trees on the hillside, chimneys on the sloping roofs) the task was to draw the crucial feature as non-perpendicular in relation to its baseline. There is plenty of evidence (Ibbotson and Bryant, 1976) that, in copying, children find it quite difficult to draw one straight line that meets another straight line, the baseline, when the line that they have to draw is obliquely oriented to that baseline (see Figure 5.2).

They tend to misrepresent the line that they are drawing either as perpendicular to the baseline or as closer to the perpendicular than it should be. There are various possible reasons for this 'perpendicular error', but at the very least it shows that children have some difficulty in representing non-perpendicular lines. The work by Piaget *et al.* establishes that the presence of stable, background features of the spatial environment, like the table top, does not help children surmount this bias.

Early spatial knowledge: logic and measurement

Inferences about space and measurement

The early spatial achievements that we have described so far are, broadly speaking, perceptual ones. Our next task is to consider how young children reason about space. We must consider whether young children are able to make logical inferences about space and can understand other people's inferential reasoning about space by the age when they first go to school.

We can start with spatial measurements. These depend on logical inferences about space. Measurement allows us to make comparisons between quantities that we cannot compare directly. We can work out whether a washing line is long enough to stretch between posts by measuring the line that we have and the distance between the posts. We compare the two lengths, the length of the line and the distance from one post to the other indirectly, by comparing both directly to the same measuring instrument – a tape measure or ruler. We combine two direct comparisons to make an indirect comparison.

When we put two pieces of information together in this way in order to produce a new conclusion, we are making a logical inference. Inferences about continua, like length, are called *transitive inferences*. We, adults, know that if A = B in length and B = C, then A is necessarily the same length as C, even though we have never seen A and C together and therefore have not been able to compare them directly. We also know, of course, that if A > B and B > C (in length), then A > C, again without making a direct comparison between A and C. In these inferences B is the independent measure through which A and C can be compared.

Piaget, Inhelder and Szeminska (1960) were the first to discuss this link between understanding logic and being able to measure in their well-known book on geometry. They argued that the main cause of the difficulties that children have in learning about measurement is that they do not understand transitive inferences. These authors' claim about the importance of transitive inferences in learning about measurement is indisputable and an extremely important one. However, their idea that young children cannot make or understand transitive inferences has always been a controversial one, and it is now clear that we must make a fundamental distinction between being able to make the inference and knowing when this inference is needed and how to put it into effect.

There are usually two consecutive parts to a transitive inference task. In the first, the child is given two premises (A = B, B = C) and in the second he or she has to try to draw an inference from these premises. For example, in Piaget's first study of transitive inferences, which was not about length but about the behaviour of some fictional people, he first told the children that 'Mary is naughtier than Sarah, and Sarah is naughtier than Jane' and then asked then 'Who was the naughtier, Mary or Jane?' Most children below the age of roughly nine years found, and still do find, this an extremely difficult question and often say that they cannot tell. The failure is a dramatic one, but there are at least two possible reasons for it.

One, favoured by Piaget himself, that the failure is a logical one – that children of this age simply cannot put two premises about quantity together logically. It is worth noting that Piaget thought that the reason that young children did not make this logical move was that they could not conceive that the middle term (B when the premises are A > B and B > C) could simultaneously have one relation to A and another, different, relation to C.

The second possible reason for children not making the transitive inference is about memory. The children may be unable to make the inference simply because they have forgotten, or because they did not bother to commit to memory in the first place, one or both of the premises. The implication here is that they would be able to make the inference if they could remember both premises at the time that they were given the inferential question.

One way to test the second hypothesis is to make sure that the children in the study do remember the premises, and also to take the precaution of measuring how well they remember these premises at the same time as testing their ability to draw a transitive inference. Bryant and Trabasso (1971) did this by repeating the information about the premises in the first part of the task until the children had learned it thoroughly, and then in the second part checking how well they remembered this information and testing how well they could answer the inferential questions at the same time. In this study even the four-year-olds were able remember the premises and they managed to put them together successfully to make the correct transitive inference on 80% of the trials. The equivalent figure for the five-year-olds was 89%.

Young children's success in this inferential task suggests that they have the ability to make the inference that underlies measurement, but we still have to find out how well they apply this ability to measurement itself. Here, the research of Piaget *et al.* (1960) on measurement provides some interesting suggestions. These researchers showed children a tower made of bricks of different sizes. The tower was placed on a small table and each child was asked to build another tower of the same height on another lower table that was usually, though not always, on the other side of the other side of a partition, so that the child had to create the replica without being able compare it directly to the original tower. Piaget *et al.* also provided the child with various possible measurement instruments, such as strips of paper and a straight stick, to help her with the task, and the main question that they asked was whether the child would use any of these as measures to compare the two towers.

Children under the age of (roughly) eight years did not take advantage of the measuring instruments. Either they tried to do the task by remembering the original while creating the replica, which did not work at all well, or they used their hands or their body as a measuring instrument. For example, some children put one hand at the bottom and the other at the top of the original tower and then walked to the other tower trying at the same time to keep their hands at a constant distance from each other. This strategy, which Piaget *et al.* called 'manual transfer', tended not to be successful either, for the practical reason that the children also had to use and move their hands to add and subtract bricks to their own tower. Older children, in contrast, were happy to use the strips of paper or the dowel rod as a makeshift ruler to compare the two towers. Piaget *et al.* claimed that the children who did not use the measuring instruments failed the task because they were unable to reason about it logically. They also argued that children's initial use of their own body was a transitional step on the way to true measurement using an 'independent middle term'.

This might be too pessimistic a conclusion. There is an alternative explanation for the reactions of the children who did not attempt to use a measure at all. It is that children not only have to be able to make an inference to do well in any measuring task: they also have to realise that a direct comparison will not do, and thus that instead they should make an indirect, inferential, comparison with the help of a reliable intervening measure.

There is some evidence to support this idea. If it is right, children should be ready to measure in a task in which it is made completely obvious that direct comparisons would not work. Bryant and Kopytynska (1976) devised a task of this sort. First, they gave a group of five- and six-year-old children a version of Piaget *et al.*'s two towers task, and all of them failed. Then, in a new task, they gave the children two blocks of wood, each with a hole sunk in the middle in such a way that it was impossible to see how deep either hole was. They asked the children to find out whether the two holes were as deep as each other or not. The children were also given a rod with coloured markings. The question was whether the children, who did not measure in Piaget *et al.*'s task, would start to use a measure in this new task in which it was clear that a direct comparison would be useless.

Nearly all the children used the rod to measure both holes in the blocks of wood at least once (they were each given four problems) and over half the children measured and produced the right answer in all four problems. It seems that children of five years or older are ready to use an intervening measure to make an indirect comparison of two quantities. Their difficulty is in knowing when to distrust direct comparisons enough to resort to measurement.

Iteration and measurement

One interesting variation in the study of measurement by Piaget *et al.* (1960) was in the length of the straight dowel rod, which was the main measuring tool in this task. The rod's length equalled the height of the original tower (R = T) in some problems but in others the rod was longer (R > T) and in others still it was shorter (R < T) than the tower.

The older children who used the rod as a measure were most successful when R = T. They were slightly less successful when R > T and they had to mark a point on the rod which coincided with the summit of the tower. In contrast, the R < T problems were particularly difficult, even for the children who tried to use the rod as a measure. The solution to such problems is iteration which, in this case, is to apply the rod more than once to the tower: the child has to mark a point to represent the length of the ruler and then to start measuring again from this point.

It is worth noting that iteration also involves a great deal of care in its execution. You must cover all the surface that you are measuring, all its length in these examples, but you must never overlap – never measure any part of the surface twice.

Iteration in measurement is interesting because the people who do it successfully are actually constructing their own measure and therefore certainly have a strong and effective understanding of measurement. Piaget *et al.* (1960) also argued that children's eventual realisation that iteration is the solution to some measuring problems is the basis for their eventual understanding of the role of standardised units such as centimetres and metres. We use these units, they argued, in an iterative way: 1 metre is made up of 100 iterations of 1 centimetre, and one kilometre consists of 1000 iterations of 1 metre. Children's first insight into this iterative system, according to Piaget *et al.*, comes from their initial experiences with R < T problems. This is an interesting causal hypothesis that has some serious educational implications. It should be tested.

Conclusions about children's early spatial knowledge

- Children have a well-developed and effective relational knowledge of shape, position, distance, spatial orientation and direction long before they go to school. This knowledge may be implicit and non-numerical for the most part, but it is certainly knowledge that is related to geometry.

- The mistakes that children make in drawing horizontal and vertical lines are probably due to preferring to concentrate on relations between lines close to each other (liquid in a glass is perpendicular to the sides of the glass) rather than to separated lines (liquid in a glass is parallel to horizontal surfaces like table tops). This is a mistake not in relational perception, but in picking the right relation.

- Children are also able to understand and to make transitive inferences, which are the basic logical move that underlies measurement, several years before being taught about geometry.

- We do not yet know how well they can cope with the notion of iteration in the school years.

- There is no research on the possible causal links between these impressive early perceptual and logical abilities and the successes and difficulties that children have when they first learn about geometry. This is a serious gap in our knowledge about geometrical learning.

The connections between children's knowledge of space before being taught geometry and how well they learn when they are taught about geometry

To what extent does children's early spatial development predict their success in geometry later on? The question is simple, clear and overwhelmingly important. If we were dealing with some other school subject – say learning to read – we would have no difficulty in finding an answer, perhaps more than one answer, about the importance of early, informal learning and experience, because of the very large amount of work done on the subject. With geometry, however, it is different. Having established that young children do have a rich and in many ways sophisticated understanding of their spatial environment, psychologists seem to have made their excuses and left the room. Literally hundreds of longitudinal and intervention studies exist on what children already know about language and how they learn to read and spell. Yet, as far as we know, no one has made a systematic attempt, in longitudinal or intervention research, to link what children know about space to how they learn the mathematics of spatial relations, even though there are some extremely interesting and highly specific questions to research.

To take one example, what connections are there between children's knowledge of measurement before they learn about it and how well they learn to use and understand the use of rulers? To take another, we know that children have a bias towards representing angles as more perpendicular than they are: what connection is there between the extent of this bias and the success that children have in learning about angles, and is the relation a positive or a negative one? These are practicable and immensely interesting questions that could easily be answered in longitudinal studies. It is no longer a matter of what is to be done. The question that baffles us is: why are the right longitudinal and intervention studies not being done?

How can we intervene to prepare young children in the pre-school period for geometry? If there is a connection between the remarkable spatial knowledge that we find in quite young children and their successes and failures in learning about geometry later on, it should be possible to work on these early skills and enhance them in various ways that will help them learn about geometry when the time comes.

Here the situation is rather different. Educators have produced systematic programmes to prepare children for formal instruction in geometry. Some of these are ingenious and convincing, and they deserve attention. The problem in some cases is a lack of empirical evaluation.

One notable programme comes from the highly respected Freudenthal Institute in the Netherlands. A team of educational researchers there (van den Heuven-Panhuizen and Buys, 2008) have produced an ingenious and original plan for enhancing children's geometric skills before the age when they would normally be taught in a formal way about the subject. We shall concentrate here on the recommendations that van den Heuven-Panhuizen and Buys make for introducing kindergarten children to some basic geometrical concepts. However, we shall begin with the remark that, though their recommendations deserve our serious attention, the Freudenthal team offer us no empirical evidence at all that they really do work. Neither intervention studies with pre-tests and post-tests and randomly selected treatment groups, nor longitudinal predictive projects, seem to have played any part in this particular initiative.

The basic theoretical idea behind the Freudenthal team's programme for preparing children for geometry is that children's everyday life includes experiences and activities which are relevant to geometry but that the geometric knowledge that kindergarten children glean from these experiences is implicit and unsystematic. The solution that the team offers is to give these young children a systematic set of enjoyable game-like activities with familiar material and after each activity is finished to discuss and to encourage the children to reflect on what they have just done.

Some of these activities are about measurement (Buys and Veltman, 2008). In one interesting example, a teacher encourages the children to find out how many cups of liquid would fill a particular bottle and, when they have done that, to work out how many cups of liquid the bottle would provide when the bottle is not completely full. This leads to the idea of putting marks on the bottle to indicate when it contains one or more cups' worth of liquid. Thus, the children experience measurement units and also iteration. In another measurement activity children use conventional measures. They are given three rods each a metre long and are asked to measure the width of the room. Typically the children start well by forming the rods into a straight 3-metre line, but then hit the problem of measuring the remaining space: their first reaction is to ask for more rods, but the teacher then provides the suggestion that instead they try moving the first rod ahead of the third and then to move the second rod: Buys and Veltman report that the children readily follow this suggestion and apparently understand the iteration involved perfectly well.

Other exercises, equally ingenious, are about constructing and operating on shapes (van den Heuvel-Panhuizen, Veltman, Janssen and Hochstenbach, 2008). The Freudenthal team use the device of folding paper and then cutting out shapes to encourage children to think about the relationship between shapes: cutting an isosceles triangle across the fold, for example, creates a regular parallelogram when the paper is unfolded. The children also play games that take the form of four children creating a four-part figure between them with many symmetries: each child produces the mirror-image of the figure that the previous child had made (see Figure 5.3). The aim of such games is to give children systematic experience of the transformations, rotation and reflection, and to encourage them to reflect on these transformations.

If this group of researchers is right, children's early knowledge of geometric relationships and comparisons, though implicit and unsystematic, plays an important part in their eventual learning about geometry. It is a resource that can be enhanced by sensitive teaching of the kind that the Freudenthal group has pioneered. They may be right, but someone has to establish, through empirical research, how right they are.

There are a few empirical studies of ways of improving spatial skills in pre-school children. In these the children are given pre-tests which assess how well they do in spatial tasks which are suitable for children of that age, then go through intervention sessions which are designed to increase some of these skills and finally, soon after the end of this teaching, they are given post-tests to measure improvement in the same skills.

Two well designed studies carried out by Casey and her colleagues take this form (Casey, Erkut, Ceder and Young, 2008; Casey, Andrews, Schindler, Kersh and Young, 2008). In both studies the researchers were interested in how well five- and six-year-old children can learn to compose geometric shapes by combining other geometric shapes and how well they decompose shapes into component shapes, and also whether it is easier to improve this particular skill when it is couched in the context of a story than when the context is a more formal and abstract one.

The results of these two studies showed that the special instruction did, on the whole, help children to compose and decompose shapes and did have an effect on related spatial skills in the children who were taught in this way. They also showed that the narrative context added to the effect of teaching children at this age. Recently, Clements and Sarama (2007 *a*) reported a very different study of slightly younger, nursery children. These researchers were interested in the effects of a pre-school programme, called Building Blocks, the aim of which is to prepare children for mathematics in general including geometry. This programme is based on a theory about children's mathematical development: as far as

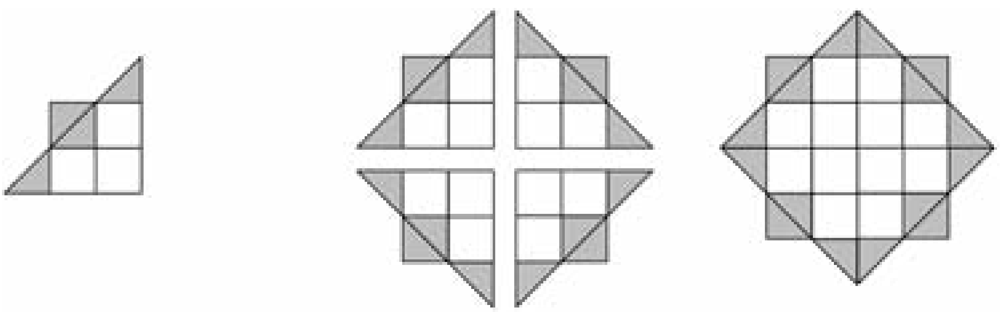

Figure 5.3: An activity devised by van den Heuvel-Panhuizen, Veltman, Janssen and Hochstenbach: four children devise a four-part shape by forming mirror-images.

geometry is concerned Clement and Sarama's strongest interest is in children's awareness of the composition of shapes and the relationship between different shapes. They also believe that the actual teaching given to individual children should be determined by their developmental levels. Thus the day-to-day instruction in their programme depends on the children's developmental trajectories. Clements and Sarama report that the young children taught in the Building Blocks programmes improved from pre-test to post-test more rapidly than children taught in other ways in tasks that involved constructing or relating shapes.

These are interesting conclusions and a good start. However, research on the question of the effects of intervention programmes designed to prepare children for geometry need to go further than this. We need studies of the effects of pre-school interventions on the progress that children make when they are eventually taught geometry at school a few years later on. We cannot be sure that the changes in the children's skills that were detected in these studies would have anything to do with their successes and failures later on in geometry.

Summary

1 Children have a well-developed and effective relational knowledge of shape, position, distance, spatial orientation and direction long before they go to school. This knowledge may be implicit and non-numerical for the most part, but it is certainly knowledge that is related to geometry.

2 The mistakes that children make in drawing horizontal and vertical lines are probably due to them preferring to concentrate on relations between lines close to each other (liquid in a glass is perpendicular to the sides of the glass) rather than to separated lines (liquid in a glass is parallel to horizontal surfaces like table tops). This is a mistake not in relational perception, but in picking the right relation.

3 Children are also able to understand and to make transitive inferences, which are the basic logical move that underlies measurement, several years before being taught about geometry.

4 We do not yet know how well they can cope with the notion of iteration in the school years.

5 There is little research on the possible causal links between these impressive early perceptual and logical abilities and the successes and difficulties that children have when they first learn about geometry. This is a serious gap in our knowledge about geometrical learning.

Learning about geometry

The aim of teaching children geometry is to show them how to reason logically and mathematically about space, shapes and the relation between shapes, using as tools conventional mathematical measures for size, angle, direction, orientation and position. In geometry classes children learn to analyse familiar spatial experiences in entirely new ways, and the experience of this novel and explicit kind of analysis should allow them to perceive and understand spatial relationships that they knew nothing about before.

In our view the aspects of analysing space geometrically that are new to children coming to the subject for the first time are:

- representing spatial relations which are already familiar to them, like length, area and position, in numbers
- learning about relations that are new to them, at any rate in terms of explicit knowledge, such as angle
- forming new categories for shapes, such as triangles, and understanding that the properties of a figure depends on its geometric shape
- understanding that there are systematic relations between shapes, for instance between rectangles and parallelograms
- understanding the relation between shapes across transformations, such as rotation, enlargement and changes in position.

Applying numbers to familiar spatial relations and forming relations between different shapes

Length measurement

Young children are clearly aware of length. They know that they grow taller as they grow older, and that some people live closer to the school than others. However, putting numbers on these changes and differences, which is one of their first geometric feats, is something new to them.

Standard units of measurement are equal subdivisions of the measuring instrument, and this means that children have to understand that this instrument, a ruler or tape measure or protractor, is not just a continuous quantity but is also subdivided into units that are exactly the same as each other. The child has to understand, for example, that by using a ruler, she can represent an object's length through an iteration of measurement units, like the centimetre.

When children measure, for example, the length of a straight line, they must relate the units on the ruler to the length that they are measuring, which is a one-to-one correspondence, but of a relatively demanding form. In order to see that the measured length is, for example, 10 cm long, they have to understand that the length that they are measuring can also be divided into the same unit and that ten of the units on the ruler are in one-to-one correspondence with ten imaginary but exactly similar units on what is being measured. This is an active form of one-to-one correspondence, since it depends on the children understanding that they are converting a continuous into a discontinuous quantity by dividing it into imaginary units. Here, is a good example of how even the simplest of mathematical analysis of space makes demands on children's imagination: they must imagine and impose divisions on undivided quantities in order to create the one-to-one correspondence which is basic to all measurement of length.

Measuring a straight line with a ruler is probably the simplest form of measurement of all, but children even make mistakes with this task and their mistakes suggest that they do not at first grasp that measuring the line takes the form of imposing one-to-one correspondence of the units on the measure with imagined units on the line. This was certainly suggested by the answers that a large number of children who were in their first three years of secondary school (11-, 12-, 13- and 14-year-olds) gave to a question about the length of a straight line, which was part of a test devised by Hart, Brown, Kerslake, Küchemann and Ruddock (1985). The children were shown a picture of straight line beside a ruler that was marked in centimetres. One end of the line was aligned with the 1 cm mark on the ruler and the other end with the 7 cm mark. The children were asked how long the line was and, in the youngest group, almost as many of them (46%) gave the answer 7 cm as gave the right answer (49%) which was 6 cm. Thus even at the comparatively late age of 11 years, after several years experience of using rulers, many children seemed not to understand, or at any rate not to understand perfectly, that it is the number of units on the ruler that the line corresponded to that decided its length.

A study by Nunes, Light and Mason (1993) gives us some insight into this apparently persistent difficulty. These experimenters asked pairs of six- to eight-year-old children to work together in a measuring task. They gave both children in each pair a piece of paper with a straight line on it, and the pair's task was to find out whether their lines were the same length or, if not whose was longer and whose shorter. Neither child could see the other's line because the two children did the task in separate rooms and could only talk to each other over a telephone. Both children in each pair were given a measure to help them compare these lines and the only difference between the pairs was in the measures that the experimenters gave them.

The pairs of children were assigned to three groups. In one group, both children in each pair were given a string with no markings: this therefore was a measure without units. In a second group, each child in every pair was given a standard ruler, marked in centimetres. In the third group the children were also given rulers marked in centimetres, but, while one of the children had a standard ruler, the other child in the pair was given a 'broken' ruler: it started at four centimetres. The child with the broken ruler could not produce the right answer just by reading out the number from the ruler which coincided with the end of the line. If the line was, for example, 7 cm long, that number would be 11 cm. The children in these pairs had to pay particular attention to the units in the ruler.

The pairs in the second group did well. They came up with the correct solution 84% of the time. The few mistakes that they made were mostly about placing the ruler or counting the units. Some children aligned one endpoint with the 1 cm point on the ruler rather than the 0 cm point and thus overestimated the length by 1 cm. This suggests that they were wrongly concentrating on the boundaries between units rather than the units themselves. Teachers should be aware that some children have this misconception. Nevertheless the ruler did, on the whole, help the children in this study since those who worked with complete rulers did much better than the children who were just given a string to measure with.

However, the broken ruler task was more difficult. The children in the standard ruler group were right 84%, and those in the broken ruler group 63%, of the time. Those who got it right despite having a broken ruler either counted the units or read off the last number (e.g. 11 cm for a 7 cm line) and then subtracted 4, and since they managed to do this more often than not, their performance established that these young children have a considerable amount of understanding of how to use the units in a measuring instrument and of what the units mean. However, on 30% of the trials the children in this group seemed not to understand the significance of the missing first four centimetres in the ruler. Either they simply read off the number that matched the line's endpoint (11 cm for 7 cm) or they did not subtract the right amount from it.

A large-scale American study (Kloosterman, Warfield, Wearne, Koc, Martin and Strutchens, 2004; Sowder, Wearne, Martin and Strutchens, 2004) later confirmed this striking difficulty. In this study, the children had to judge the length of an object which was pictured just above a ruler, though neither of its endpoints was aligned with the zero endpoint on that ruler. Less than 25% of the 4th graders (nine-year-olds) solved the problem correctly and only about 60% of the 8th graders managed to find the correct answer. This strong result, combined with those reported by Nunes *et al.*, suggests that many children may know how to use a standard ruler, but do not fully understand the nature or structure of the measurement units that they are dealing with when they do measure. Their mistake, we suggest, is not a misunderstanding of the function of a ruler: it is a failure in an active form of one-to-one correspondence – in imagining the same units on the line as on the ruler and then counting these units.

Summary

1 Measuring a straight line with a ruler is a procedure and it is also a considerable intellectual feat.

2 The procedure is to place the zero point of the ruler at one end of the straight line and to read off the number of standardised units on the ruler that corresponds to the other end of this line. There is no evidence that this procedure causes young children any consistent difficulty.

3 The intellectual feat is to understand that the ruler iterates a standardised unit (e.g. the centimetre) and that the length of the line being measured is the number of units in the part of the ruler that is in correspondence to the line. Thus measurement of length is a one-to-one correspondence problem, and the correspondence is between units that are displayed on the ruler but have to be imagined on the line itself. This act of imagination seem obvious and easy to adults but may not be so for young children.

4 Tests and experiments in which the line being measured is not aligned with zero show that initially children do not completely understand how measurement is based on imagining one-to-one correspondence of iterated units.

Measurement of area: learning about the relationship between the areas of different shapes

There is a striking contrast between young children's apparently effortless informal discriminations of size and area and the difficulties that they have in learning how to analyse and measure area geometrically. Earlier in this chapter we reported that babies are able to recognise objects by their size and can do so even when they see these objects at different distance on different occasions. Yet, many children find it difficult at first to measure or to understand the area of even the simplest and most regular of shapes.

All the intellectual requirements for understanding how to measure length, such as knowing about transitivity, iteration, and standardised units, apply as well to measuring area. The differences are that:

- area is necessarily a more complex quantity to measure than length because now children have to learn to consider and measure two dimensions and to co-ordinate these different measurements. The co-ordination is always a multiplicative one (e.g. base x height for rectangles; πr^2 for circles etc.).

- the standardised units of area – square centimetres and square metres or square inches etc. – are new to the children and need a great deal of explanation. This additional step is usually quite a hard one for children to take.

Rectangles

Youngsters are usually introduced to the measurement of area by being told about the base x height rule for rectangles. Thus, rectangles provide them with their first experience of square centimetres. The large-scale study of 11- to 14-year-old children by Hart *et al.* (1985), which we have mentioned already, demonstrates the difficulties that many children have even with this simplest of area measurements. In one question the children were shown a rectangle, drawn on squared paper, which measured 4 squares (base) by 2½ squares (height) and then were asked to draw another rectangle of the same area with a base of 5 squares. Only 44% of the 11-year-old children got this right: many judged that it was impossible to solve this problem.

We have to consider the reason for this difficulty. One reason might be that children find it hard to come to terms with a new kind of measuring unit, the square. In order to explain these new units teachers often give children 'covering' exercises. The children cover a rectangle with squares, usually 1 cm squares, arranged in columns and rows and the teacher explains that the total number of squares is a measure of the rectangle's area. The arrangement of columns and rows also provides a way of introducing children to the idea of multiplying height by width to calculate a rectangle's area. If the rectangle has five rows and four columns of squares, which means that its height is 5 cm and its width 4 cm, it is covered by 20 squares.

This might seem like an easy transition, but it has its pitfalls. These two kinds of computation are based on completely different reasoning: counting is about finding out the number that represents a quantity and involves additive reasoning whereas multiplying the base by the height involves understanding that there is a multiplicative relation between each of these measures and the area. Therefore, practice on one (counting) will not necessarily encourage the child to adopt the other formula (multiplying). Another radical difference is that the covering exercise provides the unit, the square centimetre, from the start but when the child uses a ruler to measure the sides and then to multiply height by width, she is measuring with one unit, the centimetre, but creating a new unit, the square centimetre (for further discussion, see Paper 3).

This could be an obstacle. The French psychologist, Gerard Vergnaud (1983), rightly distinguishes problems in which the question and the answer are about the same units ('A plant is 5 cm high at the beginning of the week and by the end of the week it is 2 cm higher. How high is it at the end of the week?') and those in which the question is couched in one unit and the answer in another ('The page on your book is 15 cm high and 5 cm wide. What is the area of this page?'). The answer to the second question must be in square centimetres even though the question itself is couched only in terms of centimetres. Vergnaud categorised the first kind of problem as 'isomorphism of measures' and the second as 'product of measures'. His point was that product of measures problems are intrinsically the more difficult of the two because, in order to solve such problems, the child has to understand how one kind of unit can be used to create another.

At first, even covering tasks are difficult for many young children. Outhred and Mitchelmore (2000) gave young children a rectangle to measure and just one 1 cm^2 square tile to help them to do this. The children also had pencils and were encouraged to draw on the rectangle itself. Since the children had one tile only to work with, they could only 'cover' the area by moving that tile about. Many of the younger children adopted this strategy but carried it out rather unsuccessfully. They left gaps between their different placements of the tile and there were also gaps between the squares in the drawings that they made to represent the different positions of the tiles.

These mistakes deserve attention, but they are hard to interpret because there are two quite different ways of accounting for them. One is that these particular children made a genuinely conceptual mistake about the iteration of the measuring unit. They may not have realised that gaps are not allowed – that the whole area must be covered by these standardised units. The alternative account is that this was an executive, not a conceptual, failure. The children may have known about the need for complete covering, and yet may have been unable to carry it out. Moving a tile around the rectangle, so that the tile covers every part of it without any overlap, is a complicated task, and children need a great deal of dexterity and a highly organised memory to carry it out, even if they know exactly what they have to do. These 'executive' demands may have been the source of the children's problems. Thus, we cannot say for sure what bearing this study has on Vergnaud's distinction between isomorphism and product of measures until we know whether the mistakes that children made in applying the measure were conceptual or executive ones.

Vergnaud's analysis, however, fits other data that we have on children's measurement of area quite well.

Nunes *et al.* (1993) asked pairs of eight- and nine-year-old children to work out whether two rectangles had the same area or not. The dimensions of the two rectangles were always different, even when their areas were the same (e.g. 5 x 8 and 10 x 4 cm). The experimenters gave all the children standard rulers, and also 1 cm^3 bricks to help them solve the problem.

The experimenters allowed the pairs of children to make several attempts to solve each problem until they agreed with each other about the solution. Most pairs started by using their rulers, as they had been taught to at school, but many of them then decided to use the bricks instead. Overall the children who measured with bricks were much more successful than those who relied entirely on their rulers. This clear difference is a demonstration of how difficult it is, at first, for children to use one measurement unit (centimetres) to create another (square centimetres). At this age they are happier and more successful when working just with direct representation of the measurement units that they have to calculate than when they have to use a ruler to create these units.

The success of the children who used the bricks was not due to them just counting these bricks. They hardly ever covered the area and then laboriously counted all the bricks. Much more often, they counted the rows and the columns of bricks and then either multiplied the two figures or used repeated addition or a mixture of the two to come up with the correct solution (A: *'Eight bricks in a row. And 5 rows. What's five eights?'* B: *'Two eights is 16 and 16 is 32. Four eights is 32. 32. 40'*). In fact, the children who used bricks multiplied in order to calculate the area more than three times as often as the children who used the ruler. Those who used rulers often concentrated on the perimeter: they measured the length of the sides and added lengths instead.

This confusion of area and perimeter is a serious obstacle. It can be traced back in time to a systematic bias in judgements that young children make about area long before they are taught the principles of area measurement. This bias is towards judging the area of a figure by its perimeter.

The bias was discovered independently in studies by Wilkening (1979) in Germany and Anderson and Cuneo (1978) and Cuneo (1983) in America. Both groups of researchers asked the same two questions (Wilkening and Anderson, 1982).

1 If you ask people to judge the area of different rectangles that vary both in height and in width, will their judgement be affected by both these dimensions? In other words, if you hold the width of two rectangles constant will they judge the higher of the two as larger, and if you hold their height constant will they judge the wider one as the larger? It is quite possible that young children might attend to one dimension only, and indeed Piaget's theory about spatial reasoning implies that this could happen.

2 If people take both dimensions into account, do they do so in an *additive* or a *multiplicative* way? The correct approach is the multiplicative one, because the area of a rectangle is its height multiplied by its width. This means that the difference that an increase in the rectangle's height makes to the area of the rectangle depends on its width, and vice versa. An increase of 3 cm in the height of a 6 cm wide rectangle adds another 18 cm^2 to its area, but the same increase in height to an 8 cm wide rectangle adds another 24 cm^2. The additive approach, which is wrong, would be to judge that an equal change in height to two rectangles has exactly the same effect on their areas, even if their widths differ. This is not true of area, but it is true of perimeter. To increase the height of a rectangle by 3 cm has exactly the same effect on the perimeter of a 6 cm and an 8 cm wide rectangle (and increase of 6 cm) and the same goes for increases to the width of rectangles with different heights. Also, the same increase in width has exactly the same effect on the two rectangles' perimeters, but very different effects on their area. It follows that anyone who persistently makes additive judgements about area is probably confusing area with perimeter.

The tasks that these two teams of experimenters gave to children and adults in their studies were remarkably similar, and so we will describe only Wilkening's (1979) experiment. He showed 5-, 8- and 11-year-old children and a group of adults a series of rectangles that varied both in height (6, 12 and 18 cm) and in width (again 6, 12 and 18 cm). He told the participants that these could be broken into pieces of a particular size, which he illustrated by showing them also the size of one of these pieces. The children's and adults' task was to imagine what would happen if each rectangle was broken up and the pieces were arranged in a row. How long would this row be?

The most striking contrast in the pattern of these judgments was between the five-year-old children and the adults. To put it briefly, five-year-old children made additive judgements and adults made multiplicative judgements.

The five-year-olds plainly did take both height and width into account, since they routinely judged rectangles of the same height but different widths as having different areas and they did the same with rectangles of the same widths but different heights. This is an important result, and it must be reassuring to anyone who has to teach young schoolchildren about how to measure area. They are apparently ready to take both dimensions into account.

However, the results suggest that young children often co-ordinate information about height and width in the wrong way. The typical five-year-old judged, for example, that a 6 cm difference in height would have the same effect on 12 cm and 18 cm wide rectangles. In contrast, the adults' judgements showed that they recognised that the effect would be far greater on the 18 cm than on the 12 cm wide figures. This is evidence that young children rely on the figures' perimeters, presumably implicitly, in order to judge their area. As have already seen, when children begin to use rulers many of them fall into the trap of measuring a figure's perimeter in order to work out its area, (Nunes, Light and Mason, 1993). Their habit of concentrating on the perimeter when making informal judgements about area may well be the basis for this later mistake. The existence among schoolchildren of serious confusion between area and perimeter was confirmed in later research by Dembo, Levin and Siegler (1997).

We can end this section with an interesting question. One obvious possible cause of the radical difference in the patterns of 5-year-olds' and adults' judgements might be that the 5-year-olds had not learned how to measure area while the adults had. In other words, mathematical learning could alter this aspect of people's spatial cognition. The suggestion does not seem far-fetched, especially when one also considers the performance of the older children in Wilkening's interesting study. The 5-year-olds had not been taught about measurement at all: the 8-year-olds had had some instruction, but not a great deal: the 12-year-olds were well-versed in measurement, but probably still made mistakes. Wilkening found some signs of a multiplicative pattern in the responses of the 8-year-olds, but this was slight: he found stronger signs of this pattern among the 12-year-olds, though not as pronounced as in the adult group. These changes do not prove that being taught how to measure and then becoming increasingly experienced with measuring led to this difference between the age groups, but they are certainly consistent with that idea. There is an alternative explanation, which is that adults and older children have more informal experience than 8-year-olds do of judging and comparing areas, as for example when they have to judge how much paint they need to cover different walls. Here is a significant and interesting question for research: do teachers alone change our spatial understanding of area or does informal experience play a part as well?

Summary

1 Measuring area is a multiplicative process: we usually multiply two simple measurements (e.g. base by height for rectangles) to produce a total measure of an area. The process also produces a different unit (i.e. product of measures): measuring the base and height in centimetres and then multiplying them produces a measure in terms of square centimetres.

2 Producing a new measure is a difficult step for children to make. They find it easier to measure a rectangle when they measure with units which directly instantiate square centimetres than when they use a ruler to measure its base and height in centimetres.

3 The multiplicative aspect of area measurement is also a problem for young children who show a definite bias to judge the area of a rectangle by adding its base and height rather than by multiplying them. They confuse, therefore, perimeter and area.

Parallelograms: forming relations between rectangles and parallelograms

The measurement of parallelograms takes us into one of the most exciting aspects of learning about geometry. The base-by-height rule applies to these figures as well as to rectangles. One way of justifying the base by height rule for parallelograms is that any parallelogram can be transformed into a rectangle with the same base and height measurements by adding and subtracting congruent areas to the parallelogram.

Figure 5.4 presents this justification which is a commonplace in geometry classes. It is based on the inversion principle (see Paper 2). Typically the teacher shows children a parallelogram and then creates two congruent triangles (*A* and *S*) by dropping vertical lines from the top two corners of the parallelogram and then extending the baseline to reach the new vertical that is external to the parallelogram. Triangle *A* falls outside the original parallelogram, and therefore is an addition to the figure. Triangle *S* falls inside the original parallelogram, and the nub of the teacher's demonstration is to point out that the effect of adding Triangle *A* and subtracting Triangle *S* would be to transform the figure into a rectangle with the same base and height as the original parallelogram. Triangles *A* and *S* are congruent and so their areas are equal. Therefore, adding one and subtracting the other triangle must produce a new figure (the rectangle) of exactly the same size as the original one (the parallelogram).

This is a neat demonstration, and it is an important one from our point of view, because it is our first example of the importance in geometry of understanding that there are systematic relations between shapes. Rectangles and parallelograms are different shapes but they are measured by the same base-by-height rule because one can transform any rectangle into any parallelogram, or vice versa, with the same base and height without changing the figure's area.

Some classic research by the well-known Austrian psychologist Max Wertheimer (1945) suggests that many children learn the procedure for transforming parallelograms into rectangles quite easily, but apply it inflexibly. Wertheimer witnessed a group of 11-year-old children learning from their teacher why the same base-by-height rule applied to parallelograms as well. The teacher used the justification that we have already described, which the pupils appeared to understand. However, Wertheimer was not certain whether these children really had understood the underlying idea. So, he gave them another parallelogram whose height was longer than its base (diagram 3 in Figure 5.4). When a parallelogram is oriented in this way, dropping two vertical lines from its top two corners does not create two congruent triangles. Wertheimer found that most of the children tried putting in the two vertical lines, but were at a loss when they saw the results of doing so. A few, however, did manage to solve the problem by rotating the new figure so that the base was longer than its height, which made it possible for them to repeat the teacher's demonstration.

The fact that most of the pupils did not cope with Wertheimer's new figure was a clear demonstration that they had learned more about the teacher's procedure than about the underlying idea about transformation that he had hoped to convey. Wertheimer argued that this was probably because the teaching itself concentrated too much on the procedural sequence and too little on the idea of

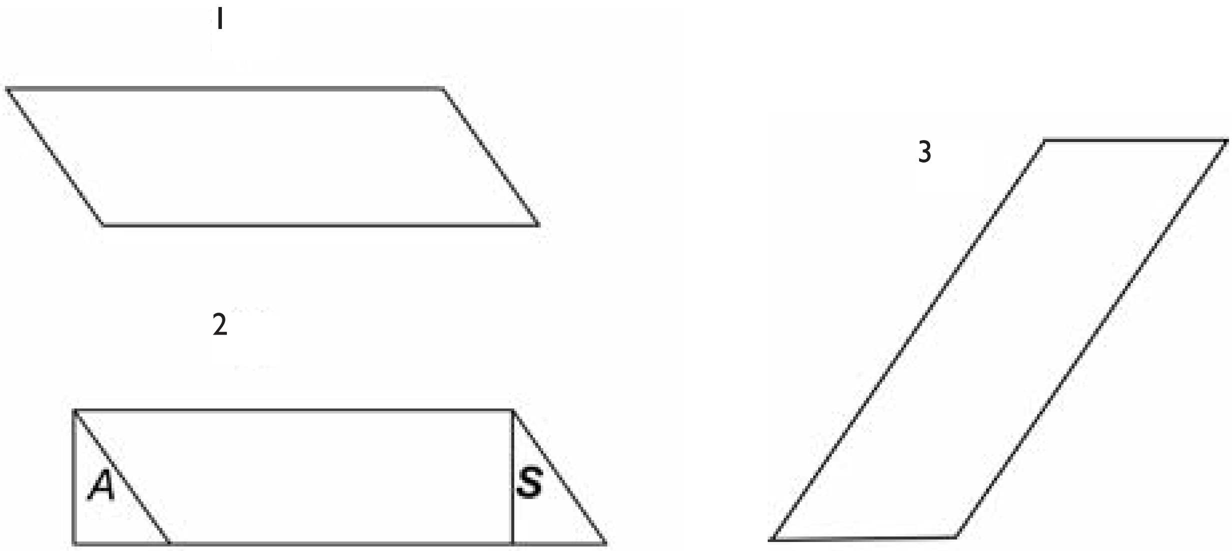

Figure 5.4: Demonstrating by transforming a parallelogram into a rectangle that the base-by-height rule applies to parallelograms as well as to rectangles.

transformation. In later work, that is only reported rather informally (Luchins and Luchins, 1970), Wertheimer showed children two figures at a time, one of which could be easily transformed into a measurable rectangle while the other could not (for example, the *A* and *B* figures in Figure 5.5). Wertheimer reported that this is an effective way of preparing children for understanding the relation between the area of parallelograms and rectangles.

It is a regrettable irony that this extraordinarily interesting and ingenious research by a leading psychologist was done so long ago and is so widely known, and yet few researchers since then have studied children's knowledge of how to transform one geometric shape into another to find its area.

In fact, Piaget *et al.* (1960) did do a relevant study, also a long time ago. They asked children to measure the area of an irregular polygon (Figure 5.6). One good way to solve this difficult problem is to partition the figure by imaging the divisions represented by the dotted lines in the right hand figure. This creates the Triangle *x* and also a rectangle which includes another Triangle *y*. Since the two triangles are congruent and Triangle *x* is part of the original polygon while Triangle *y* is not, the area of the polygon must add up to the area of the rectangle (plus Triangle *x* minus Triangle *y*).

Piaget *et al.* report that the problem flummoxed most of the children in their study, but report that some 10-year-olds did come up with the solution that we have just described. They also tell us that many children made no attempt to break up the figure but that others, more advanced, were ready to decompose the figure into smaller shapes, but did not have the idea of in effect adding to the figure by

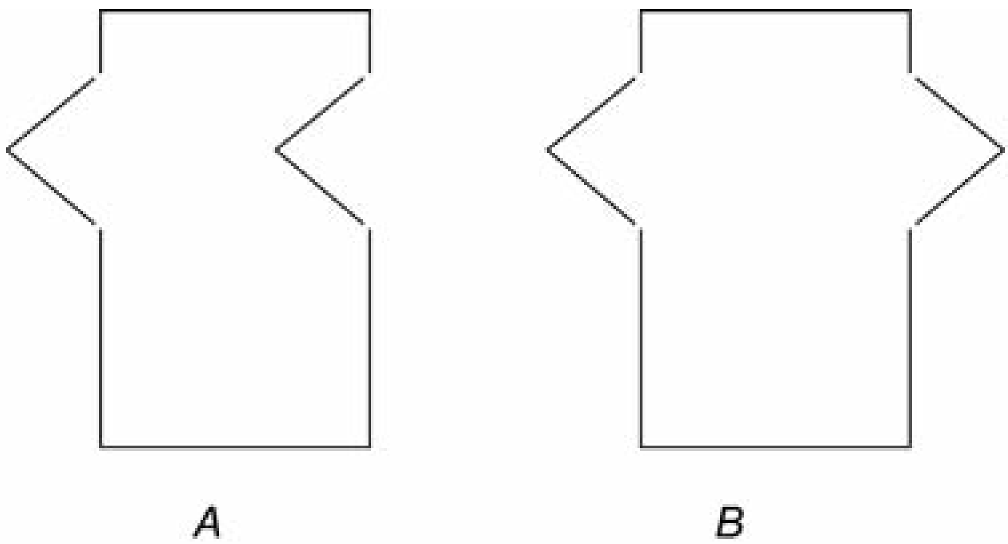

Figure 5.5: An example of Wertheimer's A and B figures. A figures could be easily transformed into a simple rectangle. B figures could not.

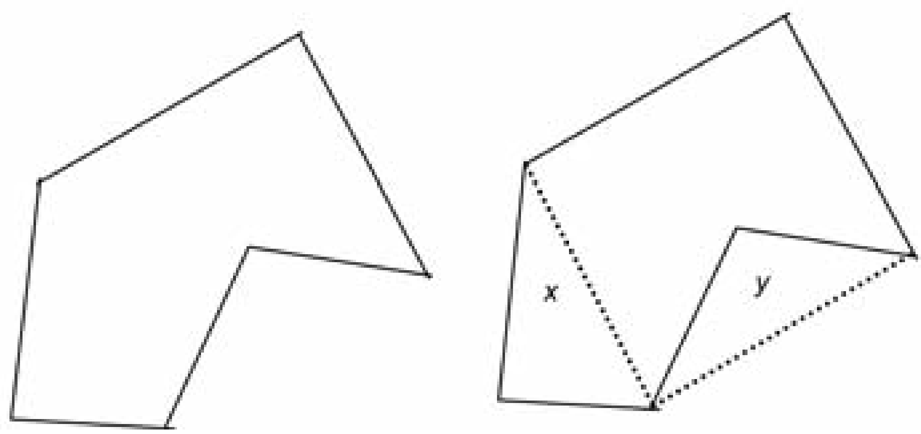

Figure 5.6: Piaget et al.'s irregular polygon whose area could be measured by decomposition.

imagining the BD line which was external to the figure. Thus, the stumbling block for these relatively advanced children was in adding to the figure. This valuable line of research, long abandoned, needs to be restarted.

Triangles: forming relations between triangles and parallelograms

Other transformations from one shape to another are equally important. Triangles can be transformed into parallelograms, or into rectangles if they are right-angle triangles, simply by being doubled (see Figure 5.7). Thus the area of a triangle is half that of a parallelogram with the same base and height.

Hart *et al.*'s (1985) study shows us that 11- to 14-year-old children's knowledge of the (base x height)/2 rule for measuring triangles is distinctly sketchy. Asked to calculate the area of a right-angle triangle with a base of 3 cm and a height of 4 cm, only 48% of the children in their third year in secondary school (13- and 14-year-olds) gave the right answer. Only 31% of the first year secondary school children (11-year-olds) succeeded, while almost an equal number of them – 29% – gave the answer 12, which means that they correctly multiplied base by height but forgot to halve the product of that multiplication.

Children are usually taught about the relationship between triangles, rectangles and parallelograms quite early on in their geometry lessons. However, we know of no direct research on how well children understand the relationships between different shapes or on the best way to teach them about these relations.

Summary

1 Learning about the measurement of the area of different shapes is a cumulative affair which is based not just on formulas for measuring particular shapes but on grasping the relationships, through transformations, of different shapes to each other. Parallelograms can be transformed into rectangles by adding and subtracting congruent triangles: triangles can be transformed into parallelograms by being doubled.

2 There is little direct research on children's understanding of the importance of the relationships between shapes in their measurement. Wertheimer's observations suggest very plausibly that their understanding depends greatly on the quality of the teaching.

3 We need more research on what are the most effective ways to teach children about these relationships.

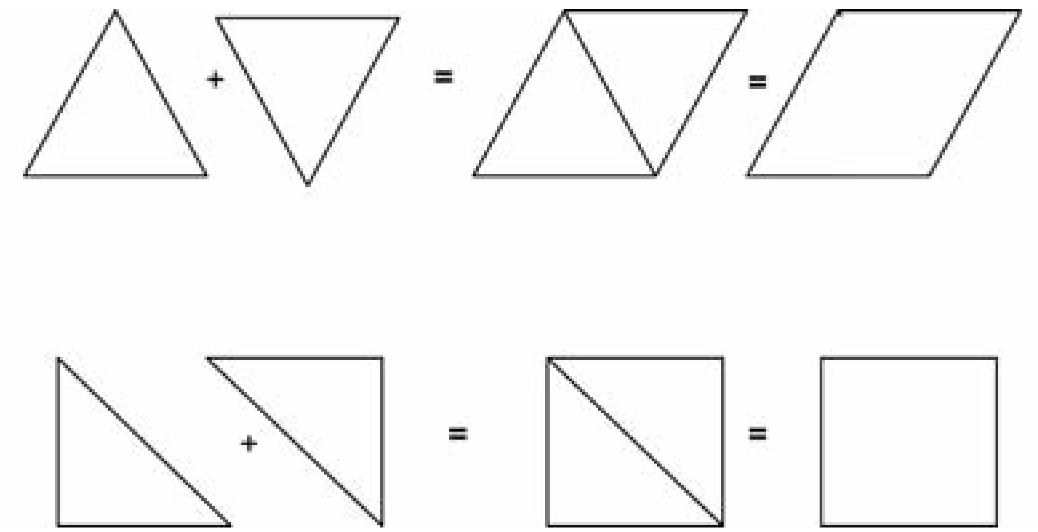

Figure 5.7: A demonstration that any two congruent triangles add up to a measurable parallelogram and any two congruent right angle triangles add up to a measurable rectangle.

Understanding new relationships: the case of angle

Angle is an abstract relation. Sometimes it is the difference between the orientations of two lines or rays, sometimes the change in your orientation from the beginning of a turn that you are making to its end, and sometimes the relation between a figure or a movement to permanent aspects of the environment: the angle at which an aeroplane rises after take-off is the relation between the slope of its path and the spatial horizontal. Understanding that angles are a way of describing such a variety of contexts is a basic part of learning plane geometry. Yet, research by psychologists on this important and fascinating topic is remarkably thin on the ground.

Most of the relatively recent studies of children's and adults' learning about angles are about the effectiveness of computer-based methods of teaching. This is estimable and valuable work, but we also need a great deal more information about children's basic knowledge about angles and about the obstacles, which undoubtedly do exist, to forming an abstract idea of what angles are.

In our everyday lives we experience angles in many different contexts, and it may not at first be easy for children to connect information about angles encountered in different ways. The obvious distinction here is between perceiving angles as configurations, such as the difference between perpendicular and non-perpendicular lines in pictures and diagrams, and as changes in movement, such as changing direction by making a turn. These forms of experiencing angles can themselves be subdivided: it may not be obvious to school children that we make the same angular change in our movements when we walk along a path with a right-angle bend as when we turn a door-knob by 90° (Mitchelmore, 1998).

Another point that children might at first find hard to grasp is that angles are relational measures. When we say that the angles in some of the figures in Figure 5.8a are 90° ones and in others 45°, we are making a statement about the relation between the orientations of the two lines in each figure and not about the absolute orientation of any of the individual lines, which vary from each other. Also, angles affect the distance between lines, but only in relation to the distance along the lines: in Figure 5.8b the distance between the lines in the figure with the larger angle is greater than in the other figure when

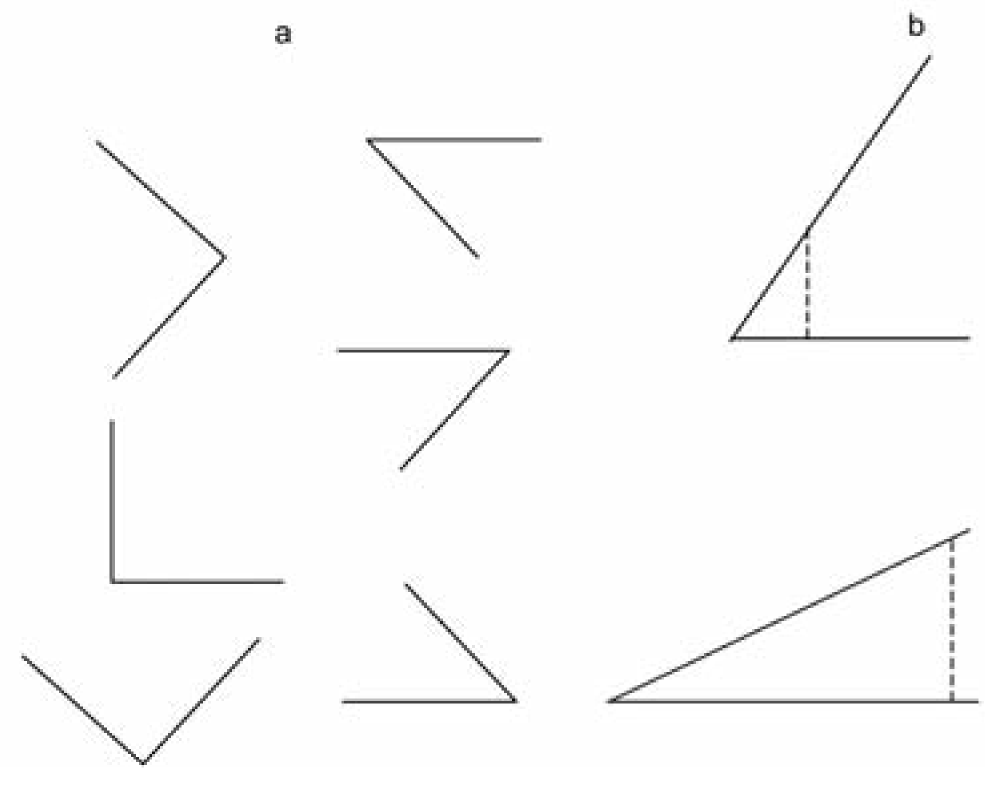

Figure 5.8: Angles as relations

the distance is measured at equivalent points along lines in the two figures, but not necessarily otherwise.

A third possible obstacle is that children might find some representations of angle more understandable than others. Angles are sometimes formed by the meeting of two clear lines, like the peak of a roof or the corner of a table. With other angles, such as the inclination of a hill, the angle is clearly represented by one line – the hill itself, but the other, a notional horizontal, is not so clear. In still other cases, such as the context of turning, there are no clearly defined lines at all: the angle is the amount of turning. Children may not see the connection between these very different perceptual situations, and they may find some much easier than others.

There are few theories of how children learn about angles, despite the importance of the topic. The most comprehensive and in many ways the most convincing of the theories that do exist was produced by Mitchelmore and White (2000). The problem that these researchers tried to solve is how children learn to abstract and classify angles despite the large variety of situations in which they experience them. They suggest that children's knowledge of angles develops in three steps:

1 **Situated angle concepts** Children first register angles in completely specific ways, according to Mitchelmore and White. They may realise that a pair of scissors, for example, can be more open or less open, and that some playground slides have steeper slopes than others but they make no link between the angles of scissors and of slides, and would not even recognise that a slide and a roof could have the same slope.

2 **Contextual angle concepts** The next step that children take is to realise that there are similarities in angles across different situations, but the connections that they do make are always restricted to particular, fairly broad, contexts. Slope, which we have mentioned already, is one of these contexts; children begin to be able to compare the slopes of hills, roofs and slides, but they do not manage to make any connection between these and the angles of, for instance, turns in a road. They begin to see the connection between angles in very different kinds of turns – in roads and in a bent nail, for example – but they do not link these to objects turning round a fixed point, like a door or a door-knob.

3 **Abstract angle concepts** Mitchelmore and White's third step is itself a series of steps. They claim that children begin to compare angles across contexts, for example, between slopes and turns, but that initially these connections across contexts are limited in scope; for example, these researchers report that even at the age of 11 years many children cannot connect angles in bends with angles in turns. So, children form one or more restricted abstract angle 'domains' (e.g. a domain that links intersections, bends, slopes and turns) before they finally develop this into a completely abstract concept of standard angles.

To test this theoretical framework, Mitchelmore and White gave children of 7, 9 and 11 years pictures of a wide range of situations (doors, scissors, bends in roads etc.) and asked them to represent the angles in these, using a bent pipe cleaner to do so, and also to compare angles in pairs of different situations. The study certainly showed different degrees of abstraction among these children and provided some evidence that abstraction about angles increases with age. This is a valuable contribution, but we certainly need more evidence about this developmental change for at least two reasons.

One is methodological. The research that we have just described was cross-sectional: the children in the different age groups were different children. A much better way of testing any hypothesis about a series of developmental cognitive changes is to do a longitudinal study of the ideas that the same children hold and then change over time as they get older. If the hypothesis is also about what makes the changes happen, one should combine this longitudinal research with an intervention study to see what provokes the development in question. We commented on the need for combining longitudinal and intervention studies in Paper 2. Once again, we commend this all-too-rarely adopted design to anyone planning to do research on children's mathematics.

The second gap in this theory is its concentration on children learning what is irrelevant rather than what is relevant to angle. The main claim is that children eventually learn, for example, that the same angles are defined by two clear lines in some cases but not in others, that some angular information is about static relations and some about movement, but that it is still exactly the same kind of information. This claim is almost certainly right, but it does not tell us what children learn instead.

One possibility, suggested by Piaget, Inhelder and Szeminska (1960), is that children need to be able to relate angles to the surrounding Euclidean framework in order to reproduce them and compare them to other angles. In one study they asked children to copy a triangle like the one in Figure 5.9.

They gave the children some rulers, strips of paper and sticks (but apparently no protractor) to help them do this. Piaget *et al.* wanted to find out how the children set about reproducing the angles CAB, ABC and ACB. In the experimenters' view the best solution was to measure all three lines, and also introduce an additional vertical line (KB in Figure 5.9) or to extend the horizontal line and then introduce a new vertical line (CK' in Figure 5.9). It is not at all surprising that children below the age of roughly ten years did not think of this Euclidean solution. Some tried to copy the triangle perceptually. Others used the rulers to measure the length of the lines but did not take any other measures. Both strategies tended to lead to inaccurate copies.

The study is interesting, but it does not establish that children have to think of angles in terms of their relationship to horizontal and vertical lines in order to be able to compare and reproduce particular angles. The fact that the children were not given the chance to use the usual conventional measure for angles – the protractor – either in this study or in Mitchelmore and White's study needs to be noted. This measure, despite being quite hard to use, may play a significant and possibly even an essential part in children's understanding of angle.

Another way of approaching children's understanding of angles is through what Mitchelmore and White called their situated angle concepts. Magina and Hoyles (1997) attempted to do this by investigating children's understanding of angle in the context of clocks and watches. They asked Brazilian children, whose ages ranged from 6 to 14 years, to show them where the hands on a clock would be in half an hour's time and also half an hour before the time it registered at that moment. Their aim was to find out if the children could judge the correct degree of turn. Magina and Hoyles report that the younger children's responses tended to be either quite unsystematic or to depend on the initial position of the minute hand: these latter children, mostly 8- to 11-year-olds, could move the minute hand to represent half an hour's difference well enough when the hand's initial position was at 6 (half past) on the clock face, but not when it was at 3 (quarter past). Thus, even in this highly familiar situation, many children seem to have an incomplete understanding of the angle as the degree of a turn. This rather disappointing result suggests that the origins of children's understanding may not lie in their informal spatial experiences.

One way in which children may learn about angles is through movement. The idea of children learning about spatial relations by monitoring their own actions in space fits well with Piaget's framework, and it is the basis for Logo, the name that Papert (1980) gave to his well-known computer-system that has often been used for teaching aspects of geometry. In Logo, children learn to write programmes to move a 'turtle' around a spatial environment. These programmes consist of a series of instructions that determine a succession of movements by the turtle. The instructions are about the length and direction of each movement, and the instructions about direction

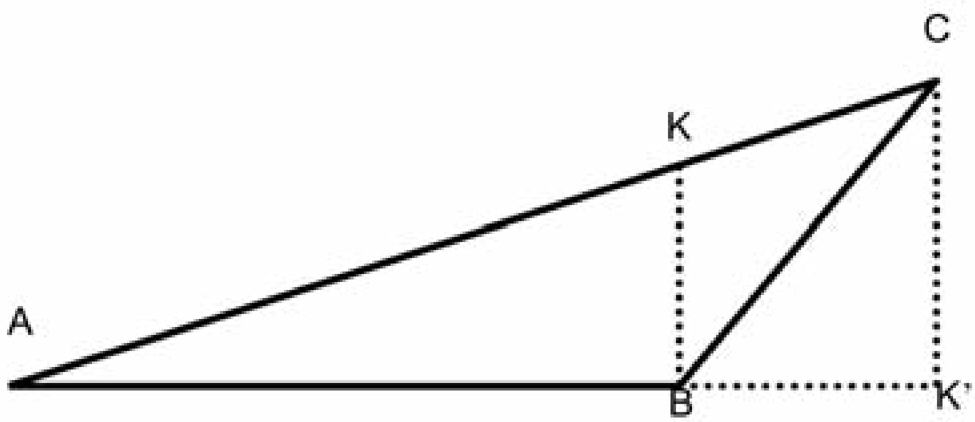

Figure 5.9: The triangle (represented by the continuous lines) that Piaget, Inhelder and Szeminska asked children to copy, with the vertical and horizontal (dotted) lines which some of the children created to help them to solve this problem.

take the form of angular changes e.g. L90 is an instruction for the turtle to make a 90° turn to the left. Since the turtle's movements leave a trace, children effectively draw shapes by writing these programmes.

There is evidence that experience with Logo does have an effect on children's learning about angles, and this in turn supports the idea that representations of movement might be one effective way of teaching this aspect of geometry. Noss (1988) gave a group of 8- to 11-year-old children, some of whom had attended Logo classes over a whole school year, a series of problems involving angles. On the whole the children who had been to the Logo classes solved these angular problems more successfully than those who had not. The relative success of the Logo group was particularly marked in a task in which the children had to compare the size of the turn that people would have to make at different points along a path, and this is not surprising since this specific task resonated with the instructions that children make when determining the direction of the turtle's movements (see Figure 5.10).

However, the Logo group also did better than the comparison in more static angular problems. This is an interesting result because it suggests that the children may have generalised what they learned about angles and movement to other angular tasks which involve no movement at all. We need more research to be sure of this conclusion and, as far as studies of the effects of Logo and other computer-based programmes are concerned, we need studies in which pre-tests are given before the children go through these programmes as well as post-tests that follow these classes.

Children do not just learn about single angles in isolation from each other. In fact, to us, the most interesting question in this area is about their learning of the relations between different angles. These relations are a basic part of geometry lessons: pupils learn quite early on in these lessons that, for example, when two straight lines intersect opposite angles are equal and that alternate angles in a Z-shape figure are equal also, but how easily this knowledge comes to them and how effectively they use it to solve geometric problems are interesting but unanswered questions (at any rate, unanswered by psychologists). Some interesting educational research by Gal and Vinner (1997) on 14-year-old students' reaction to perpendicular lines suggests that they have some difficulties in understanding the relation between the angles made by intersecting lines when the lines are perpendicular to each other. Many of the students did not realise at first that if one of the four angles made by two intersecting lines is a right angle the other three must be so as well. The underlying reason for this difficulty needs investigation.

You are walking along this path. You start at point A and you finish at point G.

- At which point would you have to turn most?
- At which point would you have to turn least?

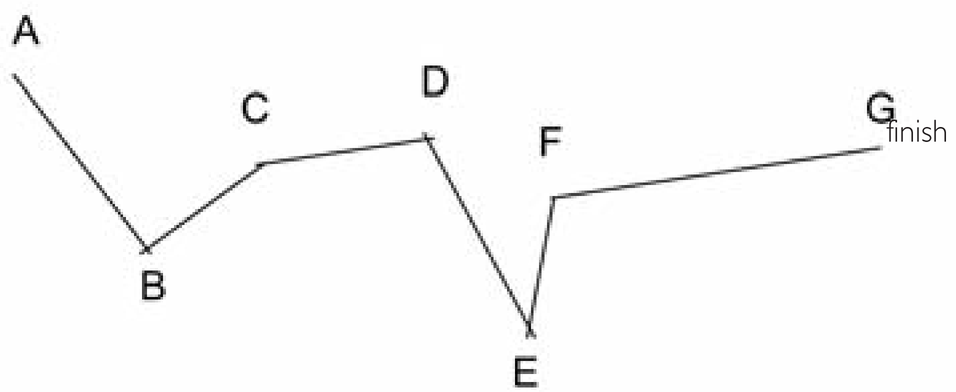

Figure 5.10: The judgement about relative amount of turns in Noss's study.

Summary

1 Although young children are aware of orientation, they seem to know little about angle (the relation between orientations) when they start on geometry.

2 The concept of angle is an abstract one that cuts across very different contexts. This is difficult for children to understand at first.

3 There is some evidence, mainly from work on Logo, that children can learn about angle through movement.

4 Children's understanding of the relation between angles within figures (e.g. when straight lines intersect, opposite angles are equivalent) is a basic part of geometry lessons, but there seems to be no research on their understanding of this kind of relation.

Spatial frameworks

Horizontal and vertical lines

Most children's formal introduction to geometry is a Euclidean one. Children are taught about straight lines, perpendicular lines, and parallel lines, and they learn how a quite complex system of geometry can be derived from a set of simple, comprehensible, axioms. It is an exercise in logic, and it must, for most children, be their first experience of a formal and explicit account of two-dimensional space. The principal feature of this account is the relation between lines such as parallel and perpendicular and intersecting lines.

These fundamental spatial relations are probably quite familiar, but in an implicit way, to the seven- and eight-year-old children when they start classes in geometry. In spatial environments, and particularly in 'carpentered' environments, there are obvious horizontal and vertical lines and surfaces, and these are at right angles (perpendicular) to each other. We reviewed the evidence on children's awareness of these spatial relations in an earlier part of this paper, when we reached the following two conclusions.

1 Although quite young children can relate the orientation of lines to stable background features and often rely on this relation to remember orientations, they do not always do this when it would help them to do so. Thus, many young children and some adults too (Howard, 1978) do not recognise that the level of liquid is parallel to horizontal features of the environment like a table top.

2 Part of the difficulty that they have in Piaget's horizontality and verticality tasks is that these depend on children being able to represent acute and obtuse angles (i.e. non-perpendicular lines). They tend to do this inaccurately, representing the line that they draw as closer to perpendicular than it actually is. This bias towards the perpendicular may get in the way of children's representation of angle.

The role of horizontal and vertical axes in the Cartesian system

The Euclidean framework makes it possible to pinpoint any position in a two-dimensional plane. We owe this insight to René Descartes, the 17th century French mathematician and philosopher, who was interested in linking Euclid's notions with algebra. Descartes devised an elegant way of plotting positions by representing them in terms of their position along two axes in a two dimensional plane. In his system one axis was vertical and the other horizontal, and so the two axes were perpendicular to each other. Descartes pointed out that all that you need to know in order to find a particular point in two-dimensional space is its position along each of these two axes. With this information you can plot the point by extrapolating an imaginary straight perpendicular line from each axis. The point at which these two lines intersect is the position in question. Figure 5.11 shows two axes, *x* and *y*, and points which are expressed as positions on these axes.

This simple idea has had a huge impact on science and technology and on all our daily lives: for example, we rely on Cartesian co-ordinates to interpret maps, graphs and block diagrams. The Cartesian co-ordinate system is a good example of a cultural tool (Vygotsky, 1978) that has transformed all our intellectual lives.

To understand and to use the Cartesian system to plot positions in two-dimensional space, one has to be able to extrapolate two imaginary perpendicular straight lines, and to co-ordinate the two in order to work out where they intersect. Is this a difficult or even an impossible barrier for young children? Teachers certainly need the answer to this question because children are introduced to graphs and block diagrams in primary school, and as we have noted these mathematical representations depend on the use of Cartesian co-ordinates.

Earlier in this section we mentioned that, in social contexts, very young children do extrapolate imaginary straight lines. They follow their mother's line of sight in order, apparently, to look at whatever it is that is attracting her attention at the time (Butterworth, 1990). If children can extrapolate straight lines in three-dimensional space, we can quite reasonably expect them to be able to do so in two-dimensional space as well. The Cartesian requirement that these extrapolated lines are perpendicular to their baselines should not be a problem either, since, as we have seen already, children usually find it easier to create perpendicular than non-perpendicular lines. The only requirement that this leaves is the ability to work out where the paths of the two imaginary straight lines intersect.

A study by Somerville and Bryant (1985) established that children as young as six years usually have this ability. In the most complex task in this study, young children were shown a fairly large square space on a screen and 16 positions were clearly marked within this space, sometimes arranged in a regular grid and sometimes less regularly than that. On the edge of the square waited two characters, each just about to set off across the square. One of these characters was standing on a vertical edge (the right or left side of the square) and the other on a horizontal edge of the square. The children were told that both characters could only walk in the direction they were facing (each character had a rather prominent nose to mark this direction, which was perpendicular to the departure line, clear), and that the two would eventually meet at one particular position in the square. It was the child's task to say which position that would be.

The task was slightly easier when the choices were arranged in a grid than when the arrangement was irregular, but in both tasks all the children chose the right position most of the time. The individual children's choices were compared to chance (if a child followed just one extrapolated line instead of co-ordinating both, he or she would be right by chance 25% of the time) and it turned out that the number of correct decisions made by every individual child was significantly above chance. Thus, all of these 6-year-old children were able to plot the intersection of two extrapolated, imaginary straight lines, which means that they were well equipped to understand Cartesian co-ordinates.

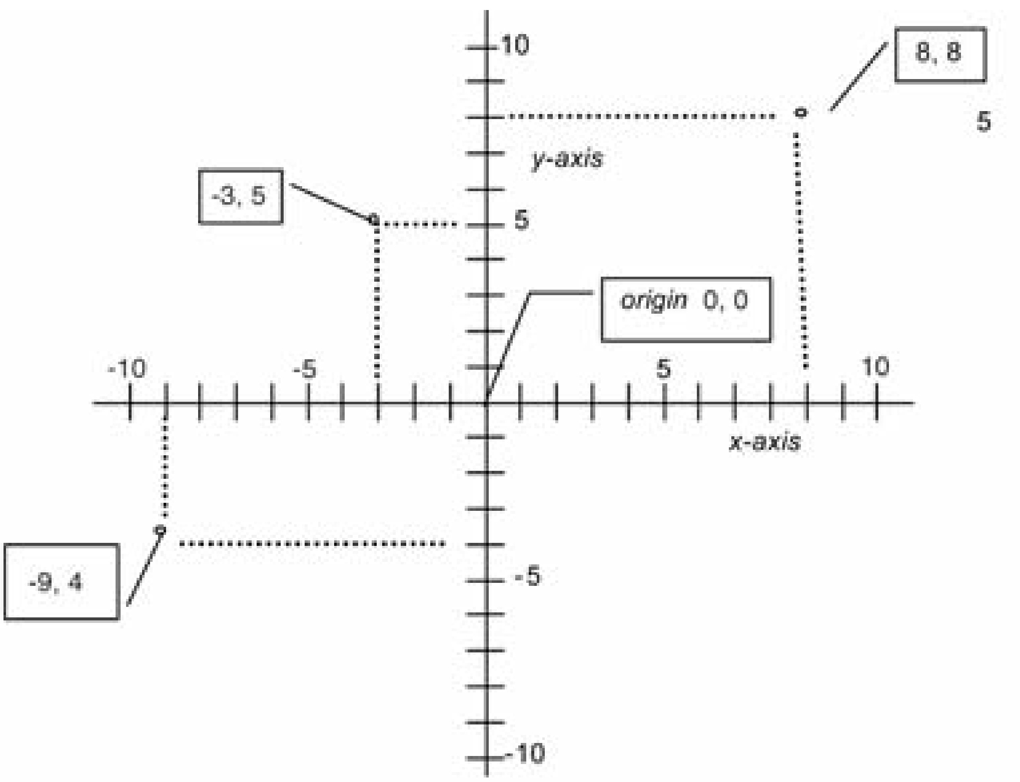

Figure 5.11: Descartes' co-ordinates: three points (8, 8), (-3, 5) and (-9, -4) are plotted by their positions on the x- and y-axes

Piaget *et al.* (1960) were less optimistic about children's grasp of co-ordinates, which they tested with a copying task. They gave each child two rectangular sheets of paper, one with a small circle on it and the other a complete blank. They also provided the children with a pencil and a ruler and strips of paper, and then they asked them to put a circle on the blank sheet in exactly the same position as the circle on the other sheet. Piaget *et al.*'s question was whether any of the children would use a co-ordinate system to plot the position of the existing circle and would then use the co-ordinates to position the circle that they had to draw on the other sheet. This was a difficult task, which children of six and seven years tended to fail. Most of them tried to put the new circle in the right place simply by looking from one piece of paper to the other, and this was a most unsuccessful strategy.

There is no conflict between the success of all the children in the Somerville and Bryant study, and the grave difficulties of children of the same age in the Piaget *et al.* task. In the Piaget *et al.* task the children had to decide that co-ordinates were needed and then had to measure in order to establish the appropriate position on each axis. In the Somerville and Bryant study, the co-ordinates were given and all that the child had to do was to use them in order to find the point where the two extrapolated lines met. So, six- and seven-year-old children can establish a position given the co-ordinates but often cannot set up these co-ordinates in the first place.

Some older children in Piaget *et al.*'s study (all the successful children given as examples in the book were eight- or-nine-years-old) did apparently spontaneously use co-ordinates. It seems unlikely to us that these children managed to invent the Cartesian system for themselves. How could eight- and nine-year-old children come up, in one experimental session, with an idea for which mankind had had to wait till Descartes had his brilliant insight in the middle of the 17th century?
A more plausible reason for these children's success is that, being among the older children in Piaget *et al.*'s sample, they had been taught about the use of Cartesian co-ordinates in maps or graphs already.

It appears that this success is not universal. Many children who have been taught about Cartesian co-ordinates fail to take advantage of them or to use them properly. Sarama, Clements, Swaminathan, McMillen and Gomez (2003) studied a group of nine-year-old children while they were being given intensive instruction, which the researchers themselves designed. In the tasks that they gave to the children Sarama *et al.* represented the x and y co-ordinates as numbers, which was a good thing to include because it is a fundamental part of the Cartesian system, and they also imposed a rectangular grid on many of the spaces that they gave the children to work with. Thus, one set of materials was a rectangular grid, presented as a map of a grid-plan city with 'streets' as the vertical and 'avenues' as the horizontal lines.

The children were given various tasks before, during and after the instruction. Some of these involved relating locations to x and y co-ordinates: the children had to locate positions given the co-ordinates and also to work out the co-ordinates of particular positions. Sarama *et al.* reported that most of the children learned about this relation quickly and well, as one might expect given their evident ability to co-ordinate extrapolated lines in a rectangular context (Somerville and Bryant, 1985). However, when they had to co-ordinate information about two or more locations, they were in greater difficulty. For example, some children found it hard to work out the distance between two locations in the grid-like city, because they thought that the number of turns in a path affected its distance, and some did not realise that the numerical differences in the x and y co-ordinate addresses between the two locations represents the distance between them.

Thus, the children understood how to find two locations, given their coordinates, but struggled with the idea that a comparison between the two pairs of co-ordinates told them about the spatial relation between these locations. Further observations showed that the problem that some of the children had in working out the relations between two co-ordinate pairs was created by a certain tension between absolute and relative information. The two co-ordinate pairs 10,30 and 5,0 represent two absolute positions: however, some children, who were given first 10,30 and then 5,0 and asked to work out a path between the two, decided that 5,0 represented the difference between the first and the second location and plotted a location five blocks to the right of 10,30. They treated absolute information about the second position as relative information about the difference between the two positions. However, most of the children in this ingenious and important study seem to have overcome this difficulty during the period of instruction, and to have learned reasonably well that co-ordinate pairs represent the relation between positions as well as the absolute positions themselves.

Finally, we should consider children's understanding of the use of Cartesian co-ordinates in graphs. Here, research seems to lead to much the same conclusion as we reached in our discussion of the use of co-ordinates to plot spatial positions. With graphs, too, children find it easy to locate single positions, but often fail to take advantage of the information that graphs provide that is based on the relation between different positions. Bryant and Somerville (1986) gave six- and nine-year-old children a graph that represented a simple linear function. Using much the same technique as in their previous research on spatial co-ordinates, these researchers measured the children's ability to plot a position on *x*-axis given the position on the *y*-axis and vice versa. This was quite an easy task for the children in both age groups, and so the main contribution of the study was to establish that children can co-ordinate extrapolated straight lines in a graph-like task fairly well even before they have had any systematic instruction about graphs.

The function line in a graph is formed from a series of positions, each of which is determined by Cartesian co-ordinates. As in the Sarama *et al.* study, it seems to be hard for children to grasp what the relation between these different positions means. An interesting study by Knuth (2000) showed that American students 'enrolled in 1st-year algebra' (Knuth does not say how old these students were) are much more likely to express linear functions as equations than graphically. We do not yet know the reasons for this preference

Summary

1 Cartesian co-ordinates seem to pose no basic intellectual difficulty for young children. They are able to extrapolate imaginary straight lines that are perpendicular to horizontal and vertical axes and to work out where these imaginary lines would meet in maps and in graphs.

2 However, it is harder for children to work out the relation between two or more positions that are plotted in this way, either in an map-like or in a graph-like task.

3 Students prefer expressing functions as equations to representing them graphically.

4 Thus, although children have the basic abilities to understand and use co-ordinates well, there seem to be obstacles that prevent them using these abilities in tasks which involve two or more plotted positions. We need research on how to teach children to surmount these obstacles.

Categorising, composing and decomposing shapes

We have chosen to end this chapter on learning about space and geometry with the question of children's ability to analyse and categorise shapes, but we could just as easily have started the section with this topic, because children are in many ways experts on shape from a very early age. They are born, apparently, with the ability to distinguish and remember abstract, geometric shapes, like squares, triangles, and circles, and with the capacity to recognise such shapes as constant even when they see them from different angles on different occasions so that the shape of the retinal image that they make varies quite radically over time. We left shape to the end because much of the learning that we have discussed already, about measurement and angle and spatial co-ordinates, undoubtedly affects and changes schoolchildren's understanding of shape.

There is nearly complete agreement among those who study mathematics education that children's knowledge about shapes undergoes a series of radical changes during their time at school. Different theories propose different changes but many of these apparent disagreements are really only semantic ones. Most claim, though in different terms, that school-children start by being able to distinguish and classify shapes in a perceptual and implicit way and eventually acquire the ability to analyse the properties of shapes conceptually and explicitly.

The model developed by the Dutch educationalist van Hiele (1986) is currently the best known theoretical account of this kind of learning. This is a good example of what we called a 'pragmatic theory' in our opening paper. Van Hiele claimed that children have to take a sequence of steps in a fixed order in their geometric learning about shape. There are five such steps in van Hiele's scheme, but he agreed that not all children get to the end of this sequence:

Level 1: Visualization/recognition Students recognise and learn to name certain geometric shapes but are usually only aware of shapes as a wholes, and not of their properties or of their components.

Level 2: Descriptive/analytic Students begin to recognise shapes by their properties.

Level 3: Abstract/relational Students begin to form definitions of shapes based on their common properties, and to understand some proofs.

Level 4: Formal deduction Students understand the significance of deduction as a way of establishing geometric theory *within* an axiomatic system, and comprehend the interrelationships and roles of axioms, definitions, theorems, and formal proof.

Level 5: Rigour Students can themselves reason formally about different geometric systems.

There have been several attempts to elaborate and refine this system. For example, Clements and Sarama (2007 b) argue that it would better to rename the Visual/recognition stage as Syncretic, given its limitations. Another development was Guttierez's (1992) sustained attempt to extend the system to 3-D figures as well as 2-D ones. However, although van Hiele's steps provide us with a useful and interesting way of assessing improvements in children's understanding of geometry, they are descriptive. The theory tells us about changes in what children do and do not understand, but not about the underlying cognitive basis for this understanding, nor about the reasons that cause children to move from one level to the next. We shall turn now to what is known and what needs to be known about these cognitive bases.

Composing and decomposing

If van Hiele is right, one of the most basic changes in children's analysis of shape is the realisation that shapes, and particularly complex shapes, can be decomposed into smaller shapes. We have already discussed one of the reasons why children need to be able to compose and decompose shapes, which is that it is an essential part of understanding the measurement of the area of different shapes. Children, as we have seen, must learn, for example, that you can compose a parallelogram by putting together two identical triangles (and thus that you can decompose any parallelogram into two identical triangles) in order to understand how to measure the area of triangles. As far as we know, there has been no direct research on the relationship between children's ability to compose and decompose shapes and their understanding of the rules for measuring simple geometric figures, though such research would be easy to do.

The Hart *et al.* (1985) study included two items which dealt with shapes that were decomposed into two parts and these parts were then re-arranged. In one item the re-arranged parts were a rectangle and two triangles, which are simple and familiar geometric shapes, and in the other they were unfamiliar and more complex shapes. In both cases the children were asked about the effect of the re-arrangement on the figure's total area: was the new figure's area bigger or smaller or the same as the area of the original one? A large proportion of the 11- to 14-year-old students in the study (over 80% in each group) gave the correct answer to the first of these two questions but the second was far harder: only 60% of the 11-year-old group understood that the area was the same after the rearrangement of parts as before it. There are two reasons for being surprised at this last result. The first is that it is hard to see why there was such a large difference in the difficulty of the two problems when the logic for solving both was exactly the same. The second is that the mistakes which the children made in such abundance with the harder problem are in effect conservation failures, and yet these children are well beyond the age when conservation of area should pose any difficulty for them. This needs further study.

The insight that understanding composition and decomposition may be a basic part of children's learning about shapes has been investigated in another way. Clements, Wilson and Sarama (2004) looked at a group of three- to seven-year-old children's ability to assemble target patterns, like the figure of a man, by assembling the right component wooden shapes. This interesting study produced evidence of some sharp developmental changes: the younger children tended to create the figures bit by bit whereas the older children tended to create units made out of several bits (an arm unit for example) and the oldest dealt in units made out of other units. The next step in this research should be to find out whether there is a link between this development and the eventual progress that children make in learning about geometry. Once again we have to make a plea for longitudinal studies (which are far too rare in research on children's geometry) and intervention studies as well.

Summary

1 Although young school children are already very familiar with shapes, they have some difficulty with the idea of decomposing these into parts, e.g. a parallelogram decomposed into two congruent triangles or an isosceles triangle decomposed into two right-angle triangles, and also with the inverse process of composing new shapes by combining two or more shapes to make a different shape.

2 The barrier here may be that these are unusual tasks for children who might learn how to carry them out easily given the right experience. This is a subject for future research.

Transforming shapes: enlargement, rotation and reflection

We have already stressed the demands that measurement of length and angle make on children's imagination, and the same holds for their learning about the basic transformations of shapes – translation, enlargement, rotation and reflection. Children have to learn to imagine how shapes would change as a result of each of these transformations and we know that this is not always easy. The work by Hart (1981) and her colleagues on children's solutions to reflection and rotation problems suggests that these transformations are not always easy for children to work out. They report that there is a great deal of change in between the ages of 11 and 16 years in students' understanding of what changes and what stays constant as a result of these two kinds of transformation.

One striking pattern reported by this research group was that the younger children in the group being studied were much more successful with rotation and reflection problems that involved horizontal and vertical figures than with sloping figures: this result may be related to the evidence, mentioned earlier, that much younger children discriminate and remember horizontal and vertical lines much better than sloping ones.

Psychologists, in contrast to educationalists, have not thrown a great deal light on children's learning about transformations, even though some research on perceptual development has come close to doing so. They have shown that children remember symmetrical figures better than asymmetrical figures (Bornstein, Ferdinandsen and Gross, 1981), and there are observations of pre-school children spontaneously constructing symmetrical figures in informal play (Seo and Ginsburg, 2004). However, the bulk of the psychological work on rotation and reflection has treated these transformations in a negative sense. The researchers (Bomba, 1984; Quinn, Siqueland and Bomba, 1985; Bryant, 1969,1974) were concerned with children's confusions between symmetrical, mirror-image figures (usually reflections around a vertical axis): they studied the development of children's ability to tell symmetrical figures apart, not to understand the relation between them. Here is another bridge still to be crossed between psychology and education.

Enlargement raises some interesting issues about children's geometric understanding. We know of no direct research on teaching children or on children learning about this transformation, at any rate in the geometrical sense of shapes being enlarged by a designated scale factor. These scale factors of course directly affect the perimeter of the shapes: the lengths of the sides of, for example, a right-angle triangle enlarged by a factor of 2 are twice those of the original triangle. But, of course, the relation between the areas of the two triangles is different: the area of the larger triangle is 4 times that of the smaller one.

There is a danger that some children, and even some adults, might confuse these different kinds of relation between two shapes, one of which is an enlargement of the other. Piaget *et al.* (1960) showed children a 3 cm x 3 cm square which they said represented a field with just enough grass for one cow, and then they asked each child to draw a larger field of the same shape which would produce enough for two cows. Since this area measured 9 cm^2 the new square would have to have an area of 18 cm.^{2} and therefore sides of roughly 4.24 cm. since 4.24 is very nearly the square root of 18. In this study most of the children under the age of 10 years either acted quite unsystematically or made the mistake of doubling the sides of the original square in the new figure that they drew, which meant that their new square (an enlargement of the original square by a scale factor of two) had 6 cm sides and an area of around 36 cm^2 which is actually 4 times the area of the first square. Older children, however, recognised the problem as a multiplicative one and calculated each of the two squares' areas my multiplying its height by its width. The younger children's difficulties echoed those of the slave who, in Plato's Meno, was lucky enough to be instructed about measuring area by Socrates himself.

The widespread existence of this apparently prevailing belief that doubling the length of the sides of a shape will double its area as well was recently confirmed by a team of psychologists in Belgium ((De Bock, Verschaffel and Janssens, 1998, 2002; De Bock, Van Dooren, Janssens and Verschaffel, 2002; De Bock, Verschaffel, Janssens, Van Dooren and Claes, 2003). There could be an educational conflict here between teaching children about scale factors on the one hand and about proportional changes in area on the other. It is possible that the misconceptions expressed by students in the studies by Piaget *et al.* (1960) and also by the Belgian team may actually have been the result of confusion between the use of scale factors in drawing and effect of doubling the sides of figures. In Paper 4 we discussed in detail the difficulties of the studies carried out by the Belgian team but we still need to find out, by research, whether scale drawing does provoke this confusion. It would be easy to do such research.

Summary

1 Understanding, and being able to work out, the familiar transformations of reflection, rotation and enlargement are a basic part of the geometry that children learn at school. They are another instance of the importance of grasping the relations between shapes in learning geometry.

2 Psychology tells us little about children's understanding of these relations, though it would be easy enough for psychologists to do empirical research on this basic topic. The reason for psychologists' neglect of transformations is that they have concentrated on children distinguishing between shapes rather than on their ability to work out the relations between them.

3 There is the possibility of a clash between learning about scale factors in enlargement and about the measurement of area. This should be investigated.

General conclusions on learning geometry

1 Geometry is about spatial relations.

2 Children have become highly familiar with some of these relations, long before they learn about them formally in geometry classes: others are new to them.

3 In the case of the spatial relations that they know about already, like length, orientation and position relations, the new thing that children have to learn is to represent them numerically. The process of making these numerical representations is not always straightforward.

4 Representing length in standard units depends on children using one-to-one correspondence between the units on the ruler and imagined units on the line being measured. This may seem to be easy to do to adults, but some children find it difficult to understand.

5 Representing the area of rectangles in standard units depends on children understanding two things: (a) why they have to multiply the base with the height in centimetres (b) why this multiplication produces a measure in different units, square centimetres. Both ideas are difficult ones for young children.

6 Understanding how to measure parallelograms and triangles depends on children learning about the relation between these shapes and rectangles. Learning about the relations between shapes is a significant part of learning about geometry and deserves attention in research done by psychologists.

7 The idea of angle seems to be new to most children at the time that they begin to learn about geometry. Research suggests that it takes children some time to a form an abstract concept of angle that cuts across different contexts. More research is needed on children's understanding of the relations between angles in particular figures.

8 Children seem well-placed to learn about the system of Cartesian co-ordinates since they are, on the whole, able to extrapolate imaginary perpendicular lines from horizontal and vertical co-ordinates and to work out where they intersect. They do, however, often find co-ordinate tasks

which involve plotting and working out the relationship between two or more positions quite difficult. We need research on the reasons for this particular difficulty.

9 There is a serious problem about the quality of the research that psychologists have done on children learning geometry. Although psychologists have carried out good work on children's spatial understanding, they have done very little to extend this work to deal with formal learning about the mathematics of space. There is a special need for longitudinal studies, combined with intervention studies, of the link between informal spatial knowledge and success in learning geometry.

References

Anderson, N. H., and Cuneo, D. (1978). The height + width rule in children's judgments of quantity. *Journal of Experimental Psychology- General, 107*, 335-378.

Bomba, P. C. (1984). The Development of Orientation Categories between 2 and 4 Months of Age. *Journal of Experimental Child Psychology, 37*, 609-636.

Bornstein, M.H., Ferdinandsen, K., and Gross, C. (1981) Perception of symmetry in infancy. *Developmental Psychology, 17*, 82-86.

Bremner, A., Bryant, P., and Mareschal, D. (2006). Object-centred spatial reference in 4-month-old infants. *Infant Behavior and Development, 29*(1), 1-10.

Bryant, P. E. (1969). Perception and memory of the orientation of visually presented lines by children. *Nature, 224*, 1331-1332.

Bryant, P. E. (1974). *Perception and understanding in young children*. London: Methuen.

Bryant, P. E., and Kopytynska, H. (1976). Spontaneous measurement by young children. *Nature, 260*, 773.

Bryant, P. E., and Somerville, S. C. (1986). The spatial demands of graphs. *British Journal of Psychology, 77*, 187-197.

Bryant, P. E., and Trabasso, T. (1971). Transitive inferences and memory in young children. *Nature, 232*, 456-458.

Butterworth, G. (2002). Joint visual attention in infancy. In G. Bremner and A. Fogel (Eds.), *Blackwell Handbook of Infant Development: Handbooks of Developmental Psychology* (pp. 213-240). Oxford: Blackwell.

Butterworth, G., and Cochran, C. (1980). Towards a mechanism of joint attention in human infancy. *International Journal of Behavioural Development, 3*, 253-272.

Butterworth, G., and Grover, L. (1988). The origins of referential communication in human infancy. In L. Weiskrantz (Ed.), *Thought wihtout language* (pp. 5-24). Oxford: Oxford University Press.

Butterworth, G., and Itakura, S. (2000). How the eyes, head and hand ser ve definite reference. *British Journal of Developmental Psychology, 18*, 25-50.

Butterworth, G., and Jarrett, N. (1991). What minds have in common is space: spatial mechanisms serving joint visual attention in infancy. *British Journal of Developmental Psychology, 9*, 55-72.

Butterworth, G., and Morisette, P. (1996). Onset of pointing and the acquisition of language in infancy. *British Journal of Develomental Psychology, 14*, 219-231.

Buys, K., and Veltman, A. (2008). Measurement in Kindergarten 1 and 2. In M. van den Heuvel and K. Buys (Eds.), *Young children learn measurement and geometry: a learning-teaching trajectory with intermediate attainment targets for the lower grades in Primary school* (pp. 15-36). Rotterdam: Sense Publishers.

Casey, B., Andrews, N., Schindler, H., Kersh, J. E., Samper, A., and Copley, J. V. (2008). The development of spatial skills through interventions involving block building activities. *Cognition and Instruction, 26*, 1-41.

Casey, B., Erkut, S., Ceder, I., and Young, J. M. (2008). Use of a stor ytelling context to improve girls' and boys' geometry skills in kindergarten. Journal of Applied Developmental Psychology, 29, 29-48.

Clements, D. H., and Sarama, J. (2007 a.). Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *.Journal for Research in Mathematics Education, 38*, 136-163

Clements, D. H., and Sarama, J. (2007 b.). Early childhood mathematics learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning .* (Vol. 1, pp. 461-555). Charlotte, NC:: Information Age Publishing.

Clements, D., Wilson, D., and Sarama, J. (2004). Young children's composition of geometric figures: a learning trajectory. *Mathematical Thinking and Learning, 6*(2), 163-184.

De Bock, D., Van Dooren, W., Janssens, D., and Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*, 311–334.

De Bock, D., Verschaffel, L., and Janssens, D. (1998). The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics, 35*, 65–83.

De Bock, D., Verschaffel, L., and Janssens, D. (2002). The effects of different problem presentations and formulations on the illusion of linearity in secondary school students. *Mathematical Thinking and Learning, 4*, 65–89.

De Bock, D., Verschaffel, L., Janssens, D., Van Dooren, W., and Claes, K. (2003). Do realistic contexts and graphical representations always have a beneficial impact on students' performance? Negative evidence from a study on modeling non-linear geometry problems. *Learning and Instruction, 13*, 441-463.

Dembo, Y., Levin, I., and Siegler, R.S. (1997). A comparison of the geometric reasoning of students attending Israeli ultra-orthodox and mainstream schools. *Developmental Psychology, 33*, 92-103.

Guttierez, A. (1992). Exploring the link between van Hiele's levels and 3-dimensional geometry. *Structural Topology, 18*, 31-48.

Hart, K. (1981). *Children's understanding of mathematics: 11-16*. London: John Murray.

Hart, K., Brown, M., Kerslake, D., Küchemann, D. and Ruddock, G. (1985) *Chelsea Diagnostic Mathematics' Test: Teacher's Guide*. Windsor: NFER-Nelson.

Howard, I. P. (1978). Recognition and knowledge of the water level principle. *Perception, 7*, 151-160.

Ibbotson, A., and Bryant, P. E. (1976). The perpendicular error and the vertical effect in children's drawing. *Perception, 5*, 319-326.

Kloosterman, P., Warfield, J., Wearne, D., Koc, Y., Martin, G., and Strutchens, M. (2004). Knowledge of learning mathematics and perceptions of learning mathematics in 4th-grade students. In P. Kloosterman and F. Lester (Eds.), *Resutlts and interpetations of the 2003 mathematics assessment of the NAEP*. Reston, VA: national Council Of Teachers of Mathematics.

Knuth, E.J. (2000) Student understanding of the Cartesian connection: an exploratory study. *Journal for Research in Mathematics Education, 31*, 500-508.

Lawrenson, W., and Bryant, P. E. (1972). Absolute and relative codes in young children. *Journal of Child Psychology and Psychiatry, 12*, 25-35.

Luchins, A. S., and Luchins, E. H. (1970). *Wertheimer's Seminars Re-visited: Problem solving and thinking Vols I to III*. Albany, N.Y.: Faculty Student Association, State University of New York.

Magina, S., and Hoyles, C. (1997). Children's understanding of turn and angle. In T. Nunes and P. Bryant (Eds.), *Learning and Teaching Mathematics. An International Perspective* (pp. 88-114). Hove (UK): Psychology Press.

Mitchelmore, M. (1998). Young students' concepts of turning. *Cognition and Instruction, 16*, 265-284.

Mitchelmore, M., and White, P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics, 41*, 209-238.

Noss, R. (1988). Children's learning of geometrical concepts through Logo. *Journal for Research in Mathematics Education, 18*(5), 343-362.

Nunes, T., Light, P., and Mason, J. (1993). Tools for thought: the measurement of lenght and area. *Learning and Instruction, 3*, 39-54.

Outhred, L., and Mitchelmore, M. (2000). Young children's intuitive understanding of rectangular area measurement. *Journal for Research in Mathematics Education, 31*(2), 144-167.

Papert, S. (1980). *Mindstorms: children, computers and powerful ideas*. Brighton: Harvester Press.

Piaget, J., and Inhelder, B. (1963). *The Child's Conception of Space*. London: Routledge. and Kegan Paul.

Piaget, J., Inhelder, B., and Szeminska, A. (1960). *The child's conception of geometry*. London: Routledge and Kegan Paul.

Quinn, P. C., Siqueland, E. R., and Bomba, P. C. (1985). Delayed recognition memory for orientation by human infants. *Journal of Experimental Child Psychology, 40*, 293-303.

Rock, I. (1970). Towards a cognitive theory of perceptual constancy. In A. Gilgen (Ed.), *Contemporary Sceintific psychology* (pp. 342-378). New York: Academic Press.

Sarama, J., Clements, D.H., Swaminathan, S., McMillen, S., and Gomez, G. (2003) Development of Mathematics Concepts of Two-Dimensional Space in Grid Environments: an exploratory study. *Cognition and Instruction, 21*, 285-324.

Seo, K-H., and Ginsburg, H. (2004) What is developmentally appropriate in early childhood mathematics education? In D.H. Clements, J. Sarama, and A-M. Dibias (Eds.) *Engaging young children in mathematics: Standards for early childhood mathematics education*. (pp.91-104). Mahwah, NJ: Erlbaum.

Slater, A. (1999). Visual organisation and perceptual constancies in early infancy. In V. Walsh and J. Kulikowski (Eds.), *Perceptual Constancy: why things look as they do* (pp. 6-30). Cambridge: C.U.P.

Slater, A., Field, T., and Hernadez-Reif, M. (2002). The development of the senses. In Slater and M. Lewis (Eds.), *Introduction to Infant Development* (pp. 83-98). Oxford: O.U.P.

Slater, A., and Lewis, M. (2002). *Introduction to infant development*. Oxford: O.U.P.

Slater, A., Mattock, A., and Brown, E. (1990). Size constancy at birth: newborn infants' responses to retinal and real size. *Journal of Experimental Child Psychology, 49*, 314-322.

Slater, A., and Morison, V. (1985). Shape constancy and slant perception at birth. *Perception, 14*, 337-344.

Somerville, S., and Bryant, P. E. (1985). Young children's use of spatial coordinates. *Child Development, 56*, 604-613.

Sowder, J., Wearne, D., Martin, G., and Strutchens, M. (2004). What grade 8 students know about mathematics: changes over a decade. In P. Kloosterman and F. Lester (Eds.), *The 1990 through 2000 mathematics assessment of the NAEP: Results and interpretations* (pp. 105-143). Reston, VA: National Councilof Teachers of Mathematics.

van den Heuvel-Panhuizen, M. and Buys, K. (2008) *Young children learn measurement and Geometry: a learning-teaching trajectory with intermediate attainment targets for the lower grades in Primary school.* Rotterdam: Sense Publishers.

van den Heuvel, M., Veltman, A., Janssen, C., and Hochstenbach, J. (2008). Geometry in Kindergarten 1 and 2. In M. van den Heuvel and K. Buys (Eds.), *Young children learn measurement and geometry: a learning-teaching trajectory with intermediate attainment targets for the lower grades in Primary school*. Rotterdam: Sense Publishers.

van Hiele, P. M. (1986). *Structure and insight*. Orlando, Fl: Academic Press.

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh and M. Landau (Eds.), *Acquisition of mathematics concepts and processes* (pp. 128-175). London: Academic Press.

Vygotsky, L. S. (1978). *Mind in Society*. Cambridge, Mass: Harvard University Press.

Wertheimer, M. (1945). *Productive Thinking.* New York: Harper and Row.

Wilkening, F. (1979). Combining of stimulus dimensions in children's and adult's judgements of area: an information integration analysis. *Developmental Psychology, 15*, 25-53.

Wilkening, F., and Anderson, N. H. (1982). Comparison of two-rule assessment methodologies for studying cognitive development and knowledge structure. *Psychological Bulletin, 92*, 215-237.

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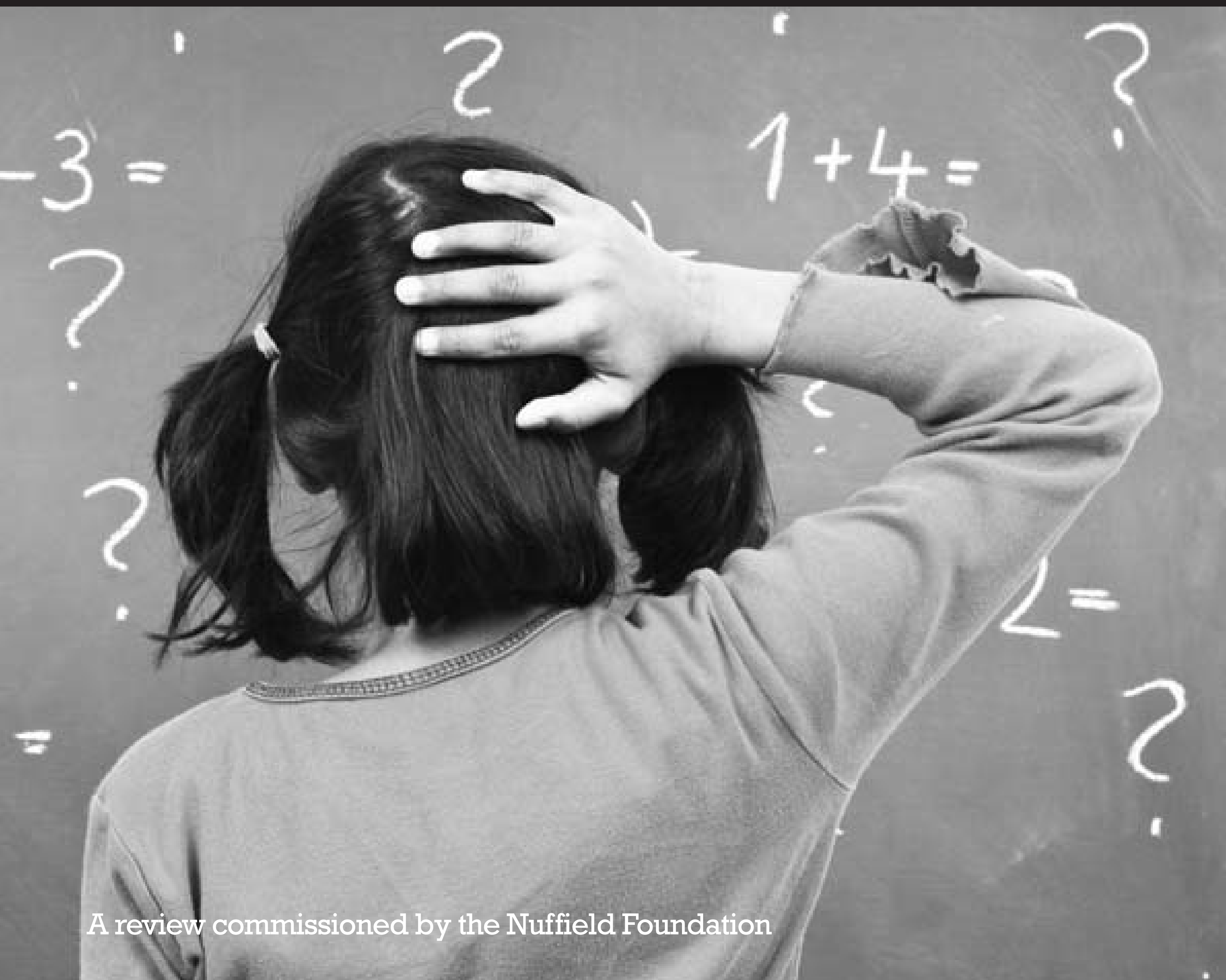
6

Key understandings in mathematics learning

Paper 6: Algebraic reasoning

By Anne Watson, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

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About the author

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 6: Algebraic reasoning

Headlines

- Algebra is the way we express generalisations about numbers, quantities, relations and functions. For this reason, good understanding of connections between numbers, quantities and relations is related to success in using algebra. In particular, students need to understand that addition and subtraction are inverses, and so are multiplication and division.

- To understand algebraic symbolisation, students have to (a) understand the underlying operations and (b) become fluent with the notational rules. These two kinds of learning, the meaning and the symbol, seem to be most successful when students know what is being expressed and have time to become fluent at using the notation.

- Students have to learn to recognise the different nature and roles of letters as: unknowns, variables, constants and parameters, and also the meanings of equality and equivalence. These meanings are not always distinct in algebra and do not relate unambiguously to arithmetical understandings, Mapping symbols to meanings is not learnt in one-off experiences.

- Students often get confused, misapply, or misremember rules for transforming expressions and solving equations. They often try to apply arithmetical meanings to algebraic expressions inappropriately. This is associated with over-emphasis on notational manipulation, or on 'generalised arithmetic', in which they may try to get concise answers.

Understanding symbolisation

The conventional symbol system is not merely an expression of generalised arithmetic; to understand it students have to understand the meanings of arithmetical operations, rather than just be able to carry them out. Students have to understand 'inverse' and know that addition and subtraction are inverses, and that division is the inverse of multiplication. Algebraic representations of relations between quantities, such as difference and ratio, encapsulate this idea of inverse. Using familiarity with symbolic expressions of these connections, rather than thinking in terms of generalising four arithmetical operations, gives students tools with which to understand commutativity and distributivity, methods of solving equations, and manipulations such as simplifying and expanding expressions.

The precise use of notation has to be learnt as well, of course, and many aspects of algebraic notation are inherently confusing (e.g. $2r$ and r^2). Over-reliance on substitution as a method of doing this can lead students to get stuck with arithmetical meanings and rules, rather than being able to recognise algebraic structures. For example, students who have been taught to see expressions such as:

$97 - 49 + 49$

as structures based on relationships between numbers, avoiding calculation, identifying variation, and having a sense of limits of variability, are able to reason with relationships more securely and at a younger age than those who have focused only on calculation. An expression such as $3x + 4$ is both the answer to a question, an object in itself, and also an algorithm or process for calculating a particular value. This has parallels in arithmetic: the answer to $3 \div 5$ is 3/5.

Time spent relating algebraic expressions to arithmetical structures, as opposed to calculations, can make a difference to students' understanding. This is especially important when understanding that apparently different expressions can be equivalent, and that the processes of manipulation (often the main focus of algebra lessons) are actually transformations between equivalent forms.

Meanings of letters and signs

Large studies of students' interpretation and use of letters have shown a well-defined set of possible actions. Learners may, according to the task and context:

- try to evaluate them using irrelevant information
- ignore them
- used as shorthand for objects, e.g. a = apple
- treat them as objects
- use a letter as a specific unknown
- use a letter as a generalised number
- use a letter as a variable.

Teachers have to understand that students may use any one of these approaches and students need to learn when these are appropriate or inappropriate. There are conventions and uses of letters throughout mathematics that have to be understood in context, and the statement 'letters stand for numbers' is too simplistic and can lead to confusion. For example:

- it is not always true that different letters have different values
- a letter can have different values in the same problem if it stands for a variable
- the same letter does not have to have the same value in different problems.

A critical shift is from seeing a letter as representing an unknown, or 'hidden', number defined within a number sentence such as:

$3 + x = 8$

to seeing it as a variable, as in $y = 3 + x$, or $3 = y - x$. Understanding x as some kind of generalized number which can take a range of values is seen by some researchers to provide a bridge from the idea of unknown to that of variables. The use of boxes to indicate unknown numbers in simple 'missing number' statements is sometimes helpful, but can also lead to confusion when used for variables, or for more than one hidden number in a statement.

Expressions linked by the 'equals' sign might be not just numerically equal, but also equivalent, yet students need to retain the 'unknown' concept when setting up and solving equations which have finite solutions. For example, $10x - 5 = 5(2x - 1)$ is a statement about equivalence, and x is a variable, but $10x - 5 = 2x + 1$ defines a value of the variable for which this equality is true. Thus x in the second case can be seen as an unknown to be found, but in the first case is a variable. Use of graphical software can show the difference visually and powerfully because the first situation is represented by one line, and the second by two intersecting lines, i.e. one point.

Misuse of rules

Students who rely only on remembered rules often misapply them, or misremember them, or do not think about the meaning of the situations in which they might be successfully applied. Many students will use guess-and-check as a first resort when solving equations, particularly when numbers are small enough to reason about 'hidden numbers' instead of 'undoing' within the algebraic structure. Although this is sometimes a successful strategy, particularly when used in conjunction with graphs, or reasoning about spatial structures, or practical situations, over-reliance can obstruct the development of algebraic understanding and more universally applicable techniques.

Large-scale studies of U.K. school children show that, despite being taught the BIDMAS rule and its equivalents, most do not know how to decide on the order of operations represented in an algebraic expression. Some researchers believe this to be due to not fully understanding the underlying operations, others that it may be due to misinterpretation of expressions. There is evidence from Australia and the United Kingdom that students who are taught to use flow diagrams, and inverse flow diagrams, to construct and reorganise expressions are better able to decide on the order implied by expressions involving combinations of operations. However, it is not known whether students taught this way can successfully apply their knowledge of order in situations in which flow diagrams are inappropriate, such as with polynomial equations, those involving the unknown on 'both sides', and those with more than one variable. To use algebra effectively, decisions about order have to be fluent and accurate.

Misapplying arithmetical meanings to algebraic expressions

Analysis of children's algebra in clinical studies with 12- to 13-year-olds found that the main problems in moving from arithmetic to algebra arose because:

- the focus of algebra is on relations rather than calculations; the relation $a + b = c$ represents three unknown quantities in an additive relationship
- students have to understand inverses as well as operations, so that a hidden value can be found even if the answer is not obvious from knowing number bonds or multiplication facts; $7 + b = 4$ can be solved using knowledge of addition, but $c + 63 = 197$ is more easily solved if subtraction is used as the inverse of addition
- some situations have to be expressed algebraically first in order to solve them. 'My brother is two years older than me, my sister is five years younger than me; she is 12, how old will my brother be in three years' time?' requires an analysis and representation of the relationships before solution. 'Algebra' in this situation means constructing a method for keeping track of the unknown as various operations act upon it.
- letters and numbers are used together, so that numbers may have to be treated as symbols in a structure, and not evaluated. For example, the structure $2(3+b)$ is different from the structure of $6 + 2b$ although they are equivalent in computational terms. Learners have to understand that sometimes it is best to leave number as an element in an algebraic structure rather than 'work it out'.
- the equals sign has an expanded meaning; in arithmetic it is often taken to mean 'calculate' but in algebra it usually means 'is equal to' or 'is equivalent to'. It takes many experiences to recognise that an algebraic equation or equivalence is a statement about relations between quantities, or between combinations of operations on quantities. Students tend to want 'closure' by compressing algebraic expressions into one term instead of understanding what is being expressed.

Expressing generalisations

In several studies it has been found that students understand how to use algebra if they have focused on generalizing with numerical and spatial representations in which counting is not an option. Attempts to introduce symbols to very young students as tools to be used when they have a *need* to express known general relationships, have been successful both for aiding their understanding of symbol use, and understanding the underlying quantitative relations being expressed. For example, some year 1 children first compare and discuss quantities of liquid in different vessels, and soon become able to use letters to stand for unknown amounts in relationships, such as $a > b$; $d = e$; and so on. In another example, older primary children could generalise the well-known questions of how many people can sit round a line of tables, given that there can be two on each side of each table and one at each of the extreme ends. The ways in which students count differ, so the forms of the general statement also differ and can be compared, such as: 'multiply the number of tables by 4 and add 2 or 'it is two times one more than the number of tables'.

The use of algebra to express *known* arithmetical generalities is successful with students who have developed advanced mental strategies for dealing with additive, multiplicative and proportional operations (e.g. compensation as in $82 – 17 = 87 – 17 – 5$). When students are allowed to use their own methods of calculation they often find algebraic structures for themselves. For example, expressing 13×7 as $10 \times 7 + 3 \times 7$, or as $2 \times 7^2 - 7$, are enactments of distributivity and learners can represent these symbolically once they know that letters can stand for numbers, though this is not trivial and needs several experiences. Explaining a general result, or structure, in words is often a helpful precursor to algebraic representation.

Fortunately, generalising from experience is a natural human propensity, but the everyday inductive reasoning we do in other contexts is not always appropriate for mathematics. Deconstruction of diagrams and physical situations, and identification of relationships between variables, have been found to be more successful methods of developing a formula than pattern-generalisation from number sequences alone. The use of verbal descriptions has been shown to enable students to bridge between observing relations and writing them algebraically.

Further aspects of algebra arise in the companion summaries, and also in the main body of *Paper 6: Algebraic reasoning*.

Recommendations

Research about mathematical learning	Recommendations for teaching
The bases for using algebraic symbolisation successfully are (a) understanding the underlying operations and relations and (b) being able to use symbolism correctly.	Emphasis should be given to reading numerical and algebraic expressions relationally, rather than computationally. For algebraic thinking, it is more important to understand how operations combine and relate to each other than how they are performed. Teachers should avoid emphasising symbolism without understanding the relations it represents.
Children interpret 'letter stands for number' in a variety of ways, according to the task. Mathematically, letters have several meanings according to context: unknown, variable, parameter, constant.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of children's possible interpretations of letters, and also that when correctly used, letters can have a range of meanings. Teachers should avoid using materials that oversimplify this variety. Hands-on ICT can provide powerful new ways to understand these differences in several representations.
Children interpret '=' to mean 'calculate'; but mathematically '=' means either 'equal to' or 'equivalent to'.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of the difficulties about the '=' sign and use multiple contexts and explicit language. Hands-on ICT can provide powerful new ways to understand these differences in several representations.
Students often forget, misremember, misinterpret situations and misapply rules.	Developers of the curriculum, advisory schemes of work and teaching methods need to take into account that algebraic understanding takes time, multiple experiences, and clarity of purpose. Teachers should emphasise situations in which generalisations can be identified and described to provide meaningful contexts for the use of algebraic expressions. Use of software which carries out algebraic manipulations should be explored.
Everyone uses 'guess-&-check' if answers are immediately obvious, once algebraic notation is understood.	Algebra is meaningful in situations for which specific arithmetic cannot be easily used, as an expression of relationships. Focusing on algebra as 'generalised arithmetic', e.g. with substitution exercises, does not give students reasons for using it.
Even very young students can use letters to represent unknowns and variables in situations where they have reasoned a general relationship by relating properties. Research on inductive generalisation from pattern sequences to develop algebra shows that moving from expressing simple additive patterns to relating properties has to be explicitly supported.	Algebraic expressions of relations should be a commonplace in mathematics lessons, particularly to express relations and equivalences. Students need to have multiple experiences of algebraic expressions of general relations based in properties, such as arithmetical rules, logical relations, and so on as well as the well-known inductive reasoning from sequences.

Recommendations for research

- The main body of *Paper 6: Algebraic reasoning* includes a number of areas for which further research would be valuable, including the following.

- How does explicit work on understanding relations between quantities enable students to move successfully from arithmetical to algebraic thinking?

- What kinds of explicit work on expressing generality enable students to use algebra?

- What are the longer-term comparative effects of different teaching approaches to early algebra on students' later use of algebraic notation and thinking?

- How do learners' synthesise their knowledge of elementary algebra to understand polynomial functions, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations?

- What useful kinds of algebraic expertise could be developed through the use of computer algebra systems in school?

Algebraic reasoning

In this review of how students learn algebra we try to balance an approach which focuses on what learners can do and how their generalising and use of symbols develop (a 'bottom up' developmental approach), and a view which states what is required in order to do higher mathematics (a 'top down' hierarchical approach). The 'top down' view often frames school algebra as a list of techniques which need to be fluent. This is manifested in research which focuses on errors made by learners in the curriculum and small-scale studies designed to ameliorate these. This research tells us about development of understanding by identifying the obstacles which have to be overcome, and also reveals how learners think. It therefore makes sense to start by outlining the different aspects of algebra. However, this is not suggesting that all mathematics teaching and learning should be directed towards preparation for higher mathematics.

By contrast a 'bottom up' view usually focuses on algebraic thinking, taken to mean the expression and use of general statements about relationships between variables. Lins (1990) sought a definition of algebraic thinking which encompassed the different kinds of engagement with algebra that run through mathematics. He concluded that algebraic thinking was an intentional shift from context (which could be 'real', or a particular mathematical case) to structure. Thus 'algebraic thinking arises when people are detecting and expressing structure, whether in the context of problem solving concerning numbers or some modelled situation, whether in the context of resolving a class of problems, or whether in the context of studying structure more generally' (Lins, 1990). Thus a complementary 'bottom up' view includes consideration of the development of students' natural ability to discern patterns and generalise them, and their growing competence in understanding and using symbols; however this would not take us very far in considering all the aspects of school algebra. The content of school algebra as the development of algebraic reasoning is expressed by Thomas and Tall (2001) as the shifts between procedure, process/concept, generalised arithmetic, expressions as evaluation processes, manipulation, towards axiomatic algebra. In this perspective it helps to see manipulation as the generation and transformation of equivalent expressions, and the identification of specific values for variables within them.

At school level, algebra can be described as:

- manipulation and transformation of symbolic statements
- generalisations of laws about numbers and patterns
- the study of structures and systems abstracted from computations and relations
- rules for transforming and solving equations
- learning about variables, functions and expressing change and relationships
- modelling the mathematical structures of situations within and outside mathematics.

Bell (1996) and Kaput (1998; 1999) emphasise the process of symbolisation, and the need to operate with symbolic statements and the use them within and outside algebra, but algebra is much more than the acquisition of a sign system with which to express known concepts. Vergnaud (1998) identifies new concepts that students will meet in algebra as: equations, formulae, functions, variables and parameters. What makes them new is that symbols are higher order objects than numbers and become mathematical objects in their own right; arithmetic has to work in algebraic systems, but symbol systems are not merely expressions of general arithmetic. Furthermore, 'the words and symbols we use to communicate do not refer directly to reality but to represented entities: objects, properties, relationships, processes, actions, and constructs, about which there is no automatic agreement' (p.167).

In this paper I draw on the research evidence about the first five of the aspects above. In the next paper I shall tackle modelling and associated issues, and their relation to mathematical reasoning and application more generally at school level.

It would be naïve to write about algebraic reasoning without reporting the considerable difficulties that students have with adopting the conventions of algebra, so the first part of this review addresses the relationship between arithmetic and algebra, and the obstacles that have to be overcome to understand the meaning of letters and expressions and to use them. The second part looks at difficulties which are evident in three approaches used to develop algebraic reasoning: expressing generalities; solving equations; and working with functions. The third part summarises the findings and makes recommendations for practice and research.

Part 1: arithmetic, algebra, letters, operations, expressions

Relationships between arithmetic and algebra

In the United States, there is a strong commitment to arithmetic, particularly fluency with fractions, to be seen as an essential precursor for algebra: 'Proficiency with whole numbers, fractions, and particular aspects of geometry and measurement are the Critical Foundation of Algebra. …The teaching of fractions must be acknowledged as critically important and improved before an increase in student achievement in Algebra can be expected.' (NMAP, 2008). While number sense precedes formal algebra in age-related developmental terms, this one-way relationship is far from obvious in mathematical terms. In the United Kingdom where secondary algebra is not taught separately from other mathematics, integration across mathematics makes a two-way relationship possible, seeing arithmetic as particular instances of algebraic structures which have the added feature that they can be calculated. For example, rather than knowing the procedures of fractions so that they can be generalised with letters and hence make algebraic fractions, it is possible for fraction calculations to be seen as enactments of relationships between rational structures, those generalised enactments being expressed as algorithms.

For this review we see number sense as preceding formal algebra in students' learning, but to imagine that algebraic understanding is merely a generalisation of arithmetic, or grows directly from it, is a misleading over-simplification.

Kieran's extensive work (e.g. 1981, 1989, 1992) involving clinical studies with ten 12- to 13-year-olds leads her to identify five inherent difficulties in making a direct shifts between arithmetic and algebra.

- The focus of algebra is on relations rather than calculations; the relation $a + b = c$ represents two unknown numbers in an additive relation, and while $3 + 5 = 8$ is such a relation it is more usually seen as a representation of 8, so that $3 + 5$ can be calculated whereas $a + b$ cannot.

- Students have to understand inverses as well as operations, so that finding a hidden number can be done even if the answer is not obvious from knowing number bonds or multiplication facts; $7 + b = 4$ can be done using knowledge of addition, but $c + 63 = 197$ is more easily done if subtraction is used as the inverse of addition[1]. Some writers claim that understanding this structure is algebraic, while others say that doing arithmetic to find an unknown is arithmetical reasoning, not algebraic reasoning.

- Some situations have to be expressed algebraically in order to solve them, rather than starting a solution straight away. 'I am 14 and my brother is 4 years older than me' can be solved by addition, but 'My brother is two years older than me, my sister is five years younger than me; she is 12, how old will my brother be in three years' time?' requires an analysis and representation of the relationships before solution. This could be with letters, so that the answer is obtained by finding k where $k - 5 = 12$ and substituting this value into $(k + 2) + 3$. Alternatively it could be done by mapping systems of points onto a numberline, or using other symbols for the unknowns. 'Algebra' in this situation means constructing a method for keeping track of the unknown as various operations act upon it.

- Letters and numbers are used together, so that numbers may have to be treated as symbols in a structure, and not evaluated. For example, the structure $2(a + b)$ is different from the structure of $2a + 2b$ although they are equivalent in computational terms.

- The equals sign has an expanded meaning; in arithmetic it often means 'calculate' but in algebra it more often means 'is equal to' or even 'is equivalent to'.

If algebra is seen solely as generalised arithmetic (we take this to mean the expression of general arithmetical rules using letters), many problems arise for learning and teaching. Some writers describe these difficulties as manifestations of a 'cognitive gap' between arithmetic and algebra (Filloy and Rojano, 1989; Herscovics and Linchevski, 1994). For example, Filloy and Rojano saw students dealing arithmetically with equations of the form $ax + b = c$, where a, b and c are numbers, using inverse operations on the numbers to complete the arithmetical statement. They saw this as 'arithmetical' because it depended only on using operations to find a 'hidden' number. The same students acted algebraically with equations such as $ax + b = cx + d$, treating each side as an expression of relationships and using direct operations not to 'undo' but to maintain the equation by manipulating the expressions and equality. If such a gap exists, we need to know if it is developmental or epistemological, i.e. do we have to wait till learners are ready, or could teaching make a difference? A bottom-up view would be that algebraic thinking is often counter-intuitive, requires good understanding of the symbol system, and abstract meanings which do not arise through normal engagement with phenomena. Nevertheless the shifts required to understand it are shifts the mind is able to make given sufficient experiences with new kinds of object and their representations. A top-down view would be that students' prior knowledge, conceptualisations and tendencies create errors in algebra. Carraher and colleagues (Carraher, Brizuela & Earnest, 2001; Carraher, Schliemann & Brizuela, 2001) show that the processes involved in shifting from an arithmetical view to an algebraic view, that is from quantifying expressions to expressing relations between variables, are repeated for new mathematical structures at higher levels of mathematics, and hence are characteristics of what it means to learn mathematics at every level rather than developmental stages of learners. This same point is made again and again by mathematics educators and philosophers who point out that such shifts are fundamental in mathematics, and that reification of new ideas, so that they can be treated as the elements for new levels of thought, is how mathematics develops both historically and cognitively. There is considerable agreement that these shifts require the action of teachers and teaching, since they all involve new ways of thinking that are unlikely to arise naturally in situations (Filloy and Sutherland, 1996).

Some of the differences reported in research rest on what is, and what is not, described as algebraic. For example, the equivalence class of fractions that represent the rational number 3/5 is all fractions of the form $3k/5k$ ($k \in N$)[2]. It is a curriculum decision, rather than a mathematical one, whether equivalent fractions are called 'arithmetic' or 'algebra' but whatever is decided, learners have to shift from seeing 3/5 as 'three cakes shared between five people' to a quantitative label for a general class of objects structured in a particular quantitative relationship. This is an example of the kind of shift learners have to make from calculating number expressions to seeing such expressions as meaningful structures.

Attempts to introduce symbols to very young students as tools to be used when they have a need to express general relationships, can be successful both for them understanding symbol use, and understanding the underlying quantitative relations being expressed (Dougherty, 1996; 2001). In Dougherty's work, students starting school mathematics first compare and discuss quantities of liquid in different vessels, and soon become able to use letters to stand for unknown amounts. Arcavi (1994) found that, with a range of students from middle school upwards over several years, symbols could be used as tools early on to express relationships in a situation. The example he uses is the well-known one of expressing how many people can sit round a line of tables, given that there can be two on each side and one at each of the extreme ends. The ways in which students count differ, so the forms of the general statement also differ, such as: 'multiply the number of tables by 4 and add two' or 'it is two times one more than the number of tables'. In Brown and Coles' work (e.g. 1999, 2001), several years of analysis of Coles' whole-class teaching showed that generalising by expressing structures was a powerful basis for students to *need* symbolic notation, which they could then use with meaning. For example, to express a number such that 'twice the number plus three' is 'three less' than 'add three and double the number' a student who has been in a class of 12-year-olds where expression of general relationships is a normal and frequent activity introduced N for himself without prompting when it is appropriate.

When students are allowed to use their own methods of calculation they often find algebraic structures for themselves. For example, expressing 13×7 as $10 \times 7 + 3 \times 7$, or as $2 \times 7^2 - 7$, are enactments of distributivity (and, implicitly, commutativity and associativity) and can be represented symbolically, though this shift is not trivial (Anghileri, Beishuizen and van Putten,2002; Lampert, 1986). On the other hand, allowing students to develop a mindset in which any method that gives a right answer is as good as any other can lock learners into additive procedures where multiplicative ones would be more generalisable, multiplicative methods where exponential methods would be more powerful, and so on. But some number-specific arithmetical methods *do* exemplify algebraic structures, such as the transformation of 13×7 described above. This can be seen either as 'deriving new number facts from known number facts' or as an instance of algebraic reasoning.

The importance of a link between the kinds of transformations necessary for mental arithmetic and algebraic thinking is demonstrated in a three-year longitudinal teaching and testing programme of 116 students aged 12 to 14 (Britt and Irwin, 2007). Students who had developed advanced mental strategies, (e.g. compensation as in $82 - 17 = 87 - 17 - 5$) for dealing with additive, multiplicative and proportional operations, could use letters in conventional algebra once they knew that they 'stood for' numbers. Those who did best at algebra were those in schools where teachers had focused on generalizing with numerical and spatial representations in situations where counting was not a sensible option.

There are differences in the meaning of notation as one shifts between arithmetic and algebra. Wong (1997) tested and interviewed four classes of secondary students to see whether they could distinguish between similar notations used for arithmetic and algebra. For example, in arithmetic the expression $3(4 + 5)$ is both a structure of operations and an invitation to calculate, but in algebra $a(b + c)$ is only a structure of operations. Thus students get confused when given mixtures such as $3(b + 5)$ because they can assume this is an invitation to calculate. This tendency to confuse what is possible with numbers and letters is subtle and depends on the expression. For example, Wong found that expressions such as $(2a^m)^n$ are harder to simplify and substitute than $(hk)^n$, possibly because the second expression seems very clearly in the realm of algebra and rules about letters. Where Booth and Kieran claim that it is not the symbolic conventions alone that create difficulties but more often a lack of understanding of the underlying operations, Wong's work helpfully foregrounds some of the inevitable confusions possible in symbolic conventions. The student has to understand when to calculate, when to leave an expression as a statement about operations, what particular kind of number (unknown, general or variable) is being denoted, and what the structure looks like with numbers and letters in combination. As an example of the last difficulty, 2^x is found to be harder to deal with than x^2 although they are visually similar in form.

The question for this review is therefore not *whether* learners can make such shifts, or *when* they make them, but *what* are the shifts they have to make, and in what circumstances do they make them.

Summary

- Algebra is not just generalised arithmetic; there are significant differences between arithmetical and algebraic approaches.

- The shifts from arithmetic to algebra are the kinds of shifts of perception made throughout mathematics, e.g. from quantifying to relationships between quantities; from operations to structures of operations.

- Mental strategies can provide a basis for understanding algebraic structures.

- Students will accept letters and symbols standing for numbers when they have quantitative relationships to express; they seem to be able to use letters to stand for 'hidden' numbers and also for 'any' number.

- Students are confused by expressions that combine numbers and letters, and by expressions in which their previous experience of combinations are reversed. They have to learn to 'read' expressions structurally even when numbers are involved.

Meaning of letters

Students' understanding of the meaning of letters in algebra, and how they use letters to express mathematical relationships, are at the root of algebraic development. Kuchemann (1981) identified several different ways adolescent students used letters in the Chelsea diagnostic test instrument (Hart, 1981). His research is based on test papers of 2900 students between 12 and 16 (see Appendix 1).

Letters were:
- evaluated in some way, e.g. $a = 1$
- ignored, e.g. $3a$ taken to be 3
- used as shorthand for objects, e.g. a = apple
- treated as objects
- used as a specific unknown
- used as a generalised number
- used as a variable.

Within his categorisation there were correct and incorrect uses, such as students who ascribed a value to a letter based on idiosyncratic decisions or past experience, e.g. $x = 4$ because it was 4 in the previous question. These interpretations appear to be task dependent, so learners had developed a sense of what sorts of question were treated in what kinds of ways, i.e. generalising (sometimes idiosyncratically) about question-types through familiarity and prior experience.

Booth (1984) interviewed 50 students aged 13 to 15 years, following up with 17 further case study students. She took a subset of Kuchemann's meanings, 'letters stand for numbers', and further unpicked it to reveal problems based on students' test answers and follow-up interviews. She identified the following issues which, for us, identify more about what students have to learn.

- It is not always true that different letters have different values; for example one solution to $3x + 5y = 8$ is that $x = y = 1$.

- A letter can have different values in the same problem, but not at the same time, if it stands for a variable (such as an equation having multiple roots, or questions such as 'find the value of $y = x^2 + x + 2$ when $x = 1, 2, 3\ldots$')

- The same letter does not have to have the same value in different problems.

- Values are not related to the alphabet ($a = 1$, $b = 2$ …; or $y > p$ because of relative alphabetic position).

- Letters do not stand for objects (a for apples) except where the objects are units (such as m for metres).

- Letters do not have to be presented in alphabetical order in algebraic expressions, although there are times when this is useful[3].

- Different symbolic rules apply in algebra and arithmetic, e.g.: '2 lots of x' is written '$2x$' but two lots of 7 are not written '27'.

As well as in Booth's study, paper and pencil tests that were administered to 2000 students in aged 11 to 15 in 24 Australian secondary schools in 1992 demonstrated all the above confusions (MacGregor and Stacey, 1997).

These problems are not resolved easily, because letters are used in mathematics in varying ways. There is no single correct way to use them. They are used as labels for objects that have no numerical value, such as vertices of shapes, or for objects that do have numerical value but are treated as general, such as lengths of sides of shapes. They denote fixed constants such as g, e or π, also non-numerical constants such as i, and also they represent unknowns which have to be found, and variables. Distinguishing between these meanings is usually not taught explicitly, and this lack of instruction might cause students some difficulty. On the other hand it is very hard to explain how to know the difference between a parameter, a constant and a variable (e.g. when asked to 'vary the constant' to explore a structure), and successful students may learn this only when it is necessary to make such distinctions in particular usage. It is particularly hard to explain that the O and E in O + E = O (to indicate odd and even numbers) are not algebraic, even though they do refer to numbers. Interpretation is therefore related to whether students understand the algebraic context, expression, equation, equivalence, function or other relation. It is not surprising that Furinghetti and Paola (1994) found that only 20 out of 199 students aged 12 to 17 could explain the difference between parameters and variables and unknowns (see also Bloedy-Vinner, 1994). Bills' (2007) longitudinal study of algebra learning in upper secondary students noticed that the letters x and y have a special status, so that these letters trigger certain kinds of behaviour (e.g. these are the variables; or (x,y) denotes the general point). Although any letter can stand for any kind of number, in practice there are conventions, such as x being an unknown; x, y, z being variables; a, b, c being parameters/coefficient or generalised lengths, and so on.

A critical shift is from seeing a letter as representing an unknown, or 'hidden', number defined within a number sentence such as:

$3 + x = 8$

to seeing it as a variable, as in $y = 3 + x$, or $3 = y - x$. While there is research to show how quasi-variables such as boxes can help students understand the use of letters in relational statements (see Carpenter and Levi, 2000) the shift from unknown to variable when similar letters are used to have different functions is not well-researched. Understanding x as some kind of generalised number which can take a range of values is seen by some researchers to provide a bridge from the idea of unknown to that of variables (Bednarz, Kieran and Lee, 1996).

The algebra of unknowns is about using solution methods to find mystery numbers; the algebra of variables is about expressing and transforming relations between numbers. These different lines of thought develop throughout school algebra. The 'variable' view depends on the idea that the expressions linked by the 'equals' sign might be not just numerically equal, but also equivalent, yet students need to retain the 'unknown' concept when setting up and solving equations which have finite solutions. For example, $10x - 5 = 5(2x - 1)$ is a statement about equivalence, and x is a variable, but $10x - 5 = 2x + 1$ defines a value of the variable for which this equality is true. Thus x in the second case can be seen as an unknown to be found.

It is possible to address some of the problems by giving particular tasks which force students to sort out the difference between parameters and variables (Drijvers, 2001). A parameter is a value that defines the structure of a relation. For example, in $y = mx + c$ the variables are x and y, while m and c define the relationship and have to be fixed before we can consider the covariation of x and y. In the United Kingdom this is dealt with implicitly, and finding the gradient and intercept in the case just described is seen as a special kind of task. At A-level, however, students have to find coefficients for partial fractions, or the coefficients of polynomials which have given roots, and after many years of 'finding x' they can find it hard to use particular values for x to identify parameters instead. By that time only those who have chosen to do mathematics need to deal with it, and those who earlier could only find the m and c in $y=mx + c$ by using formulae without comprehension may have given up maths. Fortunately, the dynamic possibilities of ICT offer tools to fully explore the variability of x and y within the constant behaviour of m and c and it is possible that more extensive use of ICT and modelling approaches might develop the notion of variable further.

Summary

- Letters standing for numbers can have many meanings.

- The ways in which operations and relationships are written in arithmetic and algebra differ.

- Learners tend to fall into well-known habits and assumptions about the use of letters.

- A particular difficulty is the difference between unknowns, variables, parameters and constants, unless these have meaning.

- Difficulties in algebra are not merely about using letters, but about understanding the underlying operations and structures.

- Students need to learn that there are different uses for different letters in mathematical conventions; for example, a, b and c are often used as parameters, or generalised lengths in geometry, and x, y and z are often used as variables.

Recognising operations

In several intervention studies and textbooks students are expected to use algebraic methods for problems for which an answer is required, and for which *ad hoc* methods work perfectly well. This arises when solving equations with one unknown on one side where the answer is a positive integer (such as $3x + 2 = 14$); in word problems which can be enacted or represented diagrammatically (such as 'I have 15 fence posts and 42 metres of wire; how far apart must the fence posts be to use all the wire and all the posts to make a straight fence?'); and in these and other situations in which trial-and-adjustment work easily. Students' choice to use non-algebraic methods in these contexts cannot be taken as evidence of problems with algebra.

In a teaching experiment with 135 students age 12 to 13, Bednarz and Janvier (1996) found that a mathematical analysis of the operations required for solution accurately predicted what students would find difficult, and they concluded that problems where one could start from what is known and work towards what is not known, as one does in arithmetical calculations, were significantly easier than problems in which there was no obvious bridge between knowns and the unknown, and the relationship had to be worked out and expressed before any calculations could be made. Many students tried to work arithmetically with these latter kinds of problem, starting with a fictional number and working forwards, generating a structure by trial and error rather than identifying what would be appropriate. This study is one of many which indicate that understanding the meaning of arithmetical operations, rather than merely being able to carry them out, is an essential precursor not only to deciding what operation is the right one to do, but also to expressing and understanding structures of relations among operations (e.g. Booth, 1984). The impact of weak arithmetical understanding is also observed at a higher level, when students can confuse the kinds of proportionality expressed in $y = k/x$ and $y = kx$, thinking the former must be linear because it involves a ratio (Baker, Hemenway and Trigueros, 2001). The ratio of k to x in the first case is specific for each value of x, but the ratio of y to x in the second case is invariant and this indicates a proportional relationship.

Booth (1984) selected 50 students from four schools to identify their most common errors and to interview those who made certain kinds of error. This led her to identify more closely how their weakness with arithmetic limited their progress with algebra. The methods they used to solve word problems were bound by context, and depended on counting, adding, and reasoning with whole and half numbers. They were unable to express how to solve problems in terms of arithmetical operations, so that algebraic expressions of such operations were of little use, being unrelated to their own methods. Similarly, their methods of recording were not conducive to algebraic expression, because the roles of different numbers and signs were not clear in the layout. For example, if students calculate as they go along, rather than maintaining the arithmetical structure of a question, much information is lost. For example, $4^2 - 2^2$ becomes $16 - 4$ and the 'difference between two squares' is lost; similarly, turning rational or irrational numbers into decimal fractions can lose both accuracy and structure.

In Booth's work it was not the use of letters that is difficult, but the underlying arithmetical understanding. This again supports the view that it is not until *ad hoc*, *number fact* and *guess-and-test* methods fail that students are likely to see a need for algebraic methods, and in a curriculum based on expressions and equations this is likely occur when solving equations with non-integer answers, where a full understanding of division expressed as fractions would be needed, and when working with the unknown on both sides of an equation. Alternatively, if students are trying to express general relationships, use of letters is essential once they realise that particular examples, while illustrating relationships, do not fully represent them. Nevertheless students' invented methods give insight into what they might know already that is formalisable, as in the 13×7 example given above.

Others have also observed the persistence of arithmetic (Kieran, 1992; Vergnaud, 1998). Vergnaud compares two student protocols in solving a distance/time problem and comments that the additive approach chosen by one is not conceptually similar to the multiplicative chosen by the other, even though the answers are the same, and that this linear approach is more natural for students than the multiplicative. Kieran (1983) conducted clinical interviews with six 13-year-old students to find out why they had difficulty with equations. The students tended to see tasks as about 'getting answers' and could not accept an expression as meaningful in itself. This was also observed by Collis (1971) and more recently by Ryan and Williams in their large scale study of students' mathematical understanding, drawing on a sample of about 15 000 U.K. students

(2007). Stacey and Macgregor (2000, p. 159) talk of the 'compulsion to calculate' and comment that at every stage students' thinking in algebraic problems was dominated by arithmetical methods, which deflected them from using algebra. Furthermore, Bednarz and Janvier (1996) showed that even those who identified structure during interviews were likely to revert to arithmetical methods minutes later. It seemed as if testing particular numbers was an approach that not only overwhelmed any attempts to be more analytical, but also prevented development of a structural method.

This suggests that too much focus on substitution in early algebra, rather than developing understanding of how structure is expressed, might allow a 'calculation' approach to persist when working with algebraic expressions. If calculation does persist, then it is only where calculation breaks down that algebraic understanding becomes crucial, or, as in Bednarz and Janvier (1996), where word problems do not yield to straightforward application of operations. For a long time in Soviet education word problems formed the core of algebra instruction. Davydov (1990) was concerned that arithmetic does not necessarily lead to awareness of generality, because the approach degenerates into 'letter arithmetic' rather than the expression of generality. He developed the approach used by Dougherty (2001) in which young students have to express relationships before using algebra to generalize arithmetic. For example, students in the first year of school compare quantities of liquid ('do you have more milk than me?') and express the relationship as, say, G < R. They understand that adding the same amount to each does not make them equal, but that they have to add some to G to make them equal. They do not use numbers until relationships between quantities are established.

Substituting values can, however, help students to understand and verify relationships: it matters if this is for an unknown: $5 = 2x - 7$ where only one value will do; or for an equation where variables will be related: $y = 2x - 7$; or to demonstrate equivalence: e.g. does $5(x + y) - 3 = 5x + 5y - 3$ or $5x + y - 3$? But using substitution to understand what expressions *mean* is not helpful. Furthermore the choice of values offered in many textbooks can exacerbate misunderstandings about the values letters can have. They can reinforce the view that a letter can only take one value in one situation, and that different letters have to have different values, and even that $a = 1$, $b = 2$ etc.

Summary

- Learners use number facts and guess-and-check rather than algebraic methods if possible.
- Doing calculations, such as in substitution and guess-and-check methods, distracts from the development of algebraic understanding.
- Substitution can be useful in exploring equivalence of expressions.
- Word problems do not, on their own, scaffold a shift to algebraic reasoning.
- Learners have to understand operations and their inverses.
- Methods of recording arithmetic can scaffold a shift to understanding operations.

What shifts have to be made between arithmetic and algebra?

Changing focus slightly, we now turn to what the learner has to see differently in order to overcome the inherent problems discussed above. A key shift which has to be made is from focusing on answers obtained in any possible way, to focusing on structure. Kieran (1989, 1992), reflecting on her long-term work with middle school students, classifies 'structure' in algebra as (1) surface structure of expression: arrangement of symbols and signs; (2) systemic: operations within an expression and their actions, order, use of brackets etc.; (3) structure of an equation: equality of expressions and equivalence.

Boero (2001) identifies transformation and anticipation as key processes in algebraic problem solving, drawing on long-term research in authentic classrooms, reconstructing learners' meanings from what they do and say. He observed two kinds of transformation, firstly the contextual arithmetical, physical and geometric transformations students do to make the problem meaningful within their current knowledge (see also Filloy, Rojano and Robio,2001); secondly, the new kinds of transformation made available by the use of algebra. If students' anticipation is locked into arithmetical activity: finding answers, calculating, proceeding step-by-step from known to unknown (see also Dettori, Garutti, and Lemut, 2001), and if their main experience of algebra is to simplify expressions, then the shift to using the new kinds of transformation afforded by algebra is hindered. Thus typical secondary school algebraic behaviour includes reaching for a formula and substituting numbers into it (Arzarello, Bazzini and Chiappini, 1994), as is often demonstrated in students' meaningless approaches to finding areas and perimeters (Dickson, 1989 *a*). Typically students will multiply every available edge length to get area, and add everything to get perimeter. These approaches might also be manifestations of learners' difficulties in understanding area (see Paper 5, Understanding space and its representation in mathematics) which cause them to rely on methods rather than meaning.

The above evidence confirms that the relationship between arithmetic and algebra is not a direct conceptual hierarchy or necessarily helpful. Claims that arithmetical understanding has to precede the teaching of algebra only make sense if the focus is on the meaning of operations and on arithmetical structures, such as inverses and fractional equivalence, rather than in correct calculation. A focus on answers and *ad hoc* methods can be a distraction unless the underlying structures of the *ad hoc* methods are generalisable and expressed structurally. Booth (1984) found that inappropriate methods were sometimes transferred from arithmetic; students often did not understanding the purpose of conventions and notations, for example not seeing a need for brackets when there are multiple operations. The possibilities of new forms of expression and transformation have to be appreciated, and the visual format of algebraic symbolism is not always obviously connected to its meanings (Wertheimer, 1960; Kirschner, 1989). For example, the meaning of index notation has to be learnt, and while y^3 can be related to its meaning in some way, $y^{1/2}$ is rather harder to interpret without understanding abstract structure.

In the U.K. context of an integrated curriculum, a non-linear view of the shift between arithmetic and algebra can be considered. Many researchers have shown that middle-school students can develop algebraic reasoning through a focus on relationships, rather than calculations[4]. Coles, Dougherty and Arcavi have already been mentioned in this respect, and Blanton and Kaput (2005) showed in an intervention-and-observation study of cohort of 20 primary teachers, in particular one self-defined as 'not a maths person' in her second year of teaching, could integrate algebraic reasoning into their teaching successfully, particularly using ICT as a medium for providing bridges between numbers and structures. Fujii and Stephens (2001, 2008) examined the role of quasi-variables (signs indicating missing values in number sentences) as a precursor to understanding generalization. Brown and Coles (1999, 2001) develop a classroom environment in a U.K. secondary school in which relationships are developed which need to be expressed structurally, and algebraic reasoning becomes a tool to make new questions and transformations possible. These studies span ages 6 to lower secondary and provide school-based evidence that the development of algebraic reasoning can happen in deliberately-designed educational contexts. In all these contexts, calculation is deliberately avoided by focusing on, quantifiable but not quantified, relationships, and using Kieran's first level of structure, surface structure, to express phenomena at her third level, equality of expressions. A study with 105 11- and 12-year-olds suggests that explaining verbally what to do in general terms is a precursor to understanding algebraic structure (Kieran's third level) (Reggiani 1994). In this section I have shown that it is possible for students to make the necessary shifts given certain circumstances, and can identify necessary experiences which can support the move.

Summary of what has to be learnt to shift from arithmetic and algebra

- Students need to focus on relations and expressions, not calculations.

- Students need to understand the meaning of operations and inverses.

- Students need to represent general relations which are manifested in situations

- In algebra letters and numbers are used together; algebra is not just letters.

- The equals sign means 'has same value as' and 'is equivalent to' – not 'calculate'.

- Arithmetic can be seen as instances of general relationships between quantities.

- Division is a tool for constructing a rational expression.

- The value of a number is less important than its relation to other numbers in an expression.

- Guessing and checking, or using known number facts, has to be put aside for more general methods.

- A letter does not always stand for a particular unknown.

Without explicit attention to these issues, learners will use their natural and quasi-intuitive reasoning to:
- try to match their use of letters to the way they use numbers
- try to calculate expressions
- try to use '=' to mean 'calculate'
- focus on value rather than relationship
- try to give letters values, often based on alphabetical assumptions.

Understanding expressions

An expression such as $3x + 4$ is both the answer to a question, an object in itself, and also an algorithm or process for calculating a particular number. This is not a new way of thinking in mathematics that only appears with algebra: it is also true that the answer to $3 \div 5$ is 3/5, something that students are expected to understand when they learn about intensive quantities and fractions. Awareness of this kind of dual meaning has been called proceptual thinking (Gray and Tall, 1994), combining the process with its outcome in the same way as a multiple is a number in itself and also the outcome of multiplication. The notions of 'procept' and 'proceptual understanding' signify that there is a need for flexibility in how we act towards mathematical expressions.

Operational understanding

Many young students understand, at least under some circumstances, the inverse relation between addition and subtraction but it takes students longer to understand the inverse relation between multiplication and division[5]. This may be particularly difficult when the division is not symbolized by the division sign ÷ but by means of a fraction, as in 1/3. Understanding division when it is symbolically indicated as a fraction would require students to realise that a symbol such as 1/3 represents not only a quantity (e.g. the amount of pizza someone ate when the pizza was cut into three parts) but also as an operation. Kerslake (1986) has shown that older primary and younger secondary students in the United Kingdom rarely understand fractions as indicating a division. A further difficulty is that multiplication, seen as repeated addition, does not provide a ready image on which to build an understanding of the inverse operation. An array can be split up vertically or horizontally; a line of repeated quantities can only be split up into commensurate lengths. The language of division in schools is usually 'sharing' or 'shared by' rather than divide, thus triggering an assignment metaphor. This is a long way from the notion of number required in order to, for example, find y when $6y = 7$. There is evidence that students understand some properties of operations better in some contexts than in others (e.g. Nunes and Bryant, 1995).

As well as knowing about operations and their inverses, students need to know that only addition and multiplication are commutative in arithmetic, so that with subtraction and division it matters which way round the numbers go. Also in subtraction and multiplication it makes a difference if an unknown number or variable is not the number being acted on in the operation. For example, if $7 - p = 4$, then to find p the appropriate inverse operation is $7 - 4$. In other words 'subtract from n' is self-inverse. A similar issue arises with 'divide into n'.

We are unconvinced by the U.S. National Mathematics Advisory Panel's suggestion that fractions must be understood before algebra is taught (NMAP, 2008). Their argument is based on a 'top-down' curriculum view and not on research about how such ideas are learnt. The problems just described are algebraic, yet contribute to a full understanding of fractions as rational structures. There is a strong argument for seeing the mathematical structure of fractions as the unifying concept which draws together parts, wholes, divisions, ratio, scalings and multiplicative relationships, but it may only be in such situations as solving equations, algebraic fractions, and so on that students need to extend their view of division and fractions, and see these as related.

To understand algebraic notation requires an understanding that terms made up of additive, multiplicative and exponential operations, e.g. $(4a^3b - 8a)$, are variables rather than instruction to calculate, and have a structure and equivalent forms. It has been suggested that spending time relating algebraic terms to arithmetical structures can provide a bridge between arithmetic and algebra (Banerjee and Subramaniam, 2004). More research is needed, but working this way round, rather than introducing terms by reverting to substitution and calculation, seems to have potential.

Summary

- Learners tend to persist in additive methods rather than using multiplicative and exponential where appropriate.
- It is hard for students to learn the nature of multiplication and division – both as inverse of multiplication and as the structure of fractions and rational numbers.
- Students have to learn that subtraction and division are non-commutative, and that their inverses are not necessarily addition and multiplication.
- Students have to learn that algebraic terms can have equivalent forms, and are not instructions to calculate. Matching terms to structures, rather than using them to practice substitution, might be useful.

Relational reasoning

Students may make shifts between arithmetic and algebra, and between operations and relations, naturally with enough experience, but research suggests that teaching can make a difference to the timing and robustness of the shift. Carpenter and Levi (2000) have worked substantially over decades to develop an approach to early algebra based on understanding equality, making generalisations explicit, representing generalisations in various ways including symbolically, and talking about justification and proof to validate generalities. Following this work, Stephens and others have demonstrated that students can be taught to see expressions such as:

97 – 49 + 49

as structures, in Kieran's second sense of relationships among operations (see also the paper on natural numbers). In international studies, students in upper primary in Japan generally tackled these relationally, that is they did not calculate all the operations but instead combined operations and inverses, at a younger age than Australian students made this shift. Chinese students generally appeared to be able to choose between rapid computation and relational thinking as appropriate, while 14-year-old English students varied between teachers in their treatment of these tasks (Fujii and Stephens, 2001, 2008; Jacobs, Franke, Carpenter, Levi and Battey, 2007). This 'seeing' relationally seems to depend on the ability to discern details (Piaget, 1969 p. xxv) and application of an intelligent sense of structure (Wertheimer, 1960) and also to know when and how to handle specifics and when to stay with structure. The power of such approaches is illustrated in the well-known story of the young Gauss' seeing a structural way to sum an arithmetic progression. In Fujii and Stephens' work, seeing patterns based on relationships between numbers, avoiding calculation, identifying variation, having a sense of limits of variability, were all found to be predictors of an ability to reason with relationships rather than numbers.

These are fundamental algebraic shifts. Seeing algebra as 'generalised arithmetic' is not achieved by inductive reasoning from special cases, but by developing a structural perspective on number sentences.

Summary

- Learners naturally generalise, they look for patterns and habits, and familiar objects.
- Inductive reasoning from several cases is a natural way to generalise, but it is often more important to look at expressions as a whole.
- Learners can shift from 'seeing' number expressions as instructions to calculate to seeing them as relationships.
- This shift can be scaffolded by teaching which encourages students not to calculate but to identify and use relations between numbers.
- Learners who are fluent in both ways of seeing expressions, as structures or as instructions to calculate, can choose which to use.

Combining operations

Problems arise when an expression contains more than one operation, as can be seen in our paper on functional relations where young children cannot understand the notion of relations between relations, such as differences of differences. In arithmetical and algebraic expressions, some relations between relations appear as combinations of operations, and learners have to decide what has to be 'done' first and how this is indicated in the notation. Carpenter and Levi (2000), Fujii and Stephens (2001, 2008), Jacobs *et al*. (2007), draw attention to this in their work on how students read number sentences. Linchevski and Herscovics (1996) studied how 12- and13-year-olds decided on the order of operations. They found that students tended to overgeneralise the order, usually giving addition priority over subtraction; or using operations in left to right order; they can show lack of awareness of possible internal cancellations; they can see brackets as merely another way to write expressions rather than an instruction to act first, for example: 926 – 167 – 167 and 926 – (167 + 167) yielded different answers (Nickson, 2000 p. 120); they also did not understand that signs were somehow attached to the following number.

Apart from flow diagrams, a common way to teach about order in the United Kingdom is to offer 'BODMAS[6]' and its variants as a rule. However, it is unclear whether such an approach adequately addresses typical errors made by students in their use of expressions.

The following expression errors were manifested in the APU tests (Foxman *et al.,* 1985). These tests involved a cohort of 12 500 students age 11 to 15 years. There is also evidence in more recent studies (see Ryan and Williams, 2007) that these are persistent, especially the first.

- Conjoining: e.g. $a + b = ab$
- Powers are interpreted as multiplication, an error made by 20% of 15-year-olds
- Not understanding that having no coefficient means the coefficient is 1
- Adding all three values when substituting in, say, $u + gt$
- Expressing the cost of a packet of sweets where x packets cost 90p as $x/90$

The most obvious explanation of the conjoining error is that conjoining is an attempt to express and 'answer' by constructing closure, or students may just not know that letters together in this notation mean 'multiply'.

Ryan and Williams (2007) found a significant number of 14-year-olds did not know what to do with an expression; they tried to 'solve' it as if it is an equation, again possibly a desire for an 'answer'. They also treated subtraction as if it is commutative, and ignored signs associated with numbers and letters. Both APU (Foxman *et al.,* 1985) and Hart, (1981) concluded that understanding operations was a greater problem than the use of symbols to indicate them, but it is clear from Ryan and Williams' study that interpretation is also significantly problematic. The prevalence of similar errors in studies 20 years apart is evidence that these are due to students' normal sense-making of algebra, given their previous experiences with arithmetic and the inherent non-obviousness of algebraic notation.

Summary

- Understanding operations and their inverses is a greater problem than understanding the use of symbols.
- Learners tend to use their rules for reading and other false priorities when combining operations, i.e. interpreting left to right, doing addition first, using language to construct expressions, etc. They need to develop new priorities.
- New rules, such as BODMAS (which can be misused), do not effectively and quickly replace old rules which are based on familiarity, habit, and arithmetic.

Equals sign

A significant body of research reports on difficulties about the meaning of the equals sign Sfard and Linchevski (1994) find that students who can do $7x + 157 = 248$ cannot do $112 = 12x + 247$, but these questions include two issues: the position and meaning of the equals sign and that algorithmic approaches lead to the temptation to subtract smaller from larger, erroneously, in the second example. They argue that the root problem is the failure to understand the inverse relation between addition and subtraction, but this research shows how conceptual difficulties, incomplete understandings and notations can combine to make multiple difficulties. If students are taught to make changes to both sides of an equation in order to solve it (i.e. transform the equation $y – 5 = 8$ into $y – 5 + 5 = 8 + 5$) and they do not see the need to maintain equivalence between the values in the two sides of the equation, then the method that they are being taught is mysterious to them, particularly as many of the cases they are offered at first can be easily solved by arithmetical methods. Booth (1984) shows that these errors combine problems with understanding operations and inverses and problems understanding equivalence.

There are two possible ways to tackle these problems: to identify all the separate problems, treat them separately, and expect learners to apply the relevant new understandings when combinations occur; or to treat algebraic statements holistically and semantically, so that the key feature of the above examples is equality. There is no research which shows conclusively that one approach is better than the other (a statement endorsed in NMAP's review (2008)).

There is semantic and syntactic confusion about the meaning of '=' that goes beyond learning a notation (Kieran, 1981; 1992). Sometimes, in algebra, it is used to mean that the two expressions are equal in a particular instance where their values are equal; other times it is used to mean that two expressions are equivalent and one can be substituted for another in every occurrence. Strictly speaking, the latter is equivalence and might be written as ' ' but we are not arguing for this to become a new 'must do' for the curriculum as this would cut across so much contextual and historical practice. Rather, the understanding of algebraic statements must be situational, and this includes learning when to use '==' to mean 'calculate'; when to use it to mean 'equal in special cases' and when to mean 'equivalent'; and when to indicate that 'these two functions are related in this way' (Saenz-Ludlow and Walgamuth, 1998). These different meanings have implications for how the letter is seen: a quantitative placeholder in a structure; a mystery number to be found to make the equality work; or a variable which co-varies with others within relationships. Saenz-Ludlow and Walgamuth showed, over a year-long study with children, that the shift towards seeing '=' to mean 'is the same as' rather than 'find the answer' could be made within arithmetic with consistent, intentional, teaching. This was a teaching experiment with eight-year-olds in which children were asked to find missing sums and addends in addition grids. The verb 'to be' was used instead of the equals sign in this and several other tasks. Another task involved finding several binary calculations whose answer was 12, this time using '='. Word problems, including some set by the children, were also used. Children also devised their own ways to represent and symbolise equality. We do not have space here to describe more of the experiment, but at the end the children had altered their initial view that '=' was an instruction to calculate. They understood '=' as giving structural information. Fujii and Stephens' (2001) research can be interpreted to show that students do get better at using 'new' meanings of the equals sign and this may be a product of repeated experience of what Boero called the 'new transformations' made possible by algebra, combined with 'new anticipations' also made possible by algebra.

Alibali and colleagues (2007) studied 81 middle school students over three years to map their understanding of equations. They found that those who had, or developed, a sophisticated understanding of the equals sign were able to deal with equivalent equations, using equivalence to transform equations and solve for unknowns. Kieran and Saldanha (2005) used a Computer Algebra System to enable five classes of upper secondary students to explore different meanings of '=' and found that given suitable tasks they were able to understand equivalence, generating for themselves two different understandings: equivalence as meaning that expressions would give them equal values for a range of input values of the variables, and equivalence as meaning that the expressions were basically transformations of the same form. Both of these understandings contribute to meaningful manipulation from one form to another. Also focusing on equivalence, Kieran and Sfard (1999) used a graphical function approach and thus enabled students to recognise that equivalent algebraic representations of functions would generate the same graphs, and hence represent the same relationships between variables.

The potential for confusion between equality and equivalence relates to confusion between finding unknowns (such as values of variables when two non-equivalent expressions are temporarily made equal) and expressing relationships between variables. Equivalence is seen when graphs coincide; equality is seen when graphs intercept.

Summary

- Learners persist in using '=' to mean 'calculate' because this is familiar and meaningful for them.

- The equals sign has different uses within mathematics; sometimes it indicates equivalence and sometimes equality; learners have to learn these differences.

- Different uses of the equals sign carry different implications for the meaning of letters: they can stand for hidden numbers, or variables, or parameters.

- Equivalence is seen when graphs coincide, and can be understood either structurally or as generation of equal outputs for every input; equality is seen when graphs intercept.

Equations and inequality

In the CMF study (Johnson, 1989), 25 classes in 21 schools in United Kingdom were tested to find out why and how students between 8 and 13 cling to guess-and-check and number-fact methods rather than new formal methods offered by teachers. The study focused on several topics, including linear equations. The findings, dependent on large scale tests and additional interviews in four schools, are summarised here and can be seen to include several tendencies already described in other, related, algebraic contexts. That the same tendencies emerge in several algebraic contexts suggest that these are natural responses to symbolic stimuli, and hence take time to overcome.

Students tended to:
- calculate each side rather than operate on them
- not use inverse operations with understanding
- use *ad hoc* number-specific methods
- interpret a box or triangle to mean 'missing number' but could not interpret a letter for this purpose
- not relate a method to the symbolic form of a method
- be unable to explain steps of their procedures
- confuse a 'changing sides' method with a 'balance' metaphor, particularly not connecting what is said to what is done, or to what is written
- test actual numbers rather than use an algebraic method
- assume different letters had different values
- think that a letter could not have the value zero.

They also found that those who used the language 'getting rid of' were more likely to engage in superficial manipulation of symbols. They singled out 'get rid of a minus' for particular comment as it has no mathematical meaning. These findings have been replicated in United Kingdom and elsewhere, and have not been refuted as evidence of common difficulties with equations.

In the same study, students were then taught using a 'function machine' approach and this led to better understanding of what an equation is and the variable nature of x. However, this approach only makes sense when an input-output model is appropriate, i.e. not for equating two functions or for higher order functions (Vergnaud 1997). Ryan and Williams (2007) found that function machines can be used by most students age 12 to 14 to solve linear equations, but only when provided. Few students chose to introduce them as a method. Most 12-year-olds could reverse operations but not their order when 'undoing' to find unknowns in this approach. Booth (1984) and Piaget and Moreau (2001) show that students who understand inversion might not understand that, when inverting a sequence of operations, the inverse operations cannot just be carried out in any order: the order in which they are carried out influences the result. Robinson, Ninowski and Gray (2006) also showed that coordinating inversion with associativity is a greater challenge than using either inversion or associativity by themselves in problem solving. Associativity is the property that $x + (y + z)$ is equal to $(x + y) + z$, so that we can add either the first two terms, and then the last, or the last two and then the first. This property applies to multiplication also. (Incidentally, note that the automatic application of BODMAS here would be unnecessary.) Students get confused about how to 'undo' such related operations, and how to undo other paired operations which are not associative. As in all such matters, teaching which is based on meaning has different outcomes (see Brown and Coles, 1999, 2001).

Once learners understand the meaning of '=' there is a range of 'intuitive' methods they use to find unknown numbers: using known facts, counting, inverse operations, and trial substitution (Kieran, 1992). These do not generalize for situations in which the unknown appears on both sides, so formal methods are taught. Formal methods each carry potential difficulties: function machines do not extend beyond 'one-sided' equations; balance methods do not work for negative signs or for non-linear equations; change-side/change-sign tends to be misapplied rather than seen as a special kind of transformation.

Many errors when solving equations appear to come from misapplication of rules and processes rather than a flawed understanding of the equals sign. Filloy describes several 'cognitive tendencies' observed over several studies of students progressing from concrete to abstract understandings (e.g. Filloy and Sutherland, 1996). These tendencies are: to cling to concrete models; to use sign systems inappropriately; to make inappropriate generalizations; to get stuck when negatives appear; to misinterpret concrete actions. Problems with the balance metaphor could be a manifestation of the general tendency to cling to concrete models (Filloy and Rojano, 1989), and the negative sign cannot be related to concrete understandings or even to some syntactic rules which may have been learnt (Vlassis, 2002). Another problem is that when the ‘unknown’ is on both sides it can no longer be treated with simple inversion techniques as finding ‘the hidden number’; $3x = 12$ entails answering the question ‘what number must I multiply 3 by to get 12?’. But when balancing ‘$4m + 3$’ with ‘$3m + 8$’ the balance metaphor can suggest testing and calculating each side until they match, rather than solving by filling-in arithmetical facts. Vlassis devised a teaching experiment with 40 lower secondary students in two classes. The first task was a word problem which would have generated two equal expressions in one variable, and students only applied trial-and-error to this. The second task was a sequence of balance problems with diagrams provided, and all students could solve these. The final task was a sequence of similar problems expressed algebraically, two of which used negative signs. These generated a range of erroneous methods, including failure to identify when to use an inverse operation, misapplication of rules, syntactical mistakes and manipulations whose meaning was hard to identify. In subsequent exercises errors of syntax and meaning diminished, but errors with negative integers persisted. Eight months later, in a delayed interview, Vlassis' students were still using correctly the principles represented in the balance model, though not using it explicitly, but still had problems when negatives were included. In Filloy and Rojano (1989) a related tendency is described, that of students creating a personal sense of concrete action (e.g. ‘I shall move this from here to here’) and using them as if they are algebraic rules (also observed by Lima and Tall, 2008). More insight into how learners understand equations is given by English and Sharry (1996) who asked students to classify equations into similar types. Some classified them according to superficial syntactic aspects, and others to underlying algebraic structure. English and Sharry draw attention to the need for students to have experience of suitable structures in order to reason analogically and identify deeper similarities.

There is little research in students' understanding of inequality in algebra. In number, children may know about ranges of smaller, or larger, or ‘between’ numbers from their position on a numberline, and children often know that adding the same quantity to two unequal quantities maintains the inequality. There are well-known confusions about relative size of decimal numbers due to misunderstandings about the notation, but beyond the scope of this review (Hart, 1981). Research by Tsamir and others describe common problems which appear to relate to a tendency to act procedurally with unequal algebraic expressions without maintaining an understanding of the inequality (Linchevski and Sfard, 1991; Tsamir and Bazzini, 2001; Tsamir and Almog, 2001). One of these studies compares the performance of 170 Italian students to that of 148 Israeli students in higher secondary school (Tsamir and Bazzini, 2001). In both countries students had been formally taught about a range of inequalities. They were asked whether statement about the set $S =\{ x \in R: x = 3\}$ could be true or not: ‘S can be the solution set of an equality and an inequality’. Only half the students understood that it could be the solution set of an inequality, and those few Italians who gave examples chose a quadratic inequality that they already knew about. Some students offered a linear inequality that could be solved to include 3 in the answer. The researchers concluded that unless an inequality question was answerable using procedural algebra it was too hard for them. Another task asked if particular solution sets satisfied $5x^4 \leq 0$. Only half were able to say that $x = 0$, the next most popular answer being $x \leq 0$. The researchers compared students' responses to both tasks. It seems that the image of ‘imbalance’ often used with algebraic inequalities is abandoned when manipulation is done. The ‘imbalance’ image does not extend to quadratic inequalities, for which a graphical image works better, but again a procedural approach is preferred by many students who then misapply it.

Summary

- Once students understand the equals sign, they are likely to use intuitive number-rules as a first resort.

- The appearance of the negative sign creates need for a major shift to abstract meanings of operations and relations, as concrete models no longer operate.

- The appearance of the unknown on both sides of an equation creates the need for a major shift towards understanding equality and variables.

- Students appear to use procedural manipulations when solving equations and inequalities without a mental image or understanding strong enough to prevent errors.

- Students appear to develop action-based rules when faced with situations which do not have obvious concrete manifestations.

- Students find it very hard to detach themselves from concrete models, images and instructions and focus on structure in equations.

Manipulatives

It is not only arithmetical habits that can cause obstacles to algebra. There are other algebraic activities in which too strong a memory for process might create obstacles for future learning. For example, a popular approach to teaching algebra is the provision of materials and diagrams which ascribe unknown numerical (dimensional) meaning to letters while facilitating their manipulation to model relationships such as commutativity and distributivity. These appear to have some success in the short term, but shifts from physical appearance to mental abstraction, and then to symbolism, are not made automatically by learners (Boulton-Lewis, Cooper, Atweh, Pillay, Wilss and Mutch, 1997). These manipulatives provide persistent images and metaphors that may be obstructions in future work. On the other hand, the original approach to dealing with variables was to represent them as spatial dimensions, so there are strong historical precedents for such methods. There are reported instances of success in teaching this, relating to Bruner's three perspectives, enactive-iconic-symbolic (1966), where detachment from the model has been understood and scaffolded by teaching (Filloy and Sutherland, 1996; Simmt and Kieren, 1999). Detachment from the model has to be made when values are negative and can no longer be represented concretely, and also with fractional values and division operations. Spatial representations have been used with success where the image is used persistently in a range of algebraic contexts, such as expressions and equations and equivalence, and where teachers use language to scaffold shifts between concrete, numerical and relational perspectives.

Use of rod or bar diagrams as in Singapore (NMAP, 2008; Greenes and Rubenstein, 2007) to represent part/whole comparisons, reasoning, and equations, appears to scaffold thinking from actual numbers to structural relationships, so long as they only involve addition and/or repeated addition. Statements in the problem are translated into equalities between lengths. These equal lengths are constructed from rods which represent both the actual and the unknown numbers. The rod arrangements or values can then be manipulated to find the value of the unknown pieces. Equations with the variable on both sides are taught to 11 and 12 year-olds in Singapore using such an approach. The introduction of such methods into classrooms where teachers are not experienced in its use has not been researched. It has some similarities to the approach based on Cuisenaire rods championed by Gattegno in Europe. Whereas use for numbers was widespread in U.K. primary schools, use for algebra was not, possibly because the curriculum focus on substitution and simplification, rather than meaning and equivalence, provided an obstacle to sustained use.

Summary

- Manipulatives can be useful for modelling algebraic relationships and structures.
- Learners might see manipulatives as 'just something else to learn'.
- Teachers can help learners connect the use of objects, the development of imagery and the use of symbols through language.
- Students have to appreciate the limitations of concrete materials and shift to mental imagery and abstract understandings.

Application of formulae within mathematics

Dickson's study with three classes of ten-year-olds (1989 *a*) into students' use of formulae and formal methods is based on using the formula for area of rectangle in various contexts. In order to be successful in such tasks, students have to understand what multiplication is and how it relates to area, e.g. through an array model, how to use the formula by substitution and how the measuring units for area are applied. Some students can then work out a formula for themselves without formal teaching. From this study, Dickson (1989*a*, 1989*b*) and her colleagues found several problems in how students approach formal methods in early secondary school. A third of her subjects did not use a formal method at all; a third used it in a test but could not explain it in interviews; a third used it and explained it. She found that they:

- may not have underlying knowledge on which to base formalisation (note that formalisation can happen spontaneously when they do have such

knowledge)
- base their reasoning on incorrect method
- have a sound strategy that may not match formal method
- may be taught methods leading up to formal, but not matching the formal method
- may retain other methods, which may have limited application
- may retain formalisation but lose meaning, then misapply a formal method in future
- pre-formal enactive or iconic experiences may have been forgotten
- might be able to use materials to explain formal method
- may interpret formal notation inadequately.

The research described above, taken as a whole, suggests that the problems students have with using formulae in subjects other than mathematics are due to: not being fluent with the notation; not understanding the underlying operations; experience of using such formulae in mathematics lessons being limited to abstract or confusing situations, or even to situations in which an algebraic formula is not necessary. In addition, of course, they may not understand the intended context.

Summary

- Learners are able to construct formulae for themselves, at least in words if not symbols, if they have sufficient understanding of the relationships and operations.

- Learners' problems using formulae have several possible root causes.
  1. Underlying knowledge of the situation or associated concepts may be weak.
  2. Existing working strategies may not match the formal method.
  3. Notational problems with understanding how to interpret and use the formula.

Part 2: problems arising in different approaches to developing algebraic reasoning

Since the CSMS study (Hart, 1981) there has been an expansion of teaching approaches to develop meaningful algebra as:
- expressing generalities which the child already knows, therefore is expressing something that has meaning, and comparing equivalent expressions
- describing relationships between expressions as equations, which can then be solved to find unknown values (as in word problems)
- a collection of techniques for transforming equations to either find unknown values or represent relationships between variables in different ways
- expressing functions and their inverses, in which inputs become outputs according to a sequence of operations; using multiple representations
- modelling situations by identifying variables and how they co-vary.

Each of these offers more success in some aspects than an approach based on rules for manipulating expressions, but also highlights further obstacles to reasoning. Research is patchy, and does not examine how students learn across contexts and materials (Rothwell-Hughes, 1979). Indeed, much of the research is specifically about learning in particular contexts and materials.

Expressing generalisations from patterns

One approach to address inherent difficulties in algebra is to draw on our natural propensity to observe patterns, and to impose patterns on disparate experiences (Reed 1972). In this approach, sequences of patterns are presented and students asked to deduce formulae to describe quantitative aspects of a general term in the sequence. The expectation is that this generates a need for algebraic symbolisation, which is then used to state what the student can already express in other ways, numerical, recursive, diagrammatically or enactively.

This approach is prevalent in the United Kingdom, Australia and parts of North America. The NMAP (2008) review finds no evidence that expressing generality contributes to algebraic understanding, yet others would say that this depends on the definition of algebraic understanding. Those we offered at the start of this chapter include expression of generality as an indication of understanding. In Australia, there are contradictory findings about the value of such tasks.

The following is an example of one of the items which was used in the large scale test administered to students by MacGregor and Stacey (reported in Mason and Sutherland, 2002).

Look at the numbers in this table and answer the questions:

x	y
1	5
2	6
3	7
4	8
5	9
6	..
7	11
8	...
....	...
...	

(i) When x is 2, what is y?
(ii) When x is 8, what is y?
(iii) When x is 800, what is y?
(iv) Describe in words how you would find y if you were told that x is ....
(v) Use algebra to write a rule connecting x and y ....

MacGregor and Stacey found performance on these items varied from school to school. The success of 14-year-old students in writing an algebraic rule ranged from 18% in one school to 73% in another. In general students searched for a term-to-term rule (e.g. Stacey, 1989). They also tested the same students with more traditional items involving substitution to show the meaning of notation and transformation, to show equivalence and finding unknowns. From this study they concluded that students taught with a pattern-based approach to algebra did no better and no worse on traditional algebra items than students taught with a more traditional approach (MacGregor and Stacey, 1993, 1995).

Redden (1994) studied the work of 1400 10- to 13-year-olds to identify the stages through which students must pass in such tasks. First they must recognise the number pattern (which might be multiplicative), then there must be a stimulus to expression, such as being asked for the next term and then the value of uncountable term; they must then express the general rule and use symbols to express it. Some students could only process one piece of data, some could process more pieces of data, some gave only a specific example, some gave the term-to-term formula and a few gave a full functional formula. A major shift of perception has to take place to express a functional formula and this is more to do with 'seeing' the functional relationship, a shift of perception, than symbolising it. Rowland and Bills (1996) describe two kinds of generalisation: empirical and structural, the first being more prevalent than the second. Amit and Neria (2007) use a similar distinction and found that students who had followed a pattern-generalisation curriculum were able to switch representations meaningfully, distinguish between variables, constants and their relationships, and shift voluntarily from additive to multiplicative reasoning when appropriate.

Moss, Beatty and Macnab (2006) worked with nine-year-old students in a longitudinal study and found that developing expressions for pattern sequences was an effective introduction to understanding the nature of rules in 'guess the rule' problems. Nearly all of the 34 students were then able to articulate general descriptions of functions in the classic handshake problem[7] which is known to be hard for students in early secondary years. By contrast, Ryan and Williams (2007) found in large-scale testing that the most prevalent error in such tasks for 12- and 14-year-olds was giving the term-to-term formula rather than the functional formula, and giving an actual value for the nth term. Cooper and Warren (2007, and Warren and Cooper, 2008), worked for three years in five elementary classrooms, using patterning and expressing patterns, to teach students to express generalisations to use various representations, and to compare expressions and structures. Their students learnt to use algebraic conventions and notations, and also understood that expressions had underlying operational meanings. Clearly, students are capable of learning these aspects of algebra in certain pedagogic conditions. Among other aspects common to most such studies, Cooper and Warren's showed the value of comparing different but equivalent expressions that arise from different ways to generalise the patterns, and also introduced inverse operations in the context of function machines, and a range of mental arithmetic methods. If other research about generalising patterns applies in this study, then it must be the combination of pattern-growth with these other aspects of algebra that made the difference in the learning of their students. They point to 'the importance of understanding and communicating aspects of representational forms which allowed commonalities to be seen across or between representations'.

As Carraher, Martinez & Schliemann (2007) show, it is important to nurture the transition from empirical (term-to-term) generalizations (called naïve induction by Radford, 2007), to generalisations that follow from explicit statements about mathematical relations between independent and dependent variables, and which might not be 'seen' in the data. Steele (2007) indicates some of the ways in which a few successful 12 to 13 year old students go about this transition when using various forms of data, pictorial, diagrammatic and numerical, but bigger studies show that this shift is not automatic and benefits from deliberate tuition. Radford further points out that once a functional relation is observed, expressing it is a further process involving integration of signs and meaning. Stephens' work (see Mason, Stephens and Watson, in press) shows that the opportunity and ability to exemplify relationships between variables as number pairs, and to express the relationship within the pairs, are necessary predictors of the ability to focus on and express a functional relationship. This research also illustrates that such abilities are developmental, and hence indicates the kind of learning experiences required to make this difficult shift.

Rivera and Becker (2007), looking longitudinally at middle school students' understanding of sequences of growing diagrammatic patterns in a teaching experiment, specify three forms of generalization that students engage with: constructive standard, constructive nonstandard, and deconstructive. It is the deconstruction of diagrams and situations that leads most easily to the functional formula, they found, rather than reasoning inductively from numbers. However, their students generally reverted to arithmetical strategies, as reported in many other studies of this and other shifts towards algebra.

Reed (1972) hypothesised that classifying is a natural act that enables us to make distinctions, clump ideas, and hence deal with large amounts of new information. It is therefore useful to think of what sort of information learners are trying to classify in these kinds of task. Reed found that people extract prototypes from the available data and then see how far other cases are from this prototype. Applying this to pattern-growth and sequence tasks makes it obvious that term-to-term descriptions are far easier and likely to be dominant when the data is expressed sequentially, such as in a table. We could legitimately ask the question: is it worth doing these kinds of activity if the shifts to seeing and then expressing functional relationships are so hard to make? Does this just add more difficulties to an already difficult subject? To answer this, we looked at some studies in which claims are made of improvements in seeing and expressing algebraic relationships, and identifying features of pedagogy or innovation which may have influenced these improvements. Yeap and Kaur (2007) in Singapore found a wider range of factors influencing success in unfamiliar generalisation tasks than has been reported in studies which focus on rehearsed procedures. In a class of 38 ten-year-old students they set tasks, then observed and interviewed students about the way they had worked on them. Their aim was to learn more about the strategies students had used and how these contributed towards success. The task was to find the sums of consecutive odd numbers: $1 + 3 + 5 + \ldots + (2n - 1)$. Students were familiar with adding integers from 1 to 100, and also with summing multiples. They were given a sequence of subtasks: a table of values to complete, to find the sum or $1 + 3 + \ldots + 99$ and to find the sum of $51 + 53 + \ldots + 99$. The researchers helped students by offering simpler versions of the same kinds of summation if necessary. Nearly all students were able to recognize and continue the pattern of sums (they turn out to be the square numbers); two-thirds were able to transfer their sense of structure to the 'sum to 99' task, but only one-third completed the 'sum from 51' task – the one most dissimilar to the table-filling tasks, requiring adaptation of methods and use of previous knowledge to make an argument. The researcher had a series of designed prompts to help them, such as to find the sum from 1 to 49, and then see what else they needed to get the sum to 99. Having found an answer, students then had to find it again using a different method. They found that success depended on:
• the ability to see structures and relationships
• prior knowledge
• metacognitive strategies

- critical-thinking strategies
- the use of organizing heuristics such as a table
- the use of simplifying heuristics such as trying out simpler cases
- task familiarity
- use of technology to do the arithmetic so that large numbers can be handled efficiently.

As with all mathematics teaching, limited experience is unhelpful. Some students only know one way to construct cases, one way to accomplish generalisation (table of values and pattern spotting), and have only ever seen simple cases used to start sequence generation, rather than deliberate choices to aid observations. Students in this situation may be unaware of the necessity for critical, reflective thinking and the value of simplifying and organising data. Furthermore, this collection of studies on expressing generality shows that construction, design, choice and comparison of various representational means does not happen spontaneously for students who are capable of using them. Choosing when and why to switch representations has long been known to be a mark of successful mathematics students (Krutetskii, 1976) and therefore this is a strategy which needs to be deliberately taught. Evidence from Blanton and Kaput's intervention study with 20 teachers (2005) is that many primary children were able to invent and solve 'missing number' sentences using letters as placeholders, symbolize quantities in patterns, devise and use graphical representations for single variables, and some could write simple relations using letters, codes, 'secret messages' or symbols. The intervention was supportive professional development which helped teachers understand what algebraic reasoning entails, and gave them resources, feedback, and other support over five years. Ainley (1996) showed that supportive technology can display the purpose of formal representations, and also remove the technical difficulties of producing new representations. Ten-year-old students in her study had worked for a few years in a computer-rich environment and used spreadsheets to collect data from purposeful experiments. They then generated graphs from the data and studied these, in relation to the data, to make conjectures and test them. One task was designed to lead to a problematic situation so that students would have to look for a shortcut, and she observed that the need to 'teach the computer' how to perform a calculation led to spontaneous formal representation of a variable.

So, if it is possible for students to learn to make these generalizations only with a great deal of pedagogic skill and technical know-how, why should it be pursued? The reason is that skill in the meaning and use of algebra enables further generalizations to be made, and transformations of mathematical relationships to be used and studied. The work required to understand the functional relationship is necessary to operate at a higher level than merely using algebra to symbolize what you do, as with term-to-term formulae. It is algebra that provides the means to building concepts upon concepts, a key aspect of secondary mathematics, by providing expression of abstract relationships in ways that can be manipulated. In algebra, the products are not answers, but structures, relationships, and information about relationships and special instances of them. These tasks provide contexts for that kind of shift, but do not guarantee that it will take place.

Assumptions, such as that which appears to be made in Redden's study, that understanding term-to-term relationships is a route to understanding functional relations contradict the experience of mathematicians that algebra expresses the structure of relations, and this can be adduced from single cases which are generic enough to illustrate the relationship through diagrams or other spatial representations. Numerical data has to be backed up with further information about relationships. For example, consider this data set:

x	y
1	1
2	4
3	9
...	...

While it is possible for these values to be examples of the function $y = x^2$ it is also possible that they exemplify $y = x^2 + (x - 1)(x - 2)(x - 3)$. Without further information, such as x being the side of a square and y being its area, we cannot deduce a functional formula, and inductive reasoning is misapplied. There is much that is mathematically interesting in the connection between term-to-term and functional formulae, such as application of the method of differences, and students have to learn how to conjecture about algebraic relationships, but to only approach generalisation from a sequence perspective is misleading and, as we have seen from these studies, very hard without the support of specially-designed tasks comparing and transforming equivalent structural generalisations.

Summary

- Learners naturally make generalisations based on what is most obviously related; this depends on the visual impact of symbols and diagrams.
- Seeing functional, abstract, relationships is hard and has to be supported by teaching.
- Deconstruction of diagrams, relationships, situations is more helpful in identifying functional relationships than pattern-generation.
- Development of heuristics to support seeing structural relationships is helpful.
- There is a further shift from seeing to expressing functional relationships.
- Learners who can express relationships correctly and algebraically can also exemplify relationships with number pairs, and express the relationships within the pairs; but not all those who can express relationships within number pairs can express the relationship algebraically.
- Learners who have combined pattern-generalisation with function machines and other ways to see relationships can become more fluent in expressing generalities in unfamiliar situations.
- Conflicting research results suggest that the nature of tasks and pedagogy make a difference to success.
- Functional relationships cannot be deduced from sequences without further information about structure.

Using an equation-centred approach to teaching algebra

There are new kinds of problem that arise in an equation-centred approach to teaching algebra in addition to those described earlier: the solution of equations to find unknown values, and the construction of equations from situations. The second of these new problems is considered in Paper 7. Here we look at difficulties that students had in teaching studies designed to focus on typical problems in finding unknowns.

Students in one class of Booth's (1984) intervention study (which took place with four classes in lower secondary school) had a teacher who emphasised throughout that letters had numerical value. These students were less likely than others to treat a letter as merely an object. In her study, discussion about the meaning of statements before formal activity seemed to be beneficial, and those students who were taught a formal method seemed to understand it better some time after the lesson, maybe after repeated experiences. However, some students did not understand it at all. As with all intervention studies, the teaching makes a difference. Linchevski and Herscovics (1996) taught six students to collect like terms and then decompose additive terms in order to focus on 'sides' or equations as expressions which needed to be equated. While this led to them being better able to deal with equations, there were lingering problems with retaining the sign preceding the letter rather than attaching the succeeding sign.

Several other intervention studies (e.g. van Ameron, 2003; Falle, 2005) confirm that the type of equation and the nature of its coefficients often make non-formal methods available to learners, even if they have had significant recent teaching in formal methods. These studies further demonstrate that students will use *ad hoc* methods if they seem more appropriate, given that they understand the meaning of an equation; where they did not understand they often misapplied formal methods. Falle's study included more evidence that the structure $a/x = b$ caused particular problems as learners interpreted 'division' as if it were commutative. As with other approaches to teaching algebra, using equations as the central focus is not trouble-free.

Summary

- As with all algebraic expressions, learners may react to the visual appearance without thinking about the meaning.
- Learners need to know what the equation is telling them.
- Learners need to know why an algebraic method is necessary; this is usually demonstrated when the unknown is negative, or fractional, and/or when the unknown is on both sides. They are likely to choose *ad hoc* arithmetic methods such as guess- and-check, use of known number facts, compensation or trial-and-adjustment if these are more convenient.

- Learners' informal methods of making the sides equal in value may not match formal methods.
- 'Undoing' methods depend on using inverse operations with understanding.
- Fluent technique may be unconnected to explaining the steps of their procedures.
- Learners can confuse the metaphors offered to 'model' solving equations, e.g. 'changing sides' with 'balance'.
- Metaphors in common use do not extend to negative coefficients or 'unknowns' or non-linear equations.
- Non-commutative and associative structures are not easily used with inverse reasoning.
- As in many other contexts, division and rational structures are problematic.

Spreadsheets

Learners have to know how to recognise structures (based on understanding arithmetical operations and what they do), express structures in symbols, and calculate particular cases (to stimulate inductive understanding of concepts) in order to use algebra effectively in other subjects and in higher mathematics. Several researchers have used spreadsheets as a medium in which to explore what students might be able to learn (e.g. Schwartz and Yerushalmy, 1992; Sutherland and Rojano, 1993; Friedlander and Tabach, 2001). The advantages of using spreadsheets are as follows.

- In order to use spreadsheets you have to know the difference between parameters (letters and numbers that structure the relationship) and variables, and the spreadsheet environment is low-risk since mistakes are private and can easily be corrected.
- The physical act of pointing the cursor provides an enactive aspect to building abstract structures.
- Graphical, tabular and symbolic representations are just a click away from each other and are updated together.
- Correspondences that are not easy to see in other media can be aligned and compared on a spreadsheet, e.g. sequences can be laid side by side, input and output values for different functions can be compared, and graphs can be related directly to numerical data.
- Large data sets can be used so that questions about patterns and generalities become more meaningful.

In Sutherland and Rojano's work, two small groups of students 10- to 11-years-old with no formal algebraic background were given some algebraic spreadsheet tasks based on area. It was found that they were less likely to use arithmetical approaches when stuck than students reported in non-spreadsheet research, possibly because these arithmetical approaches are not easily available in a spreadsheet environment. Sutherland and Rojano used three foci known to be difficult for students: the relation between functions and inverse functions, the development of equivalent expressions and word problems. The arithmetic methods used included whole/part approaches and trying to work from known to unknown. Most of the problems, however, required working from the unknown to the known to build up relationships. In a similar follow-up study 15-year-old students progressively modified the values of the unknowns until the given totals were reached (Sutherland and Rojano, 1993). There was some improvement in post-tests over pre-tests for the younger students, but most still found the tasks difficult. One of the four intervention sessions involved students constructing equivalent spreadsheet expressions. Some students started by constructing expressions that generated equality in specific cases, rather than overall equivalence. Students who had started out by using particular arithmetical approaches spontaneously derived algebraic expressions in the pencil-and-paper tasks of the post-test. This appears to confound evidence from other studies that an arithmetical approach leads to obstacles to algebraic generalization. The generation of numbers, which can be compared to the desired outputs, and adjusted through adapting the spreadsheet formula, may have made the need for a formula more obvious. The researchers concluded that comparing expressions which referred only to numbers, to those which referred to variables, appeared to have enabled students to make this critical shift.

A recent area of research is in the use of computer algebra systems (CAS) to develop algebraic reasoning. Kieran and Saldanha (2005) have had some success with getting students to deal with equations as whole meaningful objects within CAS.

Summary
Use of spreadsheets to build formulae:
- allows large data sets to be used
- provides physical enactment of formula construction
- allows learners to distinguish between variables and parameters
- gives instant feedback
- does not always lock learners into arithmetical and empirical viewpoints.

Functional approach

Authors vary in their use of the word 'function'. Technically, a function is a relationship of dependency between variables, the independent variables (input) which vary by some external means, and the dependent variables (output) which vary in accordance with the relationship. It is the relationship that is the function, not a particular representation of it, however in practice authors and teachers refer to 'the function' when indicating a graph or equation. An equivalence such as temperature conversion is not a function, because these are just different ways to express the same thing, e.g. $t = 9/5\ C + 32$ where t is temperature in degrees Fahrenheit and C temperature in degrees Celsius (Janvier 1996). Thus a teaching approach which focuses on comparing different expressions of the same generality is concerned with structure and would afford manipulation, while an approach which focuses on functions, such as using function machines or multiple representations, is concerned with relationships and change and would afford thinking about pairs of values, critical inputs and outputs, and rates of change.

Function machines
Some researchers report that students find it hard to use inverses in the right order when solving equations. However, in Booth's work (1984) with function machines she found that lower secondary students were capable of instructing the 'machine' by writing operations in order, using proper algebraic syntax where necessary, and could make the shift to understanding the whole expression. They could then reverse the flow diagram, maintaining order, to 'undo' the function.

We have discussed the use of function machines to solve equations above.

Multiple representations
A widespread attempt to overcome the obstacles of learning algebra has been to offer learners multiple representations of functions because:
- different representations express different aspects more clearly
- different representations constrain interpretations – these have to be checked out against each other
- relating representations involves identifying and understanding isomorphic structures (Goldin 2002).

By and large these methods offer graphs, equations, and tabular data and maybe a physical situation or diagram from which the data has been generated. The fundamental idea is that when the main focus is on meaningful functions, rather than mechanical manipulations, learners make sensible use of representations (Booth, 1984; Yerushalmy, 1997; Ainley, Nardi and Pratt, 1999; Hollar and Norwood, 1999).

A central issue is that in most contexts for a letter to represent anything, the student must understand what is being represented, yet it is often only by the use of a letter that what is being represented can be understood. This is an essential shift of abstraction. It may be that seeing the use of letters alongside other representations can help develop meaning, especially through isomorphisms.

This line of thought leads to a substantial body of work using multiple representations to develop understandings of functions, equations, graphs and tabular data. All these studies are teaching experiments with a range of students from upper primary to first year undergraduates. What we learn from them is a range of possibilities for learning and new problems to be overcome. Powell and Maher (2003) have suggested that students can themselves discover isomorphisms. Others have found that learners can recognise similar structures (English and Sharry, 1996) but need experience or prompts in order to go beyond surface features. This is because surface features contribute to the first impact of any situation, whether they are visual, aural, the way the situation is first 'read', or the first recognition of similarity.

Hitt (1998) claims that 'A central goal of mathematics teaching is taken to be that the students be able to pass from one representation type to another without falling into contradictions.' (p. 134). In experiments with teachers on a course he asked them to match pictures of vessels with graphs to represent the relationship between the volume and height of liquid being poured into them. The most common errors in the choice of functions were due to misinterpretation of the graphical representation, and misidentification of the independent variable in the situation. Understanding the representation, in addition to understanding the situation, was essential. The choice of representation, in addition to understanding, is also influential in success. Arzarello, Bazzini and Chiappini, (1994) gave 137 advanced mathematics students this problem: 'Show that if you add a 4-digit number to the 4-digit number you get if you reverse the digits, the answer is a multiple of eleven'. There were three strategies used by successful students, and the most-used was to devise a way to express a 4-digit number as the sum of multiples of powers of ten. This strategy leads immediately to seeing that the terms in the sum combine to show multiples of eleven. The relationship between the representation and its meaning in terms of 'eleven' was very close. 'Talk' can structure a choice of representations that most closely resemble the mathematical meaning (see also Siegler and Stern, 1998).

Even (1998) points to the ability to select, use, move between and compare representations as a crucial mathematical skill. She studied 162 early students in 8 universities (the findings are informative for secondary teaching) and found a difference between those who could only use individual data points and those who could adopt a global, functional approach. Nemirovsky (1996) demonstrates that the Cartesian relationship between graphs and values is much easier to understand pointwise, from points to line perhaps via a table of values, than holistically, every point on a line representing a particular relationship.

Some studies such as Computer-Intensive Algebra (e.g. Heid, 1996) and CARAPACE (Kieran, Boileau and Garancon,1996) go some way towards understanding how learners might see the duality of graphs and values. In a study of 14 students aged about 13, the CARAPACE environment (of graphs, data, situations and functions) seemed to support the understanding of equality and equivalence of two functions. This led to findings of a significant improvement in dealing with 'unknown on both sides' equations over groups taught more conventionally. The multiple-representation ICT environment led to better performance in word problems and applications of functions, but students needed additional teaching to become as fluent in algebra as 'conventional' students. But teaching to fluency took only six weeks compared to one year for others. This result seems to confirm that if algebra is seen to have purpose and meaning then the technical aspects are easier to learn, either because there is motivation, or because the learner has already developed meanings for algebraic expressions, or because they have begun to develop appropriate schema for symbol use. When students first had to express functions, and only then had to answer questions about particular values, they had fewer problems using symbols.

There were further benefits in the CARAPACE study: they found that their students could switch from variable to unknown correctly more easily than has been found in other studies; the students saw a single-value as special case of a function, but their justifications tended to relate to tabular data and were often numerical, not relating to the overall function or the context. The students had to consciously reach for algebraic methods, even to use their own algorithms, when the situations became harder. Even in a multi-representational environment, using functions algebraically has to be taught; this is not spontaneous as long as numerical or graphical data is available. Students preferred to move between numerical and graphical data, not symbolic representations (Brenner, Mayer, Moseley, Brar, Duran, Reed and Webb,1997). This finding must depend on task and pedagogy, because by contrast Lehrer, Strom and Confrey,(2002) give examples where coordinating quantitative and spatial representations appears to develop algebraic reasoning through representational competence. Even (1998) argues that the flexibility and ease with which we hope students will move from representation to representation depends on what general strategy students bring to mathematical situations, contextual factors and previous experience and knowledge. We will look further at this in the next paper.

Further doubts about a multiple representation approach are raised by Amit and Fried (2005) in lessons on linear equations with 13 – 14 year-olds: 'students in this class did not seem to get the idea that representations are to be selected, applied, and translated'. The detail of this is elaborated through the failed attempts of one student, who did make this link, to persuade her peers about it. Hirschhorn (1993) reports on a longitudinal comparative study at three sites in which those taught using multiple representations and meaningful contexts did significantly better in tests than others taught more conventionally, but that there was no difference in attitude to mathematics. All we really learn from this is that the confluence of opportunity, task and explanation are not sufficient for learning. Overall the research suggests that there are some gains in understanding functions as meaningful expressions of variation, but that symbolic representation is still hard and the least preferred choice.

The effects of multiple representational environments on students' problem-solving and modeling capabilities are described in the next paper.

Summary

- Learners can compare representations of a relationship in graphical, numerical, symbolic and data form.
- Conflicting research results suggest that the nature of tasks and pedagogy make a difference to success.
- The hardest of these representations for learners is the symbolic form.
- Previous experience of comparing multiple representations, and the situation being modelled, helps students understand symbolic forms.
- Learners who see 'unknowns' as special cases of equality of two expressions are able to distinguish between unknowns and variables.
- Teachers can scaffold the shifts between representations, and perceptions beyond surface features, through language.
- Some researchers claim that learners have to understand the nature of the representations in order to use them to understand functions, while others claim that if learners understand the situations, then they will understand the representations and how to use them.

What students could do if taught, but are not usually taught

Most research on algebra in secondary school is of an innovative kind, in which particular tasks or teaching approaches reveal that learners of a particular age are, in these circumstances, able to display algebraic behaviour of particular kinds. Usually these experiments contradict curriculum expectations of age, or order, or nature, of learning. For example, in a teaching experiment over several weeks with 8-year-old students, Carraher, Brizuela and Schliemann (2000) report that young learners are able to engage with problems of an algebraic nature, such as expressing and finding the unknown heights in problems such as: Tom is 4 inches taller than Maria, Maria is 6 inches shorter than Leslie; draw their heights. They found that young learners could learn to express unknown heights with letters in expressions, but were sometimes puzzled by the need to use a letter for 'any number' when they had been given a particular instance. This is a real source for confusion, since Maria can only have one height. Students can naturally generalise about operations and methods using words, diagrams and actions when given suitable support (Bastable and Schifter, 2008). They can also see operators as objects (Resnick, Lesgold and Bill, 1990). These and other studies appear to indicate that algebraic thinking can develop in primary school.

In secondary school, students can work with a wider range of examples and a greater degree of complexity using ICT and graphical approaches than when confined to paper and pencil. For example, Kieran and Sfard (1999) used graphs successfully to help 12- and 13-yearold students to appreciate the equivalence of expressions. In another example, Noss, Healy and Hoyles (1997) constructed a matchsticks microworld which requires students to build up LOGO procedures for drawing matchstick sequences. They report on how the software supported some 12- to 13-year-old students in finding alternative ways to express patterns and structures of Kieran's second and third kinds. Microworlds provide support for students' shifts from particular cases to what *has* to be true, and hence support moves towards using algebra as a reasoning tool.

In a teaching study with 11 year-old students, Noble, Nemirovsky, Wright and Tierney, (2001) suggest that students can recognise core mathematical structures by connecting all representations to personally-constructed environments of their own, relevant for the task at hand. They asked pairs of students to proceed along a linear measure, using steps of different sizes, but the same number of steps each, and record where they got to after each step. They used this data to predict where one would be after the other had taken so many steps. The aim was to compare rates of change. Two further tasks, one a number table and the other a software-supported race, were given and it was noticed from the ways in which the students talked that they were bringing to each new task the language, metaphors and competitive sense which had been generated in the previous tasks. This enabled them to progress from the measuring task to comparing rates in multiple contexts and representations. This still supports the fact that students recognize similarities and look for analogical prototypes within a task, but questions whether this is related to what the teacher expects in any obvious way. In a three-year study with 16 lower secondary students, Lamon (1998) found that a year's teaching which focused on modelling sequential situations was so effective in helping students understand how to express relationships that they could distinguish between unknowns, variables and parameters and could also choose to use algebra when appropriate – normally these aspects were not expected at this stage, but two years further on.

Lee (1996) describes a long series of teaching experiments: 50 out of 200 first year university students committed themselves to an extra study group to develop their algebraic awareness. This study has implications for secondary students, as their algebraic knowledge was until then rule-based and procedural. She forced them, from the start, to treat letters as variables, rather than as hidden numbers. By many measures this group succeeded in comparative tests, and there was also evidence of success beyond testing, improvements in attitude and enjoyment. However, the impact of commitment to extra study and 'belonging' to a special group might also have played a part. Whatever the causal factors, this study shows that the notion of variable can be taught to those who have previously failed to understand, and can form a basis for meaningful algebra.

Summary

With teaching:

- Young children can engage with missing number problems, use of letters to represent unknown numbers, and use of letters to represent generalities that they have already understood.

- Young children can appreciate operations as objects, and their inverses.

- Students can shift towards looking at relationships if encouraged and scaffolded to make the shift, through language or microworlds, for example.

- Students can shift from seeing letters as unknowns to using them as variables.

- Students will develop similarities and prototypes to make sense of their experience and support future action.

- Students can shift from seeing cases as particular to seeing algebraic representations as statement about what has to be true.

- Comparison of cases and representations can support learning about functions and learning how to use algebra to support reasoning.

Part 3: Conclusions and recommendations

Conclusions

Error research about elementary algebra and pre-algebra is uncontentious and the findings are summarised above. However, it is possible for young learners to do more than is normally expected in the curriculum, e.g. they will accept the use of letters to express generalities and relationships which they already understand. Research about secondary algebra is less coherent and more patchy, but broadly can be summarised as follows.

Teaching algebra by offering situations in which symbolic expressions make mathematical sense, and what learners have to find is mathematically meaningful (e.g. through multiple representations, expressing generality, and equating functions) is more effective in leading to algebraic thinking and skill use than the teaching of technical manipulation and solution methods as isolated skills. However, these methods need to be combined through complex pedagogy and do not in themselves bring about all the necessary learning. Technology can play a big part in this. There is a difference between using ICT in the learner's control and using ICT in the teacher's control. In the learner's control the physical actions of moving around the screen and choosing between representations can be easily connected to the effects of such moves, and feedback is personalised and instant.

There is a tenuous relationship between what it means to understand and use the affordances of algebra, as described in the previous paragraph, and understanding and using the symbolic forms of algebra. Fluency in understanding symbolic expressions seems to develop through use, and also contributes to effective use – this is a two-way process. However, this statement ignores the messages from research which is purely about procedural fluency, and which supports repetitive practice of procedures in carefully constructed varying forms. Procedural research focuses on obstacles such as dealing with negative signs and fractions, multiple operations, task complexity and cognitive load but not on meaning, use, relationships, and dealing with unfamiliar situations.

Recommendations

For teaching

These recommendations require a change from a fragmented, test-driven, system that encourages an emphasis on fluent procedure followed by application.

- Algebra is the mathematical tool for working with generalities, and hence should permeate lessons so that it is used wherever mathematical meaning is expressed. Its use should be commonplace in lessons.

- Teachers and writers must know about the research about learning algebra and take it into account, particularly research about common errors in understanding algebraic symbolisation and how they arise.

- Teachers should avoid using published and web-based materials which exacerbate the difficulties by over-simplifying the transition from arithmetic to algebraic expression, mechanising algebraic transformation, and focusing on algebra as 'arithmetic with letters'.

- The curriculum, advisory schemes of work, and teaching need to take into account how shifts from arithmetical to algebraic understanding take time, multiple experiences, and clarity of purpose.

- Students at key stage 3 need support in shifting to representations of generality, understanding relationships, and expressing these in conventional forms.

- Students have to change focus from calculation, quantities, and answers to structures of operations and relations between quantities as variables. This shift takes time and multiple experiences.

- Students should have multiple experiences of constructing algebraic expressions for structural relations, so that algebra has the purpose of expressing generality.

- The role of 'guess-the-sequence-rule' tasks in the algebra curriculum should be reviewed: it is mathematically incorrect to state that a finite number of numerical terms indicates a unique underlying generator.

- Students need multiple experiences over time to understand: the role of negative numbers and the negative sign; the role of division as inverse of multiplication and as the fundamental operation associated with rational numbers; and the meaning of equating algebraic expressions.

- Teachers of key stage 3 need to understand how hard it is for students to give up their arithmetical approaches and adopt algebraic conventions.

- Substitution should be used purposefully for exemplifying the meaning of expressions and equations, not as an exercise in itself. Matching terms to structures, rather than using them to practice substitution, might be more useful.

- The affordances of ICT should be exploited fully, in the learner's control, in the teacher's control, and in shared control, to support the shifts of understanding that have to be made including constructing objects in order to understand structure.

- Teachers should encourage the use of symbolic manipulation, using ICT, as a set of tools to support transforming expressions for mathematical understanding.

For policy

- The requirements listed above signal a training need on a national scale, focusing solely on algebra as a key component in the drive to increase mathematical competence and power.

- There are resource implications about the use of ICT. The focus on providing interactive whiteboards may have drawn attention away from the need for students to be in control of switching between representations and comparisons of symbolic expression in order to understand the syntax and the concept of functions. The United Kingdom may be lagging behind the developed world in exploring the use of CAS, spreadsheets and other software to support new kinds of algebraic thinking.

- In several other countries, researchers have been able to develop differently-sequenced curricula in which students have been able to use algebra as a way of expressing general and abstract notions as these arise. Manipulation, solution of equations, and other technical matters to do with symbols develop as well as with formal teaching, but are better understood and applied. Similar development in the United Kingdom has not been possible due to an over-prescriptive curriculum and frequent testing which forces a focus on technical manipulation.

- Textbooks which promulgate an 'arithmetic with letters' approach should be avoided; this approach leads inevitably to the standard, obvious errors and hence turns students off algebra and mathematics in favour of short-term gains.

- Symbolic manipulators, graph plotters and other algebraic software are widely available and used to allow people to focus on meaning, application and implications. Students should know how to use these and how to incorporate them into mathematical explorations and extended tasks.

- We need to be free to draw on research and explore its implications in the United Kingdom, and this may include radical re-thinking of the algebra curriculum and how it is tested. This may happen as part of the 'functional maths' agenda but its foundations need to be established when students are introduced to algebra.

For research

- Little is known about school learning of algebra in the following areas.

- The experiences that an average learner needs, in educational environments conducive to change, to shift from arithmetical to algebraic thinking.

- The relationship between understanding the nature of the representations in order to use them to understand functions, and understanding the situations as an aid to understanding the representations and how to use them.

- Whether teaching experiments using functional, multi-representational, equation or generalisation approaches have an impact on students' typical notation-related difficulties. In other words, we do not know if and how semantic-focused approaches to algebra have any impact on persistent and well-known syntactic problems.

- How learners' synthesise their knowledge to understand quadratic and other polynomials, their factorisation and roots, simultaneous equations, inequalities and other algebraic objects beyond elementary expressions and equations.

- Whether and how the use of symbolic manipulators to transform syntax supports algebraic understanding in school algebra.
- Using algebra to justify and prove generalities, rather than generate and express them.
- How students make sense of different metaphors for solving equations (balance, doing-undoing, graphical, formal manipulation).

Endnotes

1 The importance of inverses was discussed in the paper on natural numbers

2 In the paper on rational numbers we talk more about the relationship between fractions and rational numbers, and we often use these words interchangeably.

3 The advantage of this is that spotting like terms might be easier, but this can also mask some other characteristics such as physical meaning (e.g. E = mc2) and symmetry e.g. x2y + y2x.

4 This should be contrasted with the problems young learners have with expressing relations using number, described in our paper on functional relations. Knowing that relations are themselves number-like objects does not necessarily mean we have to calculate them.

5 This is discussed in detail in our papers on whole numbers and rational numbers and outlined here.

6 A very common mnemonic to remind people to do: brackets, 'of', division, multiplication, addition and subtraction in that order. It does not always work.

7 If n people all shake hands with each other, how many handshakes will there be?

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References

Ainley, J. (1996) Purposeful Contexts for Formal Notation in a Spreadsheet Environment. *Journal of Mathematical Behavior*, 15(4) 405-422

Ainley, J., Nardi, E., and Pratt, D. (1999). Constructing Meaning for Formal Notation in Active Graphing, in I. Scwank (Ed.) *Proceedings of the First Conference of the European Society for Research in Mathematics Education,* vol. I Forschungsinstitut fuer Mathematikdidaktik, Osnabrueck,

Alibali, M., Knuth, E., Hattikudur, S., Mcneil, N. and Stephens, A. (2007) A Longitudinal Examination of Middle School Students' Understanding of the Equal Sign and Equivalent Equations *Mathematical Thinking and Learning,* Volume 9, Issue 3, 2007, 221 – 247

Amit, M. and Fried, M. (2005) Multiple representations in 8th grade algebra lessons: Are learners really getting it? In Chick, H. L., and Vincent, J. L. (Eds.). (2005). *Proceedings of the International Group for the Psychology of Mathematics Education* 2-57 to 2-64 Melbourne: PME

Amit, M. and Neria, D. (2007) "Rising to the challenge": using generalization in pattern problems to unearth the algebraic skills of talented pre-algebra students *ZDM Mathematics Education* 40:111–129

Anghileri J., Beishuizen, M. and van Putten, C. (2002) From informal strategies to structured procedures: Mind the gap! *Educational Studies in Mathematics* 49(2) 149-170.

Arcavi A. (1994) Symbol sense: The informal sense-making in formal mathematics, *For the Learning of Mathematics*, 14 (3) p24-35.

Arzarello, F., Bazzini, L., Chiappini, G. (1994) The process of naming in algebraic problem solving, in J. Ponte and J. Matos (Eds.) *Proc. PME XVIII*, Lisbon, II, p40-44.

Baker, B., Hemenway, C., Trigueros, M. (2001) On transformations of basic functions. In Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*. pp.41-47 University of Melbourne, Australia, Dec 9-14, 2001.

Banerjee, R. and Subramaniam, K. (2004) *'Term' as a bridge concept between algebra and arithmetic*. Paper presented at Episteme international conference December 13-17, International Centre, Dona Paula, Goa

Bastable, V. and Schifter D. (2008) Classroom stories: examples of elementary students engaged in early algebra. In J. Kaput, D. Carraher and M. Blanton (eds.) *Algebra in the early grades.* 165-184. New York: Erlbaum

Bednarz, N. and Janvier, B. (1996). Emergence and development of algebra as a problem solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran and L. Lee, L. (Eds.), *Approaches to Algebra* (pp. 115 136). Dordrecht/Boston/London: Kluwer Academic Publishers.

Bednarz, N., Kieran C., Lee, L. (1996) Approaches to algebra: perspectives for research on teaching. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. pp.3-14 Dordrecht: Kluwer

Bell, A. (1996) Algebraic thought and the role of manipulable symbolic language. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. 151-154 Kluwer, Dordrecht

Bills, L. (2007) Stereotypes of literal symbol use in senior school algebra. In E.Pehkonen (ed.) *Proceedings of the annual conference of the Internatioanl Group for the Psychology of Mathematics Education.* (2) 73-80. Lahti: University of Helsinki.

Blanton, M and Kaput, J (2005) Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36 (5) 412-446, pp. 35-25

Bloedy-Vinner, H. (1994) The analgebraic mode of thinking: the case of the parameter. In J.Ponte and J. Matos (eds.) *Proceedings of the annual conference of the International Group for the Psychology of Mathematics Education.* (2) 88-95. Lisbon, University of Lisboa.

Boero, P. (2001). Transformation and anticipation as key processes in algebraic problem solving. In Sutherland, R., Rojano, T., Bell, A. and Lins, R.(eds.) *Perspectives on School Algebra*. pp. 99-119 Dordrecht, Kluwer.

Booth, L. (1984) *Algebra: Children's strategies and errors, a report of the strategies and errors in the secondary school project*, NFER-Nelson, London.

Boulton-Lewis, G. M., Cooper, T. J., Atweh, B., Pillay, H., Wilss, L., and Mutch, S. (1997). The transition from arithmetic to algebra: A cognitive perspective. In E. Pehkonen (ed.) *Proceedings of the International Group for the Psychology of Mathematics Education*, *21*(2), 185-192.

Brenner, M. E., Mayer, R. E., Moseley, B., Brar, T., Duran, R., Reed, B. S. and Webb, D. (1997) Learning by understanding: The role of multiple representations in learning algebra. *American Educational Research Journal*, 34(4), 663-689.

Britt, M. and Irwin, K. (2007) Algebraic thinking with and without algebraic representation: a three-year longitudinal study. *ZDM Mathematics Education* 40, 39-53

Brown, L. and Coles, A. (1999) Needing to use algebra – a case study. In O Zaslavsky (ed.) *Proceedings of the 23rd^{t} annual conference of the International Group for the Psychology of Mathematics Education*, Technion; Haifa, 2, 153-160.

Brown, L. and Coles, A. (2001) Natural algebraic activity. In Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*. pp.120-127 University of Melbourne, Australia, Dec 9-14, 2001.

Bruner, J. (1966). *Toward a Theory of Instruction.* Cambridge, MA: Harvard University Press

Carpenter, T. and Levi, L. (2000) *Developing conceptions of algebraic reasoning in the primary grades. Research Report*, Madison WI: National Center for Improving Student Learning and Achievement in Mathematics and Science. www.wcer.wisc.edu/ncisla/publications

Carraher, D. Brizuela, B. and Schliemann A. (2000) Bringing out the algebraic character of arithmetic: Instantiating variables in addition and subtraction, in T. Nakahara and M. Koyama (Eds.), *Proceedings of PME–XXIV*, Vol. 2. Hiroshima, Japan, p 145-152.

Carraher, D., Brizuela, B. M., and Earnest, D. (2001). The reification of additive differences in early algebra. In H. Chick, K. Stacey, J. Vincent, and J. Vincent (Eds.), *The future of the teaching and learning of algebra: Proceedings of the 12th ICMI Study Conference (vol. 1)*. The University of Melbourne, Australia.

Carraher, D., Martinez, M. and Schliemann, A. (2007) Early algebra and mathematical generalization *ZDM Mathematics Education* 40, 3-22

Carraher, D., Schliemann, A., and Brizuela, B. (2001). Can young students operate on unknowns? *Proceedings of the XXV Conference of the International Group for the Psychology of Mathematics Education*, Utrecht, The Netherlands. Vol. 1, 130-140

Collis, K. (1971) A study of concrete and formal reasoning in school mathematics. *Australian Journal of Psychology.* 23, 289-296.

Cooper, T. and Warren, E. (2007) The effect of different representations on Years 3 to 5 students' ability to generalize. *ZDM Mathematics Education 40* 29-37

Davydov V. (1990) *Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula.* Reston, VA: NCTM.

Dettori, G. Garutti, R. Lemut, E. (2001) From arithmetic to algebraic Thinking by using a spreadsheet, in R. Sutherland, T. Rojano, A. Bell and R. Lins (Eds) *Perspectives on School Algebra*, p 191-208, Kluwer, Dordrecht.

Dickson, L. (1989a) Area of rectangle. In D. Johnson (ed.) *Children's Mathematical Frameworks 8 – 13: a study of classroom teaching*. 88-125, Windsor: NFER-Nelson.

Dickson, L. (1989b) Equations. In D. Johnson (ed.) *Children's Mathematical Frameworks 8 to 13: a study of classroom teaching*. pp. 151-190 Windsor: NFER- Nelson

Dougherty, B. (1996) The write way: a look at journal writing in first-year algebra. *Mathematics Teacher* 89, 556-560

Dougherty, B. (2001) Access to algebra: a process approach. In Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*. pp. 207-212 University of Melbourne, Australia, Dec 9-14, 2001.

Drijvers, P. (2001) The concept of parameter in a computer algebra environment. In Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*. pp. 221-227 University of Melbourne, Australia, Dec 9-14, 2001.

English, L. and Sharry, P. (1996) Analogical reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics*, 30(2), 135-157.

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*(1), 105-121.

Falle, J. (2005) From arithmetic to algebra: novice students' strategies for solving equations. *Proceedings of annual conference of Mathematics Education Research Group of Australasia.* Downloaded from http://www.merga.net.au/search.php April 2008

Filloy E. and Rojano, T. (1989) Solving equations: the transition from arithmetic to algebra. *For the Learning of Mathematics,* 9(2) 19-25.

Filloy, E. and Sutherland, R. (1996), Designing curricula for teaching and learning algebra, in A. Bishop, K. Clements, C. Keitel, J. Kilpatrick and C. Laborde (Eds.) *International Handbook of Mathematics Education* Part 1, Chapter 4, Kluwer, Dordrecht, p 139-160.

Filloy, E. Rojano, T. and Robio, G. (2001) Propositions concerning the resolution of arithmetical-algebraic problems, in R. Sutherland (Ed.) *Perspectives on Algebra,* Kluwer, Dordrecht, p155-176.

Foxman, D., Ruddock, G., Joffe, L., Mason, K., Mitchell, P., and Sexton, B. (1985) *A Review of Monitoring in mathematics 1978 to 1982,* Assessment of Performance Unit, Department of Education and Science, London.

Friedlander, A. and Tabach, M. (2001) Developing a curriculum of beginning algebra in a spreadsheet environment. In Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*. Pp.252-257. University of Melbourne, Australia.

Fujii, T. and Stephens, M. (2001) Fostering an understanding of generalisation through numerical expressions: the role of quasi-variables. In Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*. Pp.258-264.University of Melbourne, Australia.

Fujii, T. and Stephens, M. (2008) Using number sentences to introduce the idea of variable. In.C. Greenes and R. Rubenstein (eds.) *Algebra and Algebraic Thinking in School Mathematics. 70th Yearbook*. pp. 127-140. Reston, VA: NCTM.

Furinghetti, F. and Paola, D. (1994). Parameters, unknowns and variables: a little difference?, in *Proceedings of the XVIII International Conference for the Psychology of Mathematics Education*, pp.?? University of Lisbon, Portugal.

Goldin, G. (2002) Representation in mathematical learning in English, L (ed.) *Handbook of international research in mathematics education* pp.197-218, Mahwah NJ, Erlbaum.

Gray, E. and Tall, D. (1994). Duality, ambiguity and flexibility: A proceptual view of simple arithmetic, *Journal for Research in Mathematics Education*, 26 (2), 115-141

Greenes, C. and Rubenstein, R. (2007) (eds.) *Algebra and algebraic thinking in school mathematics. 70th Yearbook*. Reston, VA: NCTM.

Hart, K. (Ed.) (1981) *Children's understanding of mathematics 11-16*, Murray, London..

Heid, M. (1996) Computer Intensive Algebra: Reflections on a functional approach to beginning algebra. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. Kluwer, Dordrecht

Herscovics, N. and Linchevski, L. (1994) A cognitive gap between arithmetic and algebra, *Educational studies in mathematics*, Vol. 27, pp. 59-78.

Hirschhorn, D. (1993) A longitudinal study of students completing four years of UCSMP mathematics *Journal for Research in Mathematics Education* 24 (2), 136-158

Hitt, F. (1998) Difficulties in the articulation of different representations linked to the concept of function. *Journal of Mathematical Behavior*, 17(1), 123-134.

Hollar, J. and Norwood, K. (1999) The effects of a graphing-approach intermediate algebra curriculum on students' understanding of function *Journal for Research in Mathematics Education* 30 (2) 220-226.

Jacobs, V., Franke, M., Carpenter, T., Levi, L., Battey, D. (2007) Professional development focused on children's algebraic reasoning in elementary school. *Journal of Research in Mathematics Education* 38(3) 258-288

Janvier, C. (1996) Modeling and the initiation into algebra. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. pp. 225-238. Kluwer, Dordrecht

Johnson, D. (1989) (ed.) *Children's Mathematical Frameworks 8 – 13: a study of classroom teaching*. Windsor: NFER-Nelson.

Kaput, J. (1998) Transforming algebra from an engine of inequity to an engine of mathematical po wer by 'algebrafying' the K–12 curriculum. In National Council of Teachers of Mathematics and Mathematical Sciences Education Board (Eds.). *The nature and role of algebra in the K–14 curriculum: Proceedings of a National Symposium* (pp. 25–26). Washington, DC: National Research Council, National Academy Press.

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema and T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133-155). Mahwah, NJ:Erlbaum.

Kerslake, D. (1986) *Fractions: Children's strategies and errors*. Windsor: NFER-Nelson.

Kieran, C. (1981) Concepts associated with the equality symbol. *Educational Studies in Mathematics* 12, 317-326

Kieran, C. (1983) Relationships between novices' views of algebraic letters and their use of symmetric and asymmetric equation-solving procedures. In J. Bergeron and B. Herscovics (eds.) *Proceedings of the International Group for the Psychology of Mathematics Education* pp.1.161-1-168, Montreal: Universite de Montreal.

Kieran, C. (1989) The early learning of algebra: a structural perspective. In S. Wagner and C. Keiran (eds.) *Research issues in the learning and teaching of algebra.* pp. 33-56, Reston, VA, NCTM.

Kieran, C. (1992). The learning and teaching of algebra, in D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning*, Macmillan, New York, p390-419.

Kieran, C. and Saldanha, L. (2005) Computer algebra systems (CAS) as a tool for coaxing the emergence of reasoning about equations of algebraic expressions. In Chick, H. L., and Vincent, J. L. (Eds.). (2005). *Proceedings of the International Group for the Psychology of Mathematics Education* pp.3-193-3-200 Melbourne: PME

Kieran, C. and Sfard, A. (1999), Seeing through symbols: The case of equivalent expressions, *Focus on Learning Problems in Mathematics*, 21 (1) p1-17

Kieran, C., Boileau, A.and Garancon, M. (1996) Introducing algebra by means of a technology-supported, functional approach. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. Pp. 257-294 Kluwer, Dordrecht

Kirshner David (1989)The Visual Syntax of Algebra *Journal for Research in Mathematics Education* 20(3) 274-287

Krutetskii, V. (1976). *The psychology of mathematical abilities in school children.* Chicago: University of Chicago Press.

Küchemann, D. (1981) Algebra, in K. Hart (Ed.) *Children's Understanding of Mathematics 11-16*, p 102-119 Murray, London..

Lamon, S. (1998) Algebra: meaning through modelling, in A. Olivier and K. Newstead (Eds) *22nd Conference of the International Group for the Psychology of Mathematics Education,* vol. 3, pp.167-174. Stellenbosch: International Group for the Psychology of Mathematics Education.

Lampert, M. (1986) Knowing, doing and teaching multiplication. *Cognition and Instruction* 3 ,305-342.

Lee, L. (1996) An incitation into algebraic culture through generalization activities. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. pp. 87-106. Kluwer, Dordrecht

Lehrer, R., Strom, D. and Confrey, J. (2002) Grounding metaphors and inscriptional resonance: children's emerging understanding of mathematical similarity. *Cognition and Instruction* 20(3) 359-398.

Lima, R. and Tall, D. (2008) Procedural embodiment and magic in linear equations. *Educational Studies in Mathematics* 67(1) 3-18

Linchevski, L., and Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. *Educational Studies in Mathematics*, 30,1, 39-65.

Linchevski, L. and Sfard, A. (1991) Rules without reasons as processes without objects: The case of equations and inequalities. In F. Furinghetti (ed.) *Proceedings of the 15th International Group for the Psychology of Mathematics Education.* 2, 317-324.

Lins, R.L. (1990). A framework of understanding what algebraic thinking is. In G. Booker, P.Cobb and T. Mendicuti (eds.) *Proceedings of the 14th Conference of the International Group for the Psychology of Mathematics Education,* 14, 2, 93-101. Oaxtepec, Mexico.

MacGregor, M. and Stacey, K. (1993) Cognitive Models Underlying Students' Formulation of Simple Linear Equations *Journal for Research in Mathematics Education* 24(3) 274-232

MacGregor, M. and Stacey, K. (1995), The Effect of Different Approaches to Algebra on Students' Perceptions of Functional Relationships, *Mathematics Education Research Journal*, 7 (1) p69-85.

MacGregor M.and Stacey K. (1997) Students' understanding of algebraic notation 11-15, *Educational Studies in Mathematics*. 33(1), 1-19

Mason, J. and Sutherland, R. (2002), *Key Aspects of Teaching Algebra in Schools*, QCA, London.

Mason, J., Stephens, M. and Watson, A. (in press) Appreciating mathematical structure for all. *Mathematics Education Research Journal.*

Moss, J., R. Beatty, L. Macnab.(2006) *Design for the development and teaching of an integrated patterning curriculum*. Paper present to AERA (April 2006.)

Nemirovsky, R. (1996) A functional approach to algebra: two issues that emerge. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. pp. 295-316. Kluwer, Dordrecht

Nickson, M. (2000) *Teaching and learning mathematics: a teacher's guide to recent research and its applications.* Cassell London

NMAP, National Mathematics Advisory Panel (2008) downloaded April 2008 from: http://www.ed.gov/about/bdscomm/list/mathpanel /index.html

Noble, T., Nemirovsky, R., Wright, T., and Tierney, C. (2001) Experiencing change: The mathematics of change in multiple environments. *Journal for Research in Mathematics Education,* 32(1) 85-108.

Noss, R., Healy, L., Hoyles, C. (1997) The construction of mathematical meanings: Connecting the visual with the symbolic. *Educational Studies in Mathematics* 33(2), 203-233

Nunes, T. and Bryant, P. (1995) Do problem situations influence children's understanding of the commutativity of multiplication? *Mathematical Cognition, 1*, 245-260.

O'Callaghan, B. (1998) Computer-Intensive Algebra and Students' Conceptual Knowledge of Functions, *Journal For Research in Mathematics Education*, 29 (1) p21-40.

Piaget, J. (1969) *The mechanisms of perception.* London: RKP

Piaget, J., & Moreau, A. (2001). The inversion of arithmetic operations (R. L. Campbell, Trans.). In J. Piaget (Ed.), *Studies in reflecting abstraction* (pp. 69–86). Hove, UK: Psychology Press.Powell, A. B. and Maher, C. A. (2003). Heuristics of Twelfth Graders' Building Isomorphism. In Pateman, N., Dougherty, B. and Zilliox, J. (eds.) *Proceedings of the 27th conference of the international group for the psychology of mathematics education 4,* pp. 23-30. Honolulu, Hawaii: University of Hawaii.

Radford, L. (2007) Iconicity and contraction: a semiotic investigation of forms of algebraic generalizations of patterns in different contexts *ZDM Mathematics Education 40* 83-96

Redden, T. (1994) Alternative pathways in the transition from arithmetical thinking to algebraic thinking. In J. da Ponte and J. Matos (eds.) *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education* vol. 4, 89-96, University of Lisbon, Portugal.

Reed, S. (1972) Pattern recognition and categorization. *Cognitive Psychology* 3, 382-407

Reggiani, M. (1994) Generalization as a basis for algebraic thinking: observations with 11-12 year old pupils *Proceedings of PME XVIII,* IV, p97 - 104.

Resnick, L. B., Lesgold, S., and Bill, V. (1990). From protoquantities to number sense. In G. Booker, J. Cobb, and T. N. de Mendicuti (Eds.), *Proceedings of the Fourteenth Psychology of Mathematics Education Conference* (Vol. 3, pp. 305-311). Mexico City, Mexico: International Group for the Psychology of Mathematics Education.

Rivera, F. and Becker, J. (2007) Middle school children's cognitive perceptions of constructive and deconstructive generalizations involving linear figural patterns *ZDM Mathematics Education 40* 65-82

Robinson, K., Ninowski, J. and Gray, M. (2006) Children's understanding of the arithmetic concepts of inversion and associativity, *Journal of Experimental Child Psychology* 94, pp. 349–362.

Rothwell-Hughes, E.(1979) *Conceptual Powers of Children: an approach through mathematics and science.* Schools Council Publications.

Rowland, T. and Bills, L. (1996) 'Examples, Generalisation and Proof' *Proceedings of British Society for Research in Learning Mathematics London Institute of Education*, pp. 1-7

Ryan, J. and Williams, J. (2007) *Children's Mathematics 4 -15: learning from errors and misconceptions* Maidenhead Open University Press.

Sáenz-Ludlow, A.and Walgamuth, C. (1998) Third Graders' Interpretations of Equality and the Equal Symbol *Educational Studies in Mathematics,* 35(2) 153-187.

Schwartz, J. and Yerushalmy, M. (1992) Getting students to function in and with algebra. In G. Harel and E. Dubinsky (eds.) *The concept of function: Aspects of epistemology and pedagogy*. MAA Notes, 25, 261-289. Washington DC: MAA.

Sfard, A. and Linchevski, L. (1994) . The gains and the pitfalls of reification: The case of algebra. *Educational Studies in Mathematics* , 26, 191-228.

Siegler, R. S., and Stern, E. (1998). Conscious and unconscious strategy discoveries: a microgenetic analysis. *Journal of Experimental Psychology* 127, 377-397.

Simmt, E. and Kieren, T. (1999). Expanding the cognitive domain: the role(s) and consequence of interaction in mathematics knowing. In F. Hitt and M. Santos (Eds.) *Proceedings of the Twenty First Annual Meeting Psychology of Mathematics Education North American Chapter,* pp.299-305.

Stacey, K. (1989) Finding and Using Patterns in Linear Generalising Problems, *Educational Studies in Mathematics,* 20, p147-164.

Stacey, K., and MacGregor, M. (2000). Learning the algebraic method of solving problems. *Journal of Mathematical Behavior*, 18(2), 149-167.

Steele, D. (2007) Seventh-grade students' representations for pictorial growth and change problems 97-110 *ZDM Mathematics Education 40* 97-110

Sutherland R. and Rojano T. (1993) A Spreadsheet Algebra Approach to Solving Algebra Problems, *Journal of Mathematical Behavior,* 12, 353-383.

Thomas, M. and Tall, D. (2001) The long-term cognitive development of symbolic algebra. In Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra* pp.590-597. University of Melbourne, Australia, Dec 9-14, 2001.

Tsamir, P., and Almog, N. (2001). Students' strategies and difficulties: The case of algebraic inequalities. *International Journal of Mathematics Education in Science and Technology,* 32, 513-524.

Tsamir, P., and Bazzini, L. (2001). Can x=3 be the solution of an inequality? A study of Italian an Israeli students. In M. van den Heuvel-Panhuizen (Ed.), Proceedings of the 25th Annual Meeting for the Psychology of Mathematics Education, Utrecht: Holland. (Vol IV, pp. 303-310).

van Ameron. B. (2003) Focusing on informal strategies when linking arithmetic to early algebra. *Educational Studies in Mathematics,* 2003, vol. 54, no. 1, p. 63-75..

Vergnaud, G. (1997) The nature of mathematical concepts. In T. Nunes and P. Bryant (eds.) *Learning and teaching mathematics: an international perspective.* pp. 5-28 Hove: Psychology Press

Vergnaud, G. (1998). A comprehensive theory of representation for mathematics *Journal of Mathematical Behavior*, 17(2), 167-181

Vlassis, J., (2002) The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics* 49(3), 341-359.

Warren, E. and Cooper, T. (2008) Patterns that support early algebraic thinking in the elementary school. In.C. Greenes and R. Rubenstein (eds.) *Algebra and Algebraic Thinking in School Mathematics. 70th Yearbook.* pp.113-126, Reston, VA: NCTM.

Wertheimer, M. (1960) *Productive thinking.* New York: Harper

Wong, M. (1997) Numbers versus letters in algebraic manipulation: which is more difficult? In E.Pehkonen (ed.) *Proceedings of the annual conference of the Internatioanl Group for the Psychology of Mathematics Education.* (4) 285-290. Lahti: University of Helsinki.

Yeap, B-H. and Kaur, B. (2007) Elementary school students engaging in making generalisation:a glimpse from a Singapore classroom *ZDM Mathematics Education 40,* 55-64

Yerushalmy, M. (1997) Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education,* 28 (4) 431-466.

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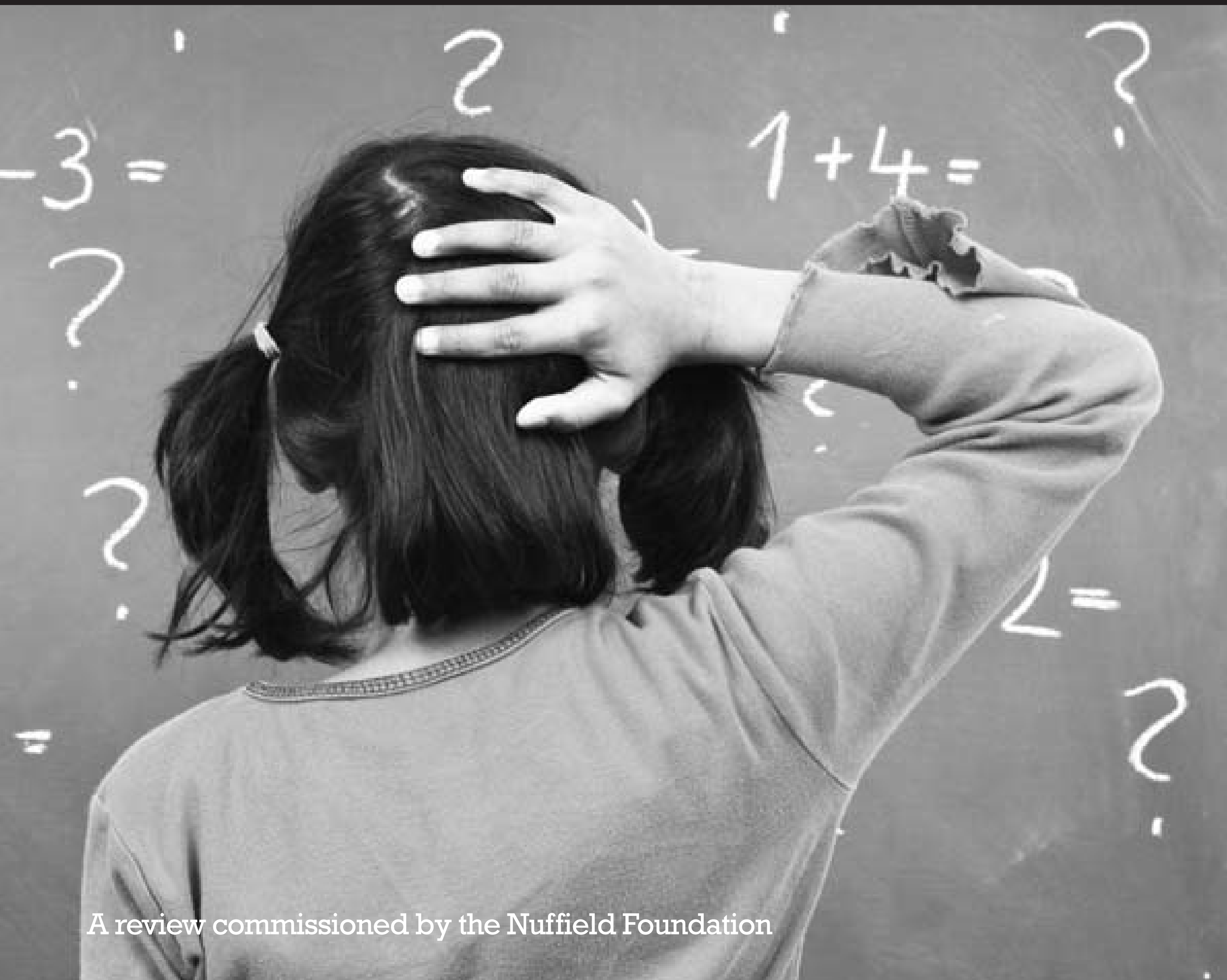
7

Key understandings in mathematics learning

Paper 7: Modelling, problem-solving and integrating concepts

By Anne Watson, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

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About the author

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Summary of paper 7: Modelling, problem-solving and integrating concepts

Headlines

We have assumed a general educational context which encourages thinking and problem-solving across subjects. A key difference about mathematics is that empirical approaches may solve individual problems, and offer directions for reasoning, but do not themselves lead to *new* mathematical knowledge or mathematical reasoning, or to the power that comes from applying an abstract idea to a situation.

In secondary mathematics, the major issue is not how children learn elementary concepts, but what experiences they have had and how these enable or limit what else can be learnt. That is why we have combined several aspects of secondary mathematics which could be exemplified by particular topics.

- Students have to be fluent in understanding methods and confident about using them to know why and when to apply them, but such application does not automatically follow learning procedures. Students have to understand the situation as well as being able to call on a familiar repertoire of ideas and methods.

- Students have to know some elementary concepts well enough to apply them and combine them to form new concepts in secondary mathematics, but little is known from research about what concepts are essential in this way. Knowledge of a range of functions is necessary for modelling situations.

- Students have to learn when and how to use informal, experiential reasoning and when to use formal, conventional, mathematical reasoning. Without special attention to meanings many students tend to apply visual reasoning, or be triggered by verbal cues, rather than to analyse situations mathematically.

- In many mathematical situations in secondary mathematics, students have to look for relations between numbers and variables and relations between relations and properties of objects, and know how to represent them.

How secondary learners tackle new situations

In new situations students first respond to familiarity in appearance, or language, or context. They bring earlier understandings to bear on new situations, sometimes erroneously. They naturally generalise from what they are offered, and they often over-generalise and apply inappropriate ideas to new situations. They can learn new mathematical concepts either as extensions or integrations of earlier concepts, and/or as inductive generalisations from examples, and/or as abstractions from solutions to problems.

Routine or context?

One question is whether mathematics is learnt better from routines, or from complex contextual situations. Analysis of research which compares how children learn mathematics through being taught routines efficiently (such as with computerised and other learning packages designed to minimise cognitive load) to learning through problem-solving in complex situations (such as through Realistic Mathematics Education) shows that the significant difference is not about the speed and retention of learning but *what* is being learnt. In each approach the main question for progression is whether the student learns new concepts well enough to use and adapt them in future learning and outside mathematics. Both approaches have inherent weaknesses in this respect. These weaknesses will become clear in what follows. However, there are several studies which show that those who develop mathematical methods of enquiry over time can then learn procedures easily and do as well, or better, in general tests.

International research into the use of ICT to provide new ways to represent situations and to see relationships, such as by comparing spreadsheets, graph plotters and dynamic images appear to speed up the process of relating representations through isomorphic reasoning about covariation, and hence the development of understanding about mathematical structures and relations.

Problem-solving and modelling

To learn mathematics one has to learn to solve mathematical problems or model situations mathematically. Studies of students' problem solving mainly focus in the successful solution of contextually-worded problems using mathematical methods, rather than using problem solving as a context for learning new concepts and developing mathematical thinking. To solve unfamiliar problems in mathematics, a meta-analysis of 487 studies concluded that for students to be maximally successful:

- problems need to be fully stated with supportive diagrams
- students need to have previous extensive experience in using the representations used
- they have to have relevant basic mathematical skills to use
- teachers have to understand problem-solving methods.

This implies that fluency with representations and skills is important, but also depends on how clearly the problem is stated. In some studies the difficulty is also to do with the underlying concept, for example, in APU tests area problems were difficult with or without diagrams.

To be able to solve problems whose wording does not indicate what to do, students have to be able to read the problem in two ways: firstly, their technical reading skills and understanding of notation have to be good enough; secondly, they have to be able to interpret it to understand the contextual and mathematical meanings. They have to decide whether and how to bring informal knowledge to bear on the situation, or, if they approach it formally, what are the variables and how do they relate. If they are approaching it formally, they then have to represent the relationships in some way and decide how to operate on them.

Application of earlier learning

Knowing methods

Students who have only routine knowledge may not recognize that it is relevant to the situation. Or they can react to verbal or visual cues without reference to context, such as 'how much?' triggering multiplication rather than division, and 'how many?' always triggering addition. A further problem is that they may not understand the underlying relationships they are using and how these relate to each other. For example, a routine approach to $2 \times 1/3 \times 3/2$ may neither exploit the meaning of fractions nor the multiplicative relation.

Students who have only experience of applying generic problem-solving skills in a range of situations sometimes do not recognize underlying mathematical structures to which they can apply methods used in the past. Indeed, given the well-documented tendency for people to use *ad hoc* arithmetical trial-and-adjustment methods wherever these will lead to reasonable results, it is possible that problem-solving experience may not result in learning new mathematical concepts or working with mathematical structures, or in becoming fluent with efficient methods.

Knowing concepts

Students who have been helped to learn concepts, and can define, recognise and exemplify elementary ideas are better able to use and combine these ideas in new situations and while learning new concepts. However, many difficulties appear to be due to having too limited a range of understanding. Their understanding may be based on examples which have irrelevant features in common, such as the parallel sides of parallelograms always being parallel to the edges of a page. Understanding is also limited by examples being similar to a prototype, rather than extreme cases. Another problem is that students may recognize examples of a concept by focusing too much on visual or verbal aspects, rather than their properties, such as believing that it is possible to construct an equilateral triangle on a nine-pin geoboard because it 'looks like one'.

Robust connections between and within earlier ideas can make it easier to engage with new ideas, but can also hinder if the earlier ideas are limited and inflexible. For example, learning trigonometry involves understanding: the definition of triangle; right-angles; recognizing them in different orientations; what angle means and how it is measured; typical units for measuring lines; what ratio means; similarity of triangles; how ratio is written as a fraction; how to manipulate a multiplicative relationship; what 'sin' (etc.) means as a symbolic representation of a function and so on. Thus knowing about ratios can support learning trigonometry, but if the understanding of 'ratio' is limited to mixing cake recipes it won't help much. To be successful students have to have had enough experience to be fluent, and enough knowledge to use methods wisely.

They become better at problem-solving and modelling when they can:

- draw on knowledge of the contextual situation to identify variables and relationships and/or, through imagery, construct mathematical representations which can be manipulated further
- draw on a repertoire of representations, functions, and methods of operation on these
- have a purpose for the modelling process, so that the relationship between manipulations in the model and changes in the situation can be meaningfully understood and checked for reasonableness.

Knowing how to approach mathematical tasks

To be able to decide when and how to use informal or formal approaches, and how to use prior knowledge, students need to be able to think mathematically about all situations in mathematics lessons. This develops best as an all-encompassing perspective in mathematics lessons, rather than through isolated experiences.

Students have to:

- learn to avoid instant reactions based in superficial visual or verbal similarity
- practice using typical methods of mathematical enquiry explicitly over time
- have experience of mulling problems over time in order to gain insight.

With suitable environments, tools, images and encouragement, learners can and do use their general perceptual, comparative and reasoning powers in mathematics lessons to:

- generalise from what is offered and experienced
- look for analogies
- identify variables
- choose the most efficient variables, those with most connections
- see simultaneous variations
- understand change
- reason verbally before symbolising
- develop mental models and other imagery
- use past experience of successful and unsuccessful attempts
- accumulate knowledge of operations and situations to do all the above successfully

Of course, all the tendencies just described can also go in unhelpful directions and in particular people tend to:

- persist in using past methods and applying procedures without meaning, if that has been their previous mathematical experience
- get locked into the specific situation and do not, by themselves, know what new mathematical ideas can be abstracted from these experiences
- be unable to interpret symbols, text, and other representations in ways the teacher expects
- use additive methods; assume that if one variable increases so will another; assume that all change is linear; confuse quantities.

RECOMMENDATIONS

Research about mathematical learning	Recommendations for teaching
Learning routine methods and learning through complex exploration lead to different kinds of knowledge and cannot be directly compared; neither method necessarily enables learning new concepts or application of powerful mathematics ideas. However, those who have the habit of complex exploration are often able to learn procedures quickly.	Developers of the curriculum, advisory schemes of work and teaching methods need to be aware of the importance of understanding new concepts, and avoid teaching solely to pass test questions, or using solely problem-solving mathematical activities which do not lead to new abstract understandings. Students should be helped to balance the need for fluency with the need to work with meaning.
Students naturally respond to familiar aspects of mathematics; try to apply prior knowledge and methods, and generalize from their experience.	Teaching should take into account students' natural ways of dealing with new perceptual and verbal information, and the likely misapplications. Schemes of work and assessment should allow enough time for students to adapt to new meanings and move on from earlier methods and conceptualizations.
Students are more successful if they have a fluent repertoire of conceptual knowledge and methods, including representations, on which to draw.	Developers of the curriculum, advisory schemes of work and teaching methods should give time for new experiences and mathematical ways of working to become familiar in several representations and contexts before moving on. Students need time and multiple experiences to develop a repertoire of appropriate functions, operations, representations and mathematical methods in order to solve problems and model situations. Teaching should ensure conceptual understanding as well as 'knowing about', 'knowing how to', and 'knowing how to use'.
Multiple experiences over time enable students to develop new ways to work on mathematical tasks, and to develop the ability to choose what and how to apply earlier learning.	Schemes of work should allow for students to have multiple experiences, with multiple representations over time to develop mathematically appropriate 'habits of mind'.
Students who work in computer-supported multiple representational contexts over time can understand and use graphs, variables, functions and the modelling process. Students who can choose to use available technology are better at problem solving, and have complex understanding of relations, and have more positive views of mathematics.	There are resource implications about the use of ICT. Students need to be in control of switching between representations and comparisons of symbolic expression in order to understand the syntax and the concept of functions. The United Kingdom may be lagging behind the developed world in exploring the use of spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

Recommendations for research

Application of research findings about problem-solving, modelling and conceptual learning to current curriculum developments in the United Kingdom suggests that there may be different outcomes in terms of students' ability to solve quantitative and spatial problems in realistic contexts. However, there is no evidence to convince us that the new National Curriculum in England will lead to better *conceptual* understanding of mathematics, either at the elementary levels, which are necessary to learn higher mathematics, or at higher levels which provide the confidence and foundation for further mathematical study. Where contextual and exploratory mathematics, integrated through the curriculum, *do* lead to further conceptual learning it is related to conceptual learning being a rigorous focus for curriculum and textbook design, and in teacher preparation, such as in China, Japan, Singapore, and the Netherlands, or in specifically designed projects based around such aims.

In the main body of *Paper 7: Modelling, solving problems and learning new concepts in secondary mathematics* there are several questions for future research, including the following.

- What are the key conceptual understandings for success in secondary mathematics, from the point of view of learning?
- How do students learn new ideas in mathematics at secondary level that depend on combinations of earlier concepts?
- What evidence is there of the characteristics of mathematics teaching at higher secondary level which contribute both to successful conceptual learning and application of mathematics?

Modelling, problem-solving and integrating concepts

Introduction

By the time students enter secondary school, they possess not only intuitive knowledge from outside mathematics and outside school, but also a range of quasi-intuitive understandings within mathematics, derived from earlier teaching and from their memory of generalisations, metaphors, images, metonymic associations and strategies that have served them well in the past. Many of these typical understandings are described in the previous chapters. Tall and Vinner (1981) called these understandings 'concept images', which are a ragbag of personal conceptual, quasi-conceptual, perceptual and other associations that relate to the language of the concept and are loosely connected by the language and observable artefacts associated with the concept. Faced with new situations, students will apply whatever familiar methods and associations come to mind relatively quickly – perhaps not realising that this can be a risky strategy. If 'doing what I think I know how to do' leads so easily to incorrect mathematics it is hardly surprising that many students end up seeing school mathematics as the acquisition and application of methods, and a site of failure, rather than as the development of a repertoire of adaptable intellectual tools.

At secondary level, new mathematical situations are usually ideas which arise through mathematics and can then be applied to other areas of activity; it is less likely that mathematics involves the formalisation of ideas which have arisen from outside experience as is common in the primary phase. Because of this difference, learning mathematics at secondary level cannot be understood only in terms of overall cognitive development. For this review, we developed a perspective which encompasses both the 'pure' and 'applied' aspects of learning at secondary level, and use research from both traditions to devise some common implications and overall recommendations for practice.

Characteristics of learning secondary mathematics

We justify the broad scope of this chapter by indicating similarities between the learning of the new concepts of secondary mathematics and learning how to apply mathematics to analyse, express and solve problems in mathematical and non-mathematical contexts. Both of these aspects of learning mathematics depend on interpreting new situations and bringing to mind a repertoire of mathematical concepts that are understood and fluent to some extent. In this review we will show that learning secondary mathematics presents core common difficulties, whatever the curriculum approach being taken, which need to be addressed through pedagogy.

In all teaching methods, when presented with a new stimulus such as a symbolic expression on the board, a physical situation, or a statement of a complex ill-defined 'real life' problem, the response of an engaged learner is to wonder:

What is this? This entails 'reading' situations, usually reading mathematical representations or words, and interpreting these in conventional mathematical ways. It involves *perception*, *attention*, understanding *representations* and being able to decipher *symbol systems*.

What is going on here? This entails identifying salient features including non-visual aspects, identifying variables, relating parts to each other, exploring what changes can be made and the effects of change, representing situations in mathematical ways, anticipating what might be the purpose of a mathematical object. It involves *attention, visualisation, modelling*, static and dynamic *representations*, understanding functional, statistical and geometrical *relationships*, focusing on what is mathematically *salient* and imagining the situation or a representation of it.

What do I know about this? This entails recognising similarities, seeking for recognisable structures beyond visual impact, identifying variables, proposing suitable functions, drawing on repertoire of past experiences and choosing what is likely to be useful. Research about *memory, problem-solving, concept images, modelling, functions, analysis* and *analogical reasoning* is likely to be helpful.

What can I do? This entails using past experience to try different approaches, heuristics, logic, controlling variables, switching between representations, transforming objects, applying manipulations and other techniques. It involves *analogical reasoning, problem-solving, tool-use, reasoning, generalisation* and *abstraction*, and so on.

Thus, students being presented with the task of understanding new ideas draw on past experience, if they engage with the task at all, just as they would if offered an unfamiliar situation and asked to express it mathematically. They may only get as far as the first step of 'reading' the stimulus. The alternative is to wait to be told what to do and treat everything as declarative, verbatim, knowledge. A full review of relevant research in all these areas is beyond the scope of this paper, and much of it is generic rather than concerned with mathematics.

We organise this Paper into three parts: Part 1 looks at what learners have to be able to do to be successful in these aspects of secondary mathematics; Part 2 considers what learners actually do when faced with new complex mathematical situations; and Part 3 reviews what happens with pedagogic intervention designed to address typical difficulties. We end with recommendations for future research, curriculum development and practice.

Part 1: What learners have to be able to do in secondary mathematics

In this chapter we describe what learners have to be able to do in order to learn new concepts, solve problems, model mathematical situations, and engage in mathematical thinking.

Learning 'new' concepts

Extension of meaning

Throughout school, students meet familiar ideas used in new contexts which include but extend their old use, often through integrating simpler concepts into more complex ideas. Sfard (1991) describes this process of development of meaning as consisting of 'interiorization' through acting on a new idea with some processes so that it becomes familiar and meaningful; understanding and expressing these processes and their effects as manageable units (condensation), and then this new structure becomes a thing in itself (reified) that can be acted on as a unit in future.

In this way, in *algebra*, letters standing for numbers become incorporated into terms and expressions which are number-like in some uses and yet cannot be calculated. Operations are combined to describe structures, and expressions of structures become objects which can be equated to each other. Variables can be related to each other in ways that represent relationships as functions, rules for mapping one variable domain to another (see the earlier chapters on functional relations and algebraic thinking).

Further, *number* develops from counting, whole numbers, and measures to include negatives, rationals, numbers of the form $a + b\sqrt{n}$ where a and b are rationals, irrationals and transcendentals, expressions, polynomials and functions which are number-like when used in expressions, and the uses of estimation which contradict earlier shifts towards accuracy. Eventually, two-dimensional complex numbers may also have to be understood. Possible discontinuities of meaning can arise between discrete and continuous quantities, monomials and polynomials, measuring and two-dimensionality, and different representations (digits, letters, expressions and functions).

Graphs are used first to compare values of various discrete categories, then are used to express two-dimensional discrete and continuous data as in scatter-graphs, or algebraic relationships between continuous variables, and later such relationships, especially linear ones, might be fitted to statistical representations.

Shapes which were familiar in primary school have to be defined and classified in new ways, and new properties explored, new geometric configurations become important and descriptive reasoning based on characteristics has to give way to logical deductive reasoning based on relational properties. Finally, all this has to be applied in the three-dimensional contexts of everyday life.

The processes of learning are sometimes said to follow historical development, but a better analogy would be to compare learning trajectories with the conceptual connections, inclusions and distinctions of mathematics itself.

Integration of concepts

As well as this kind of extension, there are new 'topics' that draw together a range of earlier mathematics. Typical examples of secondary topics are quadratic functions and trigonometry. Understanding each of these depends to some extent on understanding a range of concepts met earlier.

Quadratic functions: Learning about quadratic functions includes understanding:

- the meaning of letters and algebraic syntax;
- when letters are variables and when they can be treated as unknown numbers;
- algebraic terms and expressions;
- squaring and square rooting;
- the conventions of coordinates and graphing functions;
- the meaning of graphs as representing sets of points that follow an algebraic rule;
- the meaning of '=';
- translation of curves and the ways in which they can change shape;
- that for a product to equal zero at least one of its terms must equal zero and so on.

Trigonometry: Learning this includes knowing:

- the definition of triangle;
- about right-angles including recognizing them in different orientations;
- what angle means and how it is measured;
- typical units for measuring lines;
- what ratio means;
- similarity of triangles;
- how ratio is written as a fraction;
- how to manipulate a multiplicative relationship;
- what 'sin' (etc.) means as a symbolic representation of a function and so on.

New concepts therefore develop both through extension of meaning and combination of concepts. In each of these the knowledge learners bring to the new topic has to be adaptable and usable, not so strongly attached to previous contexts in which it has been used that it cannot be adapted. A hierarchical 'top down' view of learning mathematics would lead to thinking that all contributory concepts need to be fully understood before tackling new topics (this is the view taken in the NMAP review (2008) but is unsupported by research as far as we can tell from their document). By contrast, if we take learners' developing cognition into account we see that 'full understanding' is too vague an aim; it is the processes of applying and extending prior knowledge in the context of working on new ideas that contribute to understanding.

Whichever view is taken, learners have to bring existing understanding to bear on new mathematical contexts. There are conflicting research conclusions about the process of bringing existing ideas to bear on new stimuli: Halford (e.g. 1999) talks of conceptual chunking to describe how earlier ideas can be drawn on as packages, reducing to simpler objects ideas which are initially formed from more complex ideas, to develop further concepts and argues for such chunking to be robust before moving on. He focuses particularly on class inclusion (see Paper 5, Understanding space and its representation in mathematics) and transitivity, structures of relations between more than two objects, as ideas which are hard to deal with because they involve several levels of complexity. Examples of this difficulty were mentioned in Paper 4, Understanding relations and their graphical representation, showing how relations between relations cause problems. Chunking includes loss of access to lower level meanings, which may be useful in avoiding unnecessary detail of specific examples, but can obstruct meaningful use. Freudenthal (1991, p. 469) points out that automatic connections and actions can mask sources of insight, flexibility and creativity which arise from meanings. He observed that when students are in the flow of calculation they are not necessarily aware of what they are doing, and do

not monitor their work. It is also the case that much of the chunking that has taken place in earlier mathematics is limited and hinders and obstructs future learning, leading to confusion with contradictory experiences. For example, the expectation that multiplication will make 'things' bigger can hinder learning that it only means this sometimes – multiplication scales quantities in a variety of ways. The difficulties faced by students whose understanding of the simpler concepts learnt in primary school is later needed for secondary mathematical ideas, are theorised by, among others, Trzcieniecka-Schneider (1993) who points out that entrenched and limited conceptual ideas (including what Fischbein calls 'intuitions' (1987) and what Tall calls 'metbefores' (2004)) can hinder a student's approach to unfamiliar examples and questions and create resistance, rather than a willingness to engage with new ideas which depend on adapting or giving up strongly-held notions. This leads not only to problems understanding new concepts which depend on earlier concepts, but also makes it hard for learners to see how to apply mathematics in unfamiliar situations. On the other hand, it is important that some knowledge is fluent and easily accessible, such as number bonds, recognition of multiples, equivalent algebraic forms, the shape of graphs of common functions and so on. Learners have to know when to apply 'old' understandings to be extended, and when to give them up for new and different understandings.

Inductive generalisation

Learners can also approach new ideas by inductive generalisation from several examples. English and Halford (1995 p. 50) see this inductive process[1] as the development of a mental model which fits the available data (the range of examples and instances learners have experienced) and from which procedures and conjectures can be generated. For example, learners' understanding about what a linear graph can look like is at first a generalisation of the linear graphs they have seen that have been named as such. Similarly, learners' conjectures about the relationship between the height and volume of water in a bottle, given as a data set, depends on reasoning both from the data and from general knowledge of such changes. Leading mathematicians often remark that mathematical generalisation also commonly arises from abductive reasoning on one generic example, such as conjectures about relationships based on static geometrical diagrams. For both these processes, the examples available as data, instances, and illustrations from teachers, textbooks and other sources play a crucial role in the process. Learners have to know what features are salient and generalise from them. Often such reasoning depends on metonymic association (Holyoak and Thagard, 1995), so that choices are based on visual, linguistic and cues which might be misleading (see also earlier chapter on number) rather than mathematical meaning. As examples: the prototypical parallelogram has its parallel edges horizontal to the page; and x^2 and $2x$ are confused because it is so common to use '$x = 2$' as an example to demonstrate algebraic meaning.

Abstraction of relationships

A further way to meet new concepts is through a process of 'vertical mathematisation' (Treffers, 1987) in which experience of solving complex problems can be followed by extracting general mathematical relationships. It is unlikely that this happens naturally for any but a few students, yet school mathematics often entails this kind of abstraction. The Freudenthal Institute has developed this approach through teaching experiments and national roll-out over a considerable time, and its Realistic Mathematics Education (e.g. Gravemeijer and Doorman, 1999) sees mathematical development as

- seeing what has to be done to solve the kinds of problems that involve mathematics
- from the solutions extracting new mathematical ideas and methods to add to the repertoire
- these methods now become available for future use in similar and new situations (as with Piaget's notion of reflective abstraction and Polya's 'looking back').

Gravemeijer and Doorman show that this approach, which was developed for primary mathematics, is also applicable to higher mathematics, in this case calculus. They refer to 'the role models can play in a shift from a model of situated activity to a model for mathematical reasoning. In light of this model-of/model-for shift, it is argued that discrete functions and their graphs play a key role as an intermediary between the context problems that have to be solved and the formal calculus that is developed.'

Gravemeijer and Doorman's observation explains why, in this paper, we are treating the learning of new abstract concepts as related to the use of problem-solving and modelling as forms of mathematical activity. In all of these examples of new learning, the fundamental shift learners are expected to make, through instruction, is from informal, experiential, engagement using their existing knowledge to formal, conventional, mathematical understanding. This shift appears to have three components: construction of meaning; recognition in new contexts; playing with new ideas to build further ideas (Hershkowitz, Schwartz and Dreyfus, 2001).

It would be wrong to claim, however, that learning can only take place through this route, because there is considerable evidence that learners can acquire routine skills through programmes of carefully constructed, graded, tasks designed to deal educatively with both right answers and common errors of reasoning, giving immediate feedback (Anderson , Corbett, Koedinger and Pelletier, 1995). The acquisition of routine skills without explicit work on their meaning is not the focus of this paper, but the automatisation of routines so that learners can focus on structure and meaning by reflection later on has been a successful route for some in mathematics.

There is recent evidence from controlled trials that learning routines from abstract presentations is a more efficient way to learn about underlying mathematical structure than from contextual, concrete and story-based learning tasks (Kaminski, Sloutsky and Heckler, 2008). There are several problems with their findings, for example in one study the sample consisted of undergraduates for whom the underlying arithmetical concept being taught would not have been new, even if it had never been explicitly formalised for them before. In a similar study with 11-year-olds, addition modulo 3 was being taught. For one group a model of filling jugs with three equal doses was used; for the other group abstract symbols were used. The test task consisted of spurious combinatorics involving three unrelated objects[2]. Those who had been taught using abstract unrelated symbols did better, those who had been taught using jug-filling did not so well. While these studies suggest that abstract knowledge about structures is not less applicable than experience and *ad hoc* knowledge, they also illuminate the interpretation difficulties that students have in learning how to model phenomena mathematically, and how familiar meanings (e.g. about jug-filling) dominate over abstract engagement. What Kaminski's results say to educationists is not 'abstract rules are better' but 'be clear about the learning outcomes you are hoping to achieve and do not expect easy transfer between abstract procedures and meaningful contexts'.

Summary

- Learners have to understand new concepts as extensions or integrations of earlier concepts, as inductive generalisations from examples, and as abstractions from solutions to problems.
- Robust chunking of earlier ideas can make it easier to engage with new ideas, but can also hinder if the earlier ideas are limited and inflexible.
- Routine skills can be adopted through practise to fluency, but this does not lead to conceptual understanding, or ability to adapt to unfamiliar situations, for many students.
- Learners have to know when and how to bring earlier understandings to bear on new situations.
- Learners have to know how and when to shift between informal, experiential activity to formal, conventional, mathematical activity.
- There is no 'best way' to teach mathematical structure: it depends whether the aim is to become fluent and apply methods in new contexts, or to learn how to express structures of given situations.

Problem-solving

The phrase 'problem-solving' has many meanings and the research literature often fails to make distinctions. In much research solving word problems is seen as an end in itself and it is not clear whether the problem introduces a mathematical idea, formalises an informal idea, or is about translation of words into mathematical instructions. There are several interpretations and the ways students learn, and can learn, differ accordingly. The following are the main uses of the phrase in the literature.

1 *Word problems with arithmetical steps* used to introduce elementary concepts by harnessing informal knowledge, or as situations in which learners have to apply their knowledge of operations and order (see Paper 4, Understanding relations and their graphical representation).

These situations may be modelled with concrete materials, diagrams or mental images, or might draw on experiences outside school. The purpose may be either to learn concepts through familiar situations, or to learn to apply formal or informal mathematical methods. For example, upper primary students studied by Squire, Davies and Bryant (2004) were found to handle commutativity much better than distributivity, which they could only do if there were contextual cues to help them. For teaching purposes this indicates that distributive situations are harder to recognise and handle, and a mathematical analysis of distributivity supports this because it entails encapsulation of one operation before applying the second and recognition of the importance of order of operations.

2 *Worded contexts which require the learner to decide to use standard techniques*, such as calculating area, time, and so on. Diagrams, standard equations and graphs might offer a bridge towards deciding what to do. For instance, consider this word problem: 'The area of a triangular lawn is 20 square metres, and one side is 5 metres long. If I walk in a straight line from the vertex opposite this side, towards this side, to meet it at right angles, how far have I walked?' The student has to think of how area is calculated, recognise that she has been given a 'base' length and asked about 'height', and a diagram or mental image would help her to 'see' this. If a diagram is given some of these decisions do not have to be made, but recognition of the 'base' and 'height' (not necessarily named as such) and knowledge of area are still crucial.

   In these first two types of problem, Vergnaud's classification of three types of multiplicative problems (see Paper 4, Understanding relations and their graphical representation) can be of some help if they are straightforwardly multiplicative. But the second type often calls for application of a standard formula which requires factual knowledge about the situation, and understanding the derivation of formulae so that their components can be recognised.

3 *Worded contexts in which there is no standard relationship to apply, or algorithm to use, but an answer is expected.* Typically these require setting up an equation or formula which can then be applied and calculated. This depends on understanding the variables and relationships; these might be found using knowledge of the situation, knowledge of the meaning of operations, mental or graphical imagery. For example, consider the question, 'One side of a rectangle is reduced in length by 20%, the other side in increased by 20%; what change takes place in the area?' The student is not told exactly what to do, and has to develop a spatial, algebraic or numerical model of the situation in order to proceed. She might decide that this is about representing the changed lengths in terms of the old lengths, and that these lengths have to be multiplied to understand what happens to the area. She might ascribe some arbitrary numbers to help her do this, or some letters, or she might realise that these are not really relevant – but this realisation is quite sophisticated. Alternatively she might decide that this is an empirical problem and generate several numerical examples, then using inductive reasoning to give a general answer.

4 *Exploratory situations in which there is an ill-defined problem,* and the learner has to mathematise by identifying variables and conjecturing relationships, choosing likely representations and techniques. Knowledge of a range of possible functions may be helpful, as is mental or graphical imagery.

   In these situations the problem might have been posed as either quasi-abstract or situated. There may be no solution, for example: 'Describe the advantages and disadvantages of raising the price of cheese rolls at the school tuck shop by 5p, given that cheese prices have gone down by 5% but rolls have gone up by 6p each'. Students may even have posed the entire situation themselves. They have to treat this as a real situation, a real problem for them, and might use statistical, algebraic, logical or *ad hoc* methods.

5 *Mathematical problems* in which a situation is presented and a question posed for which there is no obvious method. This is what a mathematician means by 'problem' and the expected line of attack is to use the forms of enquiry and mathematical thinking specific to mathematics. For example: 'What happens to the relationship between the sum of squares of the two shorter sides of triangles and the square on the longer side if we allow the angle between them to vary?' We leave these kinds of question for the later section on mathematical thinking.

Learning about students' solution methods for elementary word problems has been a major focus in research on learning mathematics. This research focuses on two stages: translation into mathematical relations, and solution methods[3]. A synthesis can be found in Paper 4, Understanding relations and their graphical representation. It is not always clear in the research whether the aim is to solve the original problem, to become better at mathematising situations, or to demonstrate that the student can use algebra fluently or knows how to apply arithmetic.

Students have to understand that there will be several layers to working with worded problems and cannot expect to merely read and know immediately what to do. Problems in which linguistic structure matches mathematical structure are easier because they only require fluent replacement of words and numbers by algebra. For example, analysis according to cognitive load theory informs us that problems with fewer words, requiring fewer operations, and where the linguistic structure matches the mathematical structure closely, are easier for learners to solve algebraically (Kintsch, 1986), but this is tautologous as such problems are *necessarily* easier since they avoid the need for interpretation and translation. Such interpretation may or may not be related to mathematical understanding. This research does, however, alert us to the need for students to learn how to tackle problems which do not translate easily – simply knowing what to do with the algebraic representation is not enough. In Paper 4, Understanding relations and their graphical representation, evidence is given that rephrasing the words to make meaning more clear might hinder learning to transform the mathematical relationships in problems.

Students might start by looking at the numbers involved, thinking about what the variables are and how they relate, or by thinking of the situation and what they expect to happen in it. Whether the choice of approach is appropriate depends on curriculum aims, and this observation will crop up again and again in this chapter. It is illustrated in the assumptions behind the work of Bassler, Beers and Richardson (1975). They compared two approaches to teaching 15-year-olds how to solve verbal problems, one more conducive to constructing equations and the other more conducive to grasping the nature of the problem. Of course, different emphases in teaching led to different outcomes in the ways students approached word problems. If the aim is for students to construct symbolic equations, then strategies which involve identification of variables and relationships and understanding how to express them are the most appropriate. If the aim is for learners to solve the problem by whatever method then a more suitable approach might be for them to imagine the situation and choose from a range of representations (graphical, numerical, algebraic, diagrammatic) possibly shifting between them, which can be manipulated to achieve a solution.

Clements (1980) and others have found that with elementary students reading and comprehension account for about a quarter of the errors of lower achieving students. The initial access to such problems is therefore a separate issue before students have to anticipate and represent (as Boero (2001) describes the setting-up stage) the mathematics they are going to use. Ballew and Cunningham (1982) with a sample of 217 11-year-olds found that reading and computational weaknesses were to blame for difficulties alongside interpretation – but they may have underestimated the range of problems lurking within 'interpretation' because they did not probe any further than these two variables and the links between reading, understanding the relations, and deciding what to compute were not analysed. Verschaffel, De Corte and Vierstraete (1999) researched the problem-solving methods and difficulties experienced by 199 upper primary students with nine word problems which combined ordinal and cardinal numbers. Questions were carefully varied to require different kinds of interpretation. They found, among other characteristics, that students tended to choose operations according to the relative size of the numbers in the question and that choice of formal strategies tended to be erroneous while informal strategies were more likely to be correct. Interpretation therefore depends on understanding operations sufficiently to realize where to apply them, recognizing how variables are related, as well as reading and computational accuracy. Success also involves visualising, imagining, identifying relationships between variables. All these have to be employed before decisions about calculations can be made. (This process is described in detail in Paper 2 for the case of distinguishing between additive and multiplicative relations.) Then learners have to know which variable to choose as the independent variable, recognise how to express other variables in relation to it, have a repertoire of knowledge of operations and functions to draw on, and think to draw on them. Obviously elementary arithmetical skills are crucial, but automatisation of procedures only aids solution if the structural class is properly identified in the first place. Automatisation of techniques can hinder solution of problems that are slightly different to prior experience because it can lead to over-generalisation and misapplication, and attention to language and layout cues rather than the structural meaning of the stated problem. For example, if learners have decided that 'how many…?' questions always indicate a need to use multiplication (as in 'If five children have seven sweets each, how many do they have in total?') they may find it hard to answer the question 'If 13 players drink 10 litres of cola, how many should I buy for 22 players?' because the answer is not a straightforward application of multiplication. The 'automatic' association of 'how many' with a multiplication algorithm, whether it is taught or whether learners have somehow devised it for themselves, would lead to misapplication.

Learners may not know how and when to bring other knowledge into play; they may not have had enough experience of producing representations to think to use them; the problem may offer a representation (e.g. diagram) that does not for them have meaning which can match to the situation. If they cannot see what to do, they may decide to try possible numbers and see what happens. A difficulty with successive approximation is that young learners often limit themselves to natural numbers, and do not develop facility with fractions which appear as a result of division, nor with decimals which are necessary to deal with 'a little bit more than' and 'a little bit less than'. An area which is well-known to teachers but is under-researched is how learners shift from thinking about only about natural numbers in trial-and-adjustment situations.

Caldwell and Goldin (1987) extended what was already well-known for primary students into the secondary phase, and found that abstract problems were, as for primary, significantly harder than concrete ones for secondary students in general, but that the differences in difficulty became smaller for older students. 'Concrete problems' were those couched in terms of material objects and realistic situations, 'abstract' problems were those which contained only abstract objects and/or symbols. They analysed the scripts of over 1000 students who took a test consisting of 20 problems designed along the concrete-abstract dimension in addition to some other variables. Lower secondary students succeeded on 55% of the concrete problems and 43% abstract, whereas higher secondary students succeeded on 69% of concrete and 66% of abstract. Whether the narrowing of the gap is due to teaching (as Vygotsky might suggest) or natural maturation (as some interpretations of Piaget might suggest) we do not know. They also found that problems which required factual knowledge are easier than those requiring hypotheses for secondary students, whereas for primary students the reverse appeared to be true. This shift might be due to adolescents being less inclined to enter imaginary situations, or to adolescents knowing more facts, or it may be educative due to the emphasis teachers put on factual rather than imaginative mathematical activity. However, it is too simplistic to say 'applying facts is easy'. In this study, further analysis suggests that the questions posed may not have been comparable on a structural measure of difficulty, number of variables and operations for example, although comparing 'level of difficulty' in different question-types is not robust.

In a well-replicated result, the APU sample of 15-year-olds found area and perimeter problems equally hard both in abstract and diagrammatic presentations (Foxman *et al.* 1985). A contextual question scored 10% lower than abstract versions. The only presentation that was easier for area was 'find the number of squares in…' which virtually tells students to count squares and parts of squares. In a teaching context, this indication of method is not necessarily an over-simplification. Dickson's study of students' interpretation of area (in four schools) showed that, given the square as a measuring unit, students worked out how to evaluate area and in then went on to formalise their methods and even devise the rectangle area formula themselves (1989).

The research findings are therefore inconclusive about shifts between concrete and abstract approaches which can develop in the normal conditions of school mathematics, but the role of pedagogy indicates that more might be done to support abstract reasoning and hypothesising as important mathematical practices in secondary school.

Hembree's meta-analysis (1992) of 487 studies of problem-solving gives no surprises – the factors that contribute to success are:

- that problems are fully stated with supportive diagrams
- that students have previous extensive experience in using the representations used
- that they have relevant basic mathematical skills to use
- that teachers who understand problem-solving methods are better at teaching them
- that heuristics might help in lower secondary.

Hembree's analysis seems to say that learners get to the answer easiest if there is an obvious route to solution. While Hembree did a great service in producing this meta-analysis, it fails to help with the questions: How can students learn to create their own representations and choose between them? How can students learn to devise new methods to solve new problems? How can students learn to act mathematically in situations that are not fully defined?

How do students get the experience that makes them better at problem-solving? An alternative approach is to view problem-solving as far from clear-cut and instead to see each problem as a situation requiring modelling (see next section).

Summary

To solve problems posed for pedagogic purposes, secondary mathematics learners have to:

- be able to read and understand the problem
- know when they are expected to use formal methods
- know which methods to apply and in what order and how to carry them out
- identify variables and relationships, choosing which variable to treat as independent
- apply appropriate knowledge of situations and operations
- use mental, graphical and diagrammatic imagery
- choose representations and techniques and know how to operate with them
- know a range of useful facts, operations and functions
- decide whether to use statistical, algebraic, logical or *ad hoc* methods.

Modelling

In contrast to 'problem-solving' situations in which the aim and purpose is often ambiguous, modelling refers to the process of expressing situations in conventional mathematical representations which afford manipulation and exploration. Typically, learners are expected to construct an equation, function or diagram which represents the variables in the situation and then, perhaps, solve an equation or answer some other related question based on their model[4]. Thus modelling presents many of the opportunities and obstacles described under 'problem-solving' above but the emphasis of this section is to focus on the identification of variables and relationships and the translation of these into representations. Carpenter, Ansell, Franke, Fennema and Weisbeck (1993) show that even very young children can do far more sophisticated quantitative reasoning when modelling situations for themselves than is expected if we think of it solely as application of known operations, because they bring their knowledge of acting in similar situations to bear on their reasoning.

A typical modelling cycle involves representing a realistic situation in mathematical symbols and then using isomorphism between the model and the situation, manipulate variables either in the model or the situation and observe how such transformations re-translate between the model and the situation. This duality is encapsulated in the ideas of *model-of* and *model-for*. The situation is an instantiation of an abstract model. The abstract model becomes a model-for being used to provide new insights and possibilities for the original situation. This isomorphic duality is a more general version of Vergnaud's model described in Paper 4, Understanding relations and their graphical representation. For learners, the situation can provide insight into possibilities in the mathematics, or the mathematics can provide insights into the situation. For example, a graphical model of temperature changes can afford prediction of future temperatures, while actual temperature changes can afford understanding of continuous change as expressed by graphs.

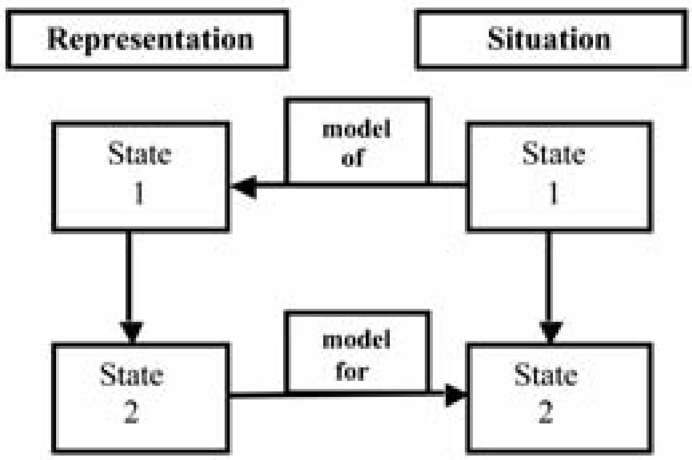

Figure 7.1: Typical modelling cycle with two-way relationship between situation and representations.

Research literature in this area gives primacy to different features. We are limited to looking at teaching experiments which are necessarily influenced by particular curriculum aims. Either the research looks at the learning of functions (that is extending the learners' repertoire of standard functions and their understanding of their features and properties) and sees modelling, interpreting and reifying functions as components of that learning (e.g. O'Callaghan, 1998), or the research sees skill in the modelling process as the goal of learning and sees knowledge of functions (their types and behaviour) as an essential component of that. In either approach there are similar difficulties. O'Callaghan, (1998) using a computer-intensive approach, found that while students did achieve a better understanding of functions through modelling than comparable students pursuing a traditional 'pure' course, and were more motivated and engaged in mathematics, they were no better at reifying what they had learnt than the traditional students. In pre- and post-tests students were asked to: model a situation using a function; interpret a function in a realistic situation; translate between representations; and use and transform algebraic functions which represent a financial situation. Students' answers improved in all but the last task which required them to understand the role of variables in the functions and the relation between the functions. In other words, they were good at modelling but not at knowing more about functions as objects in their own right.

MacGregor and Stacey (1993) (281 lower-secondary students in free response format and 1048 similar students who completed a multiple-choice item) show that the relationship between words, situations and making equations is not solely one of translating into symbols and correct algebra, rather it involves translating what is read into some kind of model developed from an existing schema and then representing the model – so there are two stages at which inappropriate relationships can be introduced, the mental model and the expressions of that model. The construction of mental models is dependent on:

- what learners know of the situation and how they imagine it
- how this influences their identification of variables, and
- their knowledge of possible ways in which variables can vary together.

What is it that students can see? Carlson and colleagues (2002) investigated students' perceptions and images of covariation, working mainly with undergraduates. The task is to work out how one variable varies in relation to another variable. Their findings have implications for younger students, because they find that their students can construct and manipulate images of how a dependent variable relates to the independent variable in dynamic events, such as when variation is positional, or visually identifiable, or can be seen to increase or decrease relative to the dependent variable, but the rate at which it changes change is harder to imagine . For our purposes, it is important to know that adolescents *can* construct images of relationships, but O'Callaghan's work shows that more is required for this facility to be used to develop knowledge of functions. When distinguishing between linear and quadratic functions, for example, rate of change is a useful indicator instead of some particular values, the turning point or symmetrical points, which may not be available in the data.

Looking at situations with a mathematical perspective is not something that can be directly taught as a topic, nor does it arise naturally out of school mathematical learning. Tanner and Jones (1994) worked with eight schools introducing modelling to their students. Their aim was not to provide a vehicle to learning about functions but to develop modelling skills as a form of mathematical enquiry. They found that modelling had to be developed over time so that learners developed a repertoire of experience of what kinds of things to focus on. Trelinski (1983) showed that of 223 graduate maths students only 9 could construct suitable mathematical models of non-mathematical situations – it was not that they did not know the relevant mathematics, but that they had never been expected to use it in modelling tasks before. It does not naturally follow that someone who is good at mathematics and knows a lot about functions automatically knows how to develop models.

So far we have only talked about what happens when learners are asked to produce models of situations. Having a use for the models, such as a problem to solve, might influence the modelling process. Campbell , Collis and Watson (1995) extended the findings of Kouba's research (1989) (reported in Paper 4) and analysed the visual images produced and used by four groups of 16-year-olds as aids to solving problems. The groups were selected to include students who had high and low scores on a test of vividness of visual imagery, and high and low scores on a test of reasoning about mathematical operations. They were then given three problems to solve: one involving drink-driving, one about cutting a painted cube into smaller cubes and one about three people consuming a large bag of apples by successively eating 1/3 of what was left in it. The images they developed differed in their levels of generality and abstraction, and success related more to students' ability to operate logically rather than to produce images, but even so there was a connection between the level of abstraction afforded by the images, logical operational facility and the use of visually based strategies. For example, graphical visualisation was a successful method in the drink-driving problem, whereas images of three men with beards sleeping in a hut and eating the apples were vivid but unhelpful. The creation of useful mathematical images needs to be learnt. In Campbell's study, questions were asked for which a model was needed, so this purpose, other than producing the model itself, may have influenced the modelling process. Models were both 'models of' and 'models for', the former being a representation to express structures and the latter being related to a further purpose (e.g. van den Heuvel-Panhuizen, 2003). Other writers have also pointed to the positive effects of purpose: Ainley, Pratt and Nardi (2001) and Friel, Curcio and Bright(2001) all found that having a purpose contributes to students' sense-making of graphs.

Summary

- Modelling can be seen as a subclass of problem-solving methods in which situations are represented in formal mathematical ways.
- Learners have to draw on knowledge of the situation to identify variables and relationships and, through imagery, construct mathematical representations which can be manipulated further.
- There is some evidence that learners are better at producing models for which they have a further purpose.
- To do this, they have to have a repertoire of mathematical representations, functions, and methods of operation on these.
- A modelling perspective develops over time and through multiple situational experiences, and can then be applied to given problems – the processes are similar to those learners do when faced with new mathematical concepts to understand.
- Modelling tasks do not necessarily lead to improved understanding of functions without the development of repertoire and deliberate pedagogy.

Functions

For learners to engage with secondary mathematics successfully they have to be able to decipher and interpret the stimuli they are offered, and this includes being orientated towards looking for relations between quantities, noticing structures, identifying change and generalising patterns of behaviour. Kieran (1992) lists these as good approaches to early algebra. They also have to know the difference between statistical and algebraic representations, such as the difference between a bar chart and an algebraic graph.

Understanding what a function is, a mapping that relates values from one space into values in another space, is not a straightforward matter for learners. In Paper 4, Understanding relations and their graphical representation, evidence that the experience of transforming between values in the same space is different from transforming between spaces is described, and for this paper we shall move on and assume that the purpose of simple additive, scalar and multiplicative functions is understood, and the task is now to understand their nature, a range of kinds of function, their uses, and the ways in which they arise and are expressed.

Whereas in early algebra learners need to shift from seeing expressions as things to be calculated to seeing them as expressing structures, they then have to shift further to seeing functions as relations between expressions, so that functions become mathematical objects in themselves and numerical 'answers' are likely to be pairs of related values (Yerushalmy and Schwartz, 1993). Similarly equations are no longer situations which hide an unknown number, but expressions of relationships between two (or more) variables. They have also to understand the difference between a point-wise view of functional relationships (as expressed by tables of values) and a holistic view (reinforced especially by graphs).

Yerushalmy and Gilead, in a teaching experiment with lower-secondary students over a few years (1999) found that knowledge of a range of functions and the nature of functions was a good basis for solving algebraic problems, particularly those that involved rate because a graph of a function allows rates to be observed and compared. Thus functions and their graphs support the focus on rate that Carlson's students found difficult in situations and diagrams. Functions appeared to provide a bridge that turned intractable word problems into modelling tasks by conjecturing which functions might 'fit' the situation. However, their students could misapply a functional approach. This seems to be an example of the well-known phenomenon of over-generalizing an approach beyond its appropriate domain of application, and arises from students paying too much attention to what has recently been taught and too little to the situation.

Students not only have to learn to think about functional relationships (and consider non-linear relationships as possibilities), which have an input to which a function is applied generating some specific output, but they also need to think about relations between relations in which there is no immediate output, rather a structure which may involve several variables. Halford's analysis (e.g.1999) closely follows Inhelder and Piaget's (1959) theories about the development of scientific reasoning in adolescence. He calls these 'quaternary' relationships because they often relate four components appearing as two pairs. Thus distributivity is quaternary, as it involves two binary operations; proportion is quaternary as it involves two ratios. So are rates of change, in which two variables are compared as they both vary in relation to something else (their functional relation, or time, for example). This complexity might contribute to explaining why Carlson and colleagues found that students could talk about covariation relationships from graphs of situations but not rates of change. Another reason could be the opacity of the way rate of change has to be read from graphs: distances in two directions have to be selected and compared to each other, a judgement or calculation made of their ratio, and then the same process has to be repeated around other points on the graphs and the ratios compared. White and Mitchelmore (1996) found that even after explicit instruction students could only identify rates of change in simple cases, and in complex cases tried to use algebraic algorithms (such as a given formula for gradient) rather than relate quantities directly.

One area for research might be to find out whether and how students connect the 'method of differences', in which rates of change are calculated from tables of values, to graphical gradients. One of the problems with understanding functions is that each representation brings certain features to the fore (Goldin, 2002). Graphical representations emphasise linearity, roots, symmetry, continuity, gradient; domain; ordered dataset representations emphasise discrete covariation and may distract students from starting conditions; algebraic representations emphasise the structure of relations between variables, and the family of functions to which a particular one might be related. To understand a function fully these have to be connected and, further, students have to think about features which are not so easy to visualize but have to be inferred from, or read into, the representation by knowing its properties, such as growth rate (Confrey and Smith, 1994; Slavit, 1997). Confrey and Smith used data sets to invite unit-by-unit comparison to focus on rate-of-change, and deduced that rate is different from ratio in the ways that it is learnt and understood (1994). Rate depends on understanding the covariation of variables, and being able to conceptualise the action of change, whereas ratio is the comparison of quantities.

Summary

To understand the use of functions to describe situations secondary mathematics learners have to:

- distinguish between statistical and algebraic representations
- extend knowledge of relations to understanding relations between relations
- extend knowledge of expressions as structures to expressions as objects
- extend knowledge of equations as defining unknown numbers to equations as expressing relationships between variables
- relate pointwise and holistic understandings and representations of functions
- see functions as a new kind of mathematical object
- emphasize mathematical meaning to avoid over-generalising
- have ways of understanding rate as covariation.

Mathematical thinking

In this section we mention mathematical problems – those that arise in the exploration of mathematics rather than problems presented to learners for them to exercise methods or develop 'problem-solving' skills. In mathematical problems, learners have to use mathematical methods of enquiry, some of which are also used in word problems and modelling situations, or in learning about new concepts. To learn mathematics in this context means two things:

to learn to use methods of mathematical enquiry and to learn mathematical ideas which arise in such enquiry.

Descriptions of what is entailed in mathematical thinking are based mainly on Polya's work (1957), in which mathematical thinking is described as a holistic habit of enquiry in which one might draw on any of about 70 tactics to make progress with a mathematical question. For example, the tactics include make an analogy, check a result, look for contradictions, change the problem, simplify, specialise, use symmetry, work backwards, and so on. Although some items in Polya's list appear in descriptions of problem-solving and modelling tactics, others are more likely to be helpful in purely mathematical contexts in which facts, logic, and known properties are more important than merely dealing with current data. Cuoco, Goldenberg and Mark (1997) have devised a typography of aspects of mathematical habits of mind. For example, mathematicians look at change, look at stability, enjoy symbolisation, invent, tinker, conjecture, experiment, relate small things to big things, and so on. The typography encompasses the perspectives which experts bring to bear on mathematics – that is they bring ideas and relationships to bear on situations rather than merely use current data and specific cases. Both of these lists contain dozens of different 'things to do' when faced with mathematics. Mason, Burton and Stacey (1982) condensed these into 'specialise-generalise; conjecture-convince' which focuses on the shifts between specific cases and general relationships and properties, and the reasoning shift between demonstrating and proving. All of these reflect the processes of mathematical enquiry undertaken by experienced mathematicians. Whereas in modelling there are clear stages of work to be done, 'mathematical thinking' is not an ordered list of procedures, rather it is a way of describing a cast of mind that views any stimulus as an object of mathematical interest, encapsulating relationships between relationships, relationships between properties, and the potential for more such relationships by varying variables, parameters and conditions.

Krutetskii (1976) conducted clinical interviews with 130 Soviet school children who had been identified as strong mathematicians. He tested them qualitatively and quantitatively on a wide range of mathematical tasks, looked for common factors in the way they tackled them, and found that those who are better at mathematics in general were faster at grasping the essence of a mathematical situation and seeing the structure through the particular surface features. They generalised more easily, omitted intermediate steps of reasoning, switched between solution methods quickly, tried to get elegant solutions, and were able to reverse trains of thought. They remembered relationships and principles of a problem and its solution rather than the details and tended to explain their actions rather than describe them. Krutetskii's methods were clinical and grounded and dependent on case studies within his sample, nevertheless his work over many years led him to form the view that such 'abilities' were educable as well as innate and drew strongly on natural propensities to reason spatially, perceptually, computationally, to make verbal analogies, mental associations with remembered experiences, and reasoning. Krutetskii, along with mathematicians reporting their own experiences, observed the need to mull, that is to leave unsolved questions alone for a while after effortful attempts, to sleep, or do other things, as this often leads to further insights when returning to them. This commonly observed phenomenon is studied in neuroscience which is beyond the scope of this paper, but does have implications for pedagogy.

Summary

- Successful mathematics learners engage in mathematical thinking in all aspects of classroom work. This means, for example, that they see what is varying and what is invariant, look for relationships, curtail or reverse chains of reasoning, switch between representations and solution methods, switch between examples and generalities, and strive for elegance.

- Mathematical 'habits of mind' draw on abilities or perception, reasoning, analogy, and mental association when the objects of study are mathematical, i.e. spatial, computational, relational, variable, invariant, structural, symbolic.

- Learners can get better at using typical methods of mathematical enquiry when these are explicitly developed over time in classrooms.

- It is a commonplace among mathematicians that mulling over time aids problem-solving and conceptualisation.

Part 2: What learners do when faced with complex situations in mathematics

In this section we collect research findings that indicate what school students typically do when faced with situations to model, solve, or make mathematical sense of.

Bringing outside knowledge to bear on mathematical problems

Real-life problems appear to *invite* solutions which are within a 'human sense' framework rather than a mathematical frame (Booth 1981). 'Wrong' approaches can therefore be seen not as errors, but as expressing a need for enculturation into what does and does not count in mathematical problem-solving. Cooper and Dunne (2000) show that in tests the appropriate use of outside knowledge and ways of reasoning, and when and when not to bring it into play, is easier for socially more advantaged students to understand than less advantaged students who may use their outside knowledge inappropriately. This is also true for students working in languages other than their first, who may only have access to formal approaches presented in standard ways. Cooper and Harries (2002) worked on this problem further and showed how typical test questions for 11- to 12-year olds could be rewritten in ways which encourage more of them to reason about the mathematics, rather than dive into using handy but inappropriate procedures. Vicente, Orrantia and Verschaffel (2007) studied over 200 primary school students' responses to word problems and found that elaborated information about the situation was much less effective in improving success than elaborating the conceptual information. Wording of questions, as well as the test environment, is therefore significant in determining whether students can or can not solve unfamiliar word problems in appropriate ways. Contrary to a common assumption that giving mathematical problems in some context helps learners understand the mathematics, analysis of learners' responses in these research studies shows that 'real-life' contexts can:

- lead to linguistic confusion
- create artificial problems that do not fit with their experience
- be hard to visualise because of unfamiliarity, social or emotional obstacles
- structure mathematical reasoning in ways which are different from abstract mathematics
- obscure the intended mathematical generalisation
- invite *ad hoc* rather than formal solution methods
- confuse students who are not skilled in deciding what 'outside' knowledge they can bring to the situation.

Clearly students (and their teachers) need to be clear about how to distinguish between situations in which everyday knowledge is, or is not, preferable to formal knowledge and how these relate. In Boaler's comparative study of two schools (1997) some students at the school, in which mathematics was taught in exploratory ways, were able to recognise these differences and decisions. However, it is also true that students' outside knowledge used appropriately might:

- enable them to visualise a situation and thus identify variables and relationships
- enable them to exemplify abstract relationships as they are manifested in reality
- enable them to see similar structures in different situations, and different structures in similar situations
- be engaged to generate practical, rather than formal, solutions
- be consciously put aside in order to perform as mathematically expected.

Information processing

In this section we will look at issues about cognitive load, attention, and mental representations. At the start of the paper we posed questions about what a learner has to do at first when faced with a new situation of any kind. Information processing theories and research are helpful but there is little research in this area within mathematics teaching except in terms of cognitive load, and as we have said before it is not helpful for cognitive load to be minimized if the aim is to learn how to work with complex situations. For example, Sweller and Leung-Martin (1997) used four experiments to find out what combinations of equations and words were more effective for students to deal successfully with equivalent information. Of course, in mathematics learning students have to be able to do both and all kinds of combinations, but the researchers did find that students who had achieved fluency with algebraic manipulations were slowed down by having to read text. If the aim is merely to do algebraic manipulations, then text is an extra load. Automaticity, such as fluency in algebraic manipulation, is achievable efficiently if differences between practice examples are minimized. Automaticity also frees up working memory for other tasks, but as Freudenthal and others have pointed out, automaticity is not always a suitable goal because it can lead to thoughtless application of methods. We would expect a learner to read text carefully if they are to choose methods meaningfully in the context. The information-processing tutors developed in the work of Anderson and his colleagues (e.g. 1995) focused mainly on mathematical techniques and processes, but included understanding the effects of such processes. We are not arguing for adopting his methods, but we do suggest that information processing has something to offer in the achievement of fluency, and the generation of multiple examples on which the learner can then reflect to understand the patterns generated by mathematical phenomena.

Most of the research on attention in mathematics education takes an affective and motivational view, which is beyond the scope of this paper (see NMAP, 2008). However, there is much that can be done about attention from a mathematical perspective. The deliberate use of variation in examples offered to students can guide their focus towards particular variables and differences. Learners have to know when to discern parts or wholes of what is offered and which parts are most critical; manipulation of variables and layouts can help direct attention. What is available to be learnt differs if different relations are emphasised by different variations. For example, students learning about gradients of straight line functions might be offered exercises as follows:

Gradient exercise 1: find the gradients between each of the following pairs of points.

(4, 3) and (8, 12)	(-2, -1) and (-10, 1)
(7, 4) and (-4, 8)	(8, -7) and (11, -1)
(6, -4) and (6, 7)	(-5, 2) and (10, 6)
(-5, 2) and (-3, -9)	(-6, -9) and (-6, -8)

Gradient exercise 2:

(4, 3) and (8, 12)	(4, 3) and (4, 12)
(4, 3) and (7, 12)	(4, 3) and (3, 12)
(4, 3) and (6, 12)	(4, 3) and (2, 12)
(4, 3) and (5, 12)	(4, 3) and (1, 12)

In the first type, learners will typically focus on the methods of calculation and dealing with negative numbers; in the second type, learners typically gesture to indicate the changes in gradient. Research in this area shows how learners can be directed towards different aspects by manipulating variables (Runesson and Mok, 2004; Chik and Lo, 2003).

Theories of mental representations claim that declarative knowledge, procedural knowledge and conceptual knowledge are stored in different ways in the brain and also draw distinctions between verbatim memory and gist memory (e.g. Brainerd and Reyna, 1993). Such theories are not much help with mathematics teaching and learning, because most mathematical knowledge is a combination of all three kinds, and in a typical mathematical situation both verbatim and gist memory would be employed. At best, this knowledge reminds us that providing 'knowledge' only in verbatim and declarative form is unlikely to help learners become adaptable mathematical problem-solvers. Learners have to handle different kinds of representation and know which different representations represent different ideas, different aspects of the same ideas, and afford different interpretations.

Summary

- Learners' attention to what is offered depends on variation in examples and experiences.
- Learners' attention can be focused on critical aspects by deliberate variation.
- Automaticity can be helpful, but can also hinder thought.
- If information is only presented as declarative knowledge then learners are unlikely to develop conceptual understanding, or adaptive reasoning.
- The form of representation is a critical influence on interpretation.

What learners do naturally that obstructs mathematical understanding

Most of the research in secondary mathematics is about student errors. These are persistent over time, those being found by Ryan and Williams (2007) being similar to those found by APU in the late 1970s. Errors do not autocorrect because of maturation, experience or assessment. Rather they are inherent in the ways learners engage with mathematics through its formal representations.

Persistence of 'child-methods' pervades mathematics at secondary level (Booth, 1981). Whether 'child-methods' are seen as intuitive, quasi-intuitive, educated, or as over-generalisations beyond the domain of applicability, the implication for teaching is that students have to experience, repeatedly, that new-to-them formal methods are more widely applicable and offer more possibilities, and that earlier ideas have to be extended and, perhaps, abandoned. If students have to adopt new methods without understanding why they need to abandon earlier ones, they are likely to become confused and even disaffected, but it is possible to demonstrate this need by offering particular examples that do not yield to child-methods. To change naïve conceptualisations is harder as the next four 'persistences' show.

Persistence of additive methods

This 'child method' is worthy of separate treatment because it is so pervasive. The negative effects of the persistence of additive methods show up again and again in research. Bednarz and Janvier (1996) conducted a teaching experiment with 135 12- to 13-year-olds before they had any algebra teaching to see what they would make of word problems which required several operations: those with multiplicative composition of relationships turned out to be much harder than those which involved composing mainly additive operations. The tendency to use additive reasoning is also found in reasoning about ratios and proportion (Hart, 1981), and in students' expectations about relationships between variables and sequential predictions. That it occurs naturally even when students know about a variety of other relationships is an example of how intuitive understandings persist even when more formal alternatives are available (Fischbein, 1987)

Persistence of more-more, same-same intuitions

Research on the interference from intuitive rules gives varied results. Tirosh and Stavy (1999) found that their identification of the intuitive rules 'more-more' and 'same-same' had a strong predictive power for students' errors and their deduction accords with the general finding that rules which generally work at primary level persist. For example, students assume that shapes with larger perimeters must have larger areas; decimals with more digits must be larger than decimals with fewer digits, and so on. Van Dooren, De Bok, Weyers and Verschaffel (2004), with a sample of 172 students from upper secondary found that, contrary to the findings of Tirosh and Stavy, students' errors were not in general due to consistent application of an intuitive rule of 'more-more' 'same-same'. Indeed the more errors a student made the less systematic their errors were. This was in a multiple-choice context, and we may question the assumption that students who make a large number of errors in such contexts are engaging in any mathematical reasoning. However, they also sampled written calculations and justifications and found that errors which looked as if they might be due to 'more-more' and 'same-same' intuitions were often due to other errors and misconceptions. Zazkis, however (1999), showed that this intuition persisted when thinking about how many factors a number might have, large numbers being assumed to have more factors.

Persistence of confusions between different kinds of quantity, counting and measuring

As well as the persistence of additive approaches to multiplication, being taught ideas and being subsequently able to use them are not immediately connected. Vergnaud (1983) explains that the conceptual field of intensive quantities, those expressed as ratio or in terms of other units (see Paper 3, Understanding rational numbers and intensive quantities), and multiplicative relationship development continues into adulthood. Nesher and Sukenik (1991) found that only 10% of students used a model based on understanding ratio after being taught to do so formally, and then only for harder examples.

Persistence of the linearity assumption

Throughout upper primary and secondary students act as if relationships are always linear, such as believing that if length is multiplied by *m* then so is area, or if the 10th term in a sequence is 32, then the 100th must be 320 (De Bock, Verschaffel and Janssens1998; Van Dooren, De Bock, Janssens and Verschaffel, 2004, 2007). Results of a teaching experiment with 93 upper-primary students in the Netherlands showed that, while linearity is persistent, a non-linear realistic context did not yield this error. Their conclusion was that the linguistic structure of word problems might invite linearity as a first, flawed response. They also found that a single experience is not enough to change this habit. A related assumption is that functions increase as the independent variable increases (Kieran, Boileau and Garancon 1996). Students' habitual ways of attacking mathematical questions and problems also cause problems.

Persistence with informal and language-based approaches

Macgregor and Stacey (1993) tested over 1300 upper-secondary students in total (in a range of studies) to see how they mathematised situations. They found that students tend first of all to try to express directly from the natural language of a situation, focusing on in equalities between quantities. Engaging with the underlying *mathematical* meaning is not a natural response. Students wanted the algebraic expression to be some kind of linguistic code, rather than a relational expression. Many researchers claim this is to do with translating word order inappropriately into symbols (such as 'there are six students to each professor, so $6s = p$') but Macgregor and Stacey suggest that the cause is more to with inadequate models of multiplicative relationships and ratio. However, it is easy to see that this is compounded by an unfortunate choice of letters as shorthand for objects rather than as variables. For example, Wollman (1983) and Clement (1982) demonstrate that students make this classic 'professor-student' error because of haste, failure to check that the meaning of the equation matches the meaning of the sentence, over-reliance on linguistic structure, use of non-algebraic symbols (such as p for professor instead of p for number of professors) and other reasons. At least one of these is a processing error which could be resolved with a 'read out loud' strategy for algebra.

Persistence of qualitative judgements in modelling

Lesh and Doerr (2003) reported on the modelling methods employed by students who had not had specific direction in what to do. In the absence of specific instructions, students repeat patterns of learning that have enabled them to succeed in other situations over time. They tend to start on each problem with qualitative judgements based on the particular context, then shift to additive reasoning, then form relationships by pattern recognition or repeated addition, and then shift to proportional and relational thinking. At each stage their students resorted to checking their arithmetic if answers conflicted rather than adapting their reasoning by seeing if answers made sense or not. This repetition of naïve strategies until they break down is inefficient and not what the most successful mathematics students do.

In modelling and problem-solving students confuse formal methods with contextual methods; they cling strongly to limited prototypes; they over-generalise; they read left-to-right instead of interpreting the meaning of symbolic expressions. In word problems they misread; miscomprehend; make errors in transformation into operations; errors in processes; and misinterpret the solution in the problem context (see also Ryan and Williams 2007).

Persistent application of procedures.

Students can progress from a manipulative approach to algebra to understanding it as a tool for problem-solving over time, but still tend over-rely on automatic procedures (Knuth, 2000). Knuth's sample of 178 first-year undergraduates' knowledge of the relationship between algebraic and graphical representations was superficial, and that they reached for algebra to do automatised manipulations rather than use graphical representations, even when the latter were more appropriate.

Summary

Learners can create obstacles for themselves by responding to stimuli in particular ways:

- persistence of past methods, child methods, and application of procedures without meaning
- not being able to interpret symbols and other representations
- having limited views of mathematics from their past experience
- confusion between formal and contextual aspects
- inadequate past experience of a range of examples and meanings
- over-reliance on visual or linguistic cues, and on application of procedures
- persistent assumptions about addition, more-more/same-same, linearity, confusions about quantities
- preferring arithmetical approaches to those based on meaning.

What learners do naturally that is useful

Students *can* be guided to explore situations in a systematic way, learning how to use a typically mathematical mode of enquiry, although it is hard to understand phenomena and change in dynamic situations. Carlson, Jacobs, Coe, Larson and Hsu. (2002) and Yerushalmy (e.g. 1997) have presented consistent bodies of work about modelling and covariation activities and their work, with that of Kaput (e.g. 1991), has found that this is not an inherently maturation problem, but that with suitable tools and representations such as those available in SimCalc children can learn not only to understand change by working with dynamic images and models, but also to create tools to analyse change. Carlson and her colleagues in teaching experiments have developed a framework for describing how students learn about this kind of co-variation. First they learn how to identify variables; then they form an image of how the variables simultaneously vary. Next, one variable has to be held still while the change in another is observed. This last move is at the core of mathematics and physics, and is essential in constructing mathematical models of multivariate situations, as Inhelder and Piaget also argued more generally.

In these supported situations, students appear to reason verbally before they can operate symbolically (Nathan and Koedinger, 2000). The usual 'order' of teaching suggested in most curricula (arithmetic, algebra, problem-solving) does not match students' development of competence in which verbal modes take precedence[5]. This fits well with Swafford and Langrall's study of ten 11-year-olds (2000) in which it was clear that even without formal teaching about algebra, students could identify variables and articulate the features of situations as equations where they were familiar with the underlying operations. Students were asked to work on six tasks in interviews. The tasks were realistic problems that could be represented by direct proportion, linear relations in a numerical context, linear relations in a geometric context, arithmetic sequences, exponential relations and inverse proportion. Each task consisted of subtasks which progressed from structured exploration of the situation, verbal description of how to find some unknown value, write an equation to express this given certain letters to represent variables, and use the equation to find out something else. The ability to express their verbal descriptions as equations was demonstrated across the tasks; everyone was able to do at least one of these successfully and most did more than one. The only situation for which no one produced an equation was the exponential one. The study also showed that, given suitably-structured tasks, students can avoid the usual assumptions of linearity. This shows some intuitive algebraic thinking, and that formal symbolisation can therefore be introduced as a tool to express relationships which are already understood from situations. Of course, as with all teaching experiments, this finding is specific to the teaching and task and would not automatically translate to other contexts, but as well as supporting the teaching of algebra as the way to express generality (see Paper 6 Algebraic reasoning) it contributes to the substantial practical knowledge of the value of starting with what students see and getting them to articulate this as a foundation for learning.

It is by looking at the capabilities of successful students that we learn more about what it takes to learn mathematics. In Krutetskii's study of such students, to which we referred earlier, (1976) he found that they exhibited what he called a 'mathematical cast of mind' which had analytical, geometric, and harmonic (a combination of the two) aspects. Successful students focused on structure and relationships rather than particular numbers of a situation. A key result is that memory about past successful mathematical work, and its associated structures, is a stronger indicator of mathematical success than memory about facts and techniques. He did not find any common aspects in their computational ability.

Silver (1981) reconstructed Krutetskii's claim that 67 lower-secondary high-achieving mathematics students remembered structural information about mathematics rather than contextual information. He asked students to sort 16 problems into groups that were mathematically-related. They were then given two problems to work on and asked to write down afterwards what they recalled about the problems. The 'writing down' task was repeated the next day, and again about four weeks later. There was a correlation between success in solving problems and a tendency to focus on underlying mathematical structure in the sorting task. In addition, students who recalled the structure of the problems were the more successful ones, but others who had performed near average on the problems could talk about them structurally immediately after discussion. The latter effect did not last in the four-week recall task however. Silver showed, by these and other similar tasks, that structural memory aided transfer of methods and solutions to new, mathematically similar, situations. A question arises, whether this is teachable or not, given the results of the four-week recall. Given that we know that mathematical strategies can be taught in general (Vos, 1976; Schoenfeld, 1979; 1982, and others) it seems likely that structural awareness might be teachable, however this may have to be sustained over time and students also need knowledge of a repertoire of structures to look for.

We also know something about how students identify relationships between variables. While many will choose a variable which has the most connections within the problem as the independent one, and tended also to start by dealing with the largest values, thus showing that they can anticipate efficiency, there are some who prefer the least value as the starting point. Nesher, Hershkovitz and Novotna, (2003) found these tendencies in the modelling strategies of 167 teachers and 132 15-year-old students in twelve situations which all had three variables and a comparative multiplication relationship with an additive constraint. This is a relatively large sample with a high number of slightly-varied situations for such studies and could provide a model for further research, rather than small studies with a few highly varied tasks.

Whatever the disposition towards identifying structures, variables and relationships, it is widely agreed that the more you know, the better equipped you are to tackle such tasks. Alexander and others (1997) worked with very young children (26 three-to five-year-olds) and found that they could reason analogically so long as they had the necessary conceptual knowledge of objects and situations to recognise possible patterns. Analogical reasoning appears to be a natural everyday power even for very young children (Holyoak and Thagard, 1995) and it is a valuable source of hypotheses, techniques, and possible translations and transformations. Construction of analogies appears to help with transfer, since seeking or constructing an analogy requires engagement with structure, and it is structure which is then sought in new situations thus enabling methods to be 'carried' into new uses. English and Sharry (1996) provide a good description of the processes of analogical reasoning: first seeing or working out what relations are entailed in the examples or instances being offered (abductively or inductively), this relational structure is extracted and represented as a model, mental, algebraic, graphical i.e. constructing an analogy in some familiar, relationally similar form. They observed, in a small sample, that some students act 'pseudo structurally' i.e. emphasising syntax hindered them seeking and recognising relational mappings. A critical shift is from focusing on visual or contextual similarity to structural similarity, and this has to be supported. Without this, the use of analogies can become two things to learn instead of one.

Past experience is also valuable in the interpretation of symbols and symbolic expressions, as well as what attracts their attention and the inter-relation between the two (Sfard and Linchevski, 1994). In addition to past experience and the effects of layout and familiarity, there is also a difference in readings made possible by whether the student perceives a statement to be operational (what has to be calculated), relational (what can be expressed algebraically) or structural (what can be generalized). Generalisation will depend on what students see and how they see it, what they look for and what they notice. Scheme-theory suggests that what they look for and notice is related to the ways they have already constructed connections between past mathematical experiences and the concept images and example spaces they have also constructed and which come to mind in the current situation. Thus generalisations intended by the teacher are not necessarily what will be noticed and constructed by students (Steele and Johanning, 2004).

Summary

There is evidence to show that, with suitable environments, tools, images and encouragement, learners can and do:

- generalise from what is offered and experienced
- look for analogies
- identify variables
- choose the most efficient variables, those with most connections
- see simultaneous variations
- observe and analyse change
- reason verbally before symbolising
- develop mental models and other imagery
- use past experience
- need knowledge of operations and situations to do all the above successfully
- particularly gifted mathematics students also:
    - quickly grasp the essence of a problem
    - see structure through surface features
    - switch between solution methods
    - reverse trains of thought
    - remembered the relationships and principles of a problem
    - do not necessarily display computational expertise.

Part 3: What happens with pedagogic intervention designed to address typical difficulties?

We have described what successful and unsuccessful learners do when faced with new and complex situations in mathematics. For this section we show how particular kinds of teaching aim to tackle the typical problems of teaching at this level. This depends on reports of teaching experiments and, as with the different approaches taken to algebra in the earlier paper, they show what it is possible for secondary students to learn in particular pedagogic contexts.

It is worth looking at the successes and new difficulties introduced by researchers and developers who have explored ways to influence learning without exacerbating the difficulties described above. We found broadly five approaches, though there are overlaps between them: focusing on development of mathematical thinking; task design; metacognitive strategies; the teaching of heuristics; and the use of ICT.

Focusing on mathematical thinking

Experts and novices see problems differently; and see different similarities and differences between problems, because experts have a wider repertoire of things to look for, and more experience about what is, and is not, worthwhile mathematically. Pedagogic intervention is needed to enable all learners to look for underlying structure or relationships, or to devise subgoals and reflect on the outcomes of pursuing these as successful students do. In a three-year course for 12- to 15-year-olds, Lamon educated learners to understand quantitative relationships and to mathematise experience by developing the habits of identifying quantities, making assumptions, describing relationships, representing relationships and classifying situations (1998). It is worth emphasising that this development of habits took place over three years, not over a few lessons or a few tasks.

- Students *can* develop habits of identifying quantities and relationships in situations, given extended experience.

Research which addresses development of mathematical thinking in school mathematics includes: descriptive longitudinal studies of cohorts of students who have been taught in ways which encourage mathematical enquiry and proof and comparative studies between classes taught in through enquiry methods and traditional methods. Most of these studies focus on the development of classroom practices and discourse, and how social aspects of the classroom influence the nature of mathematical knowledge. Other studies are of students being encouraged to use specific mathematical thinking skills, such as exemplification, conjecturing, and proof of the effects of a focus on mathematical thinking over time. These focused studies all suffer to some extent from the typical 'teaching experiment' problem of being designed to encourage X and students then are observed to do X. Research over time would be needed to demonstrate the effects of a focus on mathematical thinking on the nature of long-term learning. Longitudinal studies emphasise development of mathematical practices, but the value of these is assumed so they are outside the scope of this paper. However, it is worth mentioning that the CAME initiative appeared to influence the development of analytical and complex thinking both within mathematics and also in other subjects, evidenced in national test scores rather than only in study-specific tests (Johnson, Adhami and Shayer, 1988; Shayer, Johnson and Adhami, 1999). In this initiative teachers were trained to use materials which had been designed to encourage cycles of investigation: problem familiarity, investigating the problem, synthesising outcomes of investigation, abstracting the outcomes, applying this new abstraction to a further problem, and so on.

- Students can get better at thinking about and analysing mathematical situations, given suitable teaching.

Task design

Many studies of the complexity of tasks and the effect of this on solving appear to us to be the wrong way round when they state that problems are easier to solve if the tasks are stated more simply. For a mathematics curriculum the purpose of problem-solving is usually to learn how to mathematise, how to choose methods and representations, and how to contact big mathematical ideas – this cannot be achieved by simplifying problems so that it is obvious what to do to solve them.

Students who have spent time on complex mathematical activity, such as modelling and problem-solving, are not disadvantaged when they are tested on procedural questions against students who have had more preparation for these. This well-known result arises from several studies, such as that of Thompson and Senk (2001) in connection with the University of Chicago School Mathematics Project: those given a curriculum based on problems and a variety of exploratory activities did better on open-ended and complex, multistage tasks, than comparable groups taught in more conventional ways, and also did just as well on traditional questions. Senk and Thompson (2003) went on to collect similar results from eight mathematics teaching projects in the United States in which they looked specifically for students' development of 'basic skills' alongside problem-solving capabilities. The skills they looked for at secondary level included traditional areas of difficulty such as fractions computations and algebraic competence. Each project evaluated its findings differently, but overall the result was that students did as well or better than comparative students in basic mathematical skills at the appropriate level, and were better at applying their knowledge in complex situations. Additionally, several projects reported improved attainment for students of previously low attainment or who were 'at risk' in some sense. In one case, algebraic manipulation was not as advanced as a comparison group taught from a traditional textbook but teachers were able to make adjustments and restore this in subsequent cohorts without returning to a more limited approach. New research applying one of these curriculum projects in the United Kingdom is showing similar findings (Eade and Dickenson, 2006 *a*; Eade and Dickenson, Hough and Gough, 2006 *b*). A U.K. research project comparing two similar schools, in which the GCSE results of matched samples were compared, also showed that those who were taught through complex mathematical activity, solving problems and enquiring into mathematics, did better than students who were taught more procedurally and from a textbook. The GCSE scripts showed that the former group was more willing to tackle unfamiliar mathematics questions as problems to be solved, where the latter group tended to not attempt anything they had not been taught explicitly (Boaler, 1997). Other research also supports these results (Hembree, 1992; Watson and De Geest, 2005).

- Students who spend most of their time on complex problems can also work out how to do 'ordinary' maths questions.

Recent work by Swan (2006) shows how task design, based on introducing information which might conflict with students' current schema and which also includes pedagogic design to enable these conflicts to be explored collaboratively, can make a significant difference to learning. Students who had previously been failing in mathematics were able to resolve conflict through discussion with others in matching, sorting, relating and generating tasks. This led directly to improvements in conceptual understanding in a variety of traditionally problematic domains.

- Students can sort out conceptual confusions with others if the tasks encourage them to confront their confusion through contradiction.

Metacognitive strategies

Success in complex mathematical tasks is associated with a range of metacognitive orientation and execution decisions, but mostly with deliberate evaluating the effects of certain actions (Stillman and Galbraith, 1998). Reflecting on the effects of activity (to use Piaget's articulation) makes sense in the mathematics context, because often the ultimate goal is to understand relationships between independent and dependent variables. It makes sense, therefore, to wonder if teaching these strategies explicitly makes a difference to learning. Kramarski, Mevarech and Arami (2002) showed that explicitness about metacognitive strategies is important in success not only in complex authentic tasks but also in quite ordinary mathematical tasks. Kramarski (2004) went on to show that explicit metacognitive instruction to small groups provided them with ways to question their approach to graphing tasks. They were taught to discuss interpretations of the problem, predict the outcomes of using various strategies, and decide if their answers were reasonable. The groups who had been taught metacognitive methods engaged in discussions that were more mathematically focused, and did better on post-tests of graph interpretation and construction, than control groups. Discussion appeared to be a factor in their success. The value of metacognitive prompts also appears to be stronger if students are asked to write about their responses; students in a randomized trial tried more strategies if they were asked to write about them than those who were asked to engage in think-aloud strategies (Pugalee, 2004). In both these studies, the requirement and opportunity to express metacognitive observations turned out to be important. Kapa (2001) studied 441 students in four computer-instruction environments which offered different kinds of metacognitive prompting while they were working on mathematical questions: during the solution process, during and after the process, after the process, none at all. Those with prompts during the process were more successful, and the prompts made more difference to those with lower previous knowledge than to others. While this was an artificial environment with special problems to solve, the finding appears to support the view that teaching (in the form of metacognitive reminders and support) is important and that students with low prior knowledge can do better if encouraged to reflect on and monitor the effects of their activity. An alternative to explicit teaching and requests to apply metacognitive strategies is to incorporate them implicitly into the ways mathematics is done in classrooms. While there is research about this, it tends to be in studies enquiring into whether such habits are adopted by learners or not, rather than whether they lead to better learning of mathematics.

- Students can sometimes do better if they are helped to use metacognitive strategies.

- Use of metacognitive strategies may be enhanced: in small group discussion; if students are asked to write about them; and/or if they are prompted throughout the work.

Teaching problem-solving heuristics

The main way in which educators and researchers have explored the question of how students can get better at problem solving is by constructing descriptions of problem-solving heuristics, teaching these explicitly, and comparing the test performance of students who have and have not received this explicit teaching. In general, they have found that students do learn to apply such heuristics, and become better at problem-solving than those who have not had such teaching (e.g. Lucas, 1974). This should not surprise us.

A collection of clinical projects in the 1980s (e.g. Kantowski, 1977; Lee, 1982) which appear to show that students who are taught problem-solving heuristics get better at using them, and those who use problem-solving heuristics get better at problem solving. These results are not entirely tautologous if we question whether heuristics are useful for solving problems. The evidence suggests that they are (e.g. Webb, 1979 found that 13% of variance among 40 students was due to heuristic use), yet we do not know enough about how these help or hinder approaches to *unfamiliar* problems. For example, a heuristic which involves planning is no use if the situation is so unfamiliar that the students cannot plan. For this situation, a heuristic which involves collecting possible useful knowledge together (e.g. 'What do I know? What do I want?' Mason, Burton and Stacey , 1982) may be more useful but requires some initiative and effort and imagination to apply. The ultimate heuristic approach was probably Schoenfeld's (1982) study of seven students in which he elaborated heuristics in a multi-layered way, thus showing the things one can do while doing mathematical problem-solving to be fractal in nature, impossible to learn as a list, so that true mathematical problem-solving is a creative task involving a mathematical cast of mind (Krutetskii, 1976) and range of mathematical habits of mind (Cuoco, Goldenberg and Mark, 1997) rather than a list of processes.

Schoenfeld (1979; 1982) and Vos (1976) found that learners taught explicit problem-solving strategies are likely to use them in new situations compared to similar students who are expected to abstract processes for themselves in practice examples. There is a clear tension here between explicit teaching and the development of general mathematical awareness. Heuristics are little use without knowledge of when, why and how to use them. What is certainly true is that if learners perform learnt procedures, then we do not know if they are acting meaningfully or not. Vinner (1997) calls this 'the cognitive approach fallacy' – assuming that one can analyse learnt behavioural procedures as if they are meaningful, when perhaps they are only imitative or gap-filling processes.

Application of learnt heuristics can be seen as merely procedural if the heuristics do not require any interpretation that draws on mathematical repertoire, example spaces, concept images and so on. This means that too close a procedural approach to conceptualization and analysis of mathematical contexts is merely what Vinner calls 'pseudo'. There is no 'problem' if what is presented can be processed by heuristics which are so specific they can be applied like algorithmically. For example, finding formulae for typical spatial-numeric sequences (a common feature of the U.K. curriculum) is often taught using the heuristic 'generate a sequence of specific examples and look for patterns'. No initial analysis of the situation, its variables, and relevant choice of strategy is involved.

On the other hand, how are students to learn how to tackle problems if not given ideas about tactics and strategies? And if they are taught, then it is likely that some will misapply them as they do any learnt algorithm. This issue is unresolved, but working with unfamiliar situations and being helped to reflect on the effects of particular choices seem to be useful ways forwards.

There is little research evidence that students taught a new topic using problems with the explicit use of taught heuristics learn better, but Lucas (1974) did this with 30 students learning early calculus and they did do significantly better that a 'normal' group when tested. Learning core curriculum concepts through problems is under-researched. A recent finding reported by Kaminski, Sloutsky and Heckler (2006; 2008) is that learning procedurally can give faster access to underlying structure than working through problems. Our reading of their study suggests that this is not a robust result, since the way they categorise contextual problems and formal approaches differs from those used by the research they seek to refute.

- Students can apply taught problem-solving heuristics, but this is not always helpful in unfamiliar situations if their learning has been procedural.

One puzzle which arose in the U.S. Task Panel's review of comparative studies of students taught in different ways (NMAP, 2008) is that those who have pursued what is often called a 'problem-solving curriculum' turn out to be better at tackling unfamiliar situations using problem-solving strategies, but not better at dealing with 'simple' given word problems. How students can be better at mathematising real world problems and resolving them, but not better at solving given word problems? This comment conceals three important issues: firstly, 'word problems', as we have shown, can be of a variety of kinds, and the 'simple' kinds call on different skills than complex realistic situations; secondly, that according to the studies reported in Senk and Thompson (2003) performance on 'other aspects' of mathematics such as solving word problems may not have improved, but neither did it decline; thirdly, that interpretation of these findings as good or bad depends on curriculum aims[6]. Furthermore, the panel confined its enquiries to the U.S. context and did not take into account the Netherlands research in which the outcomes of 'realistic' activity are scaffolded towards formality. The familiar phrase 'use of real world problems' is vague and can include a range of practices.

The importance of the difference in curriculum aims is illustrated by Huntley, Rasmussen, Villarubi, Sangtog and Fey (2000) who show, along with other studies, that students following a curriculum focusing on algebraic problem solving are better at problem solving, especially with support of graphical calculators, but comparable students who have followed a traditional course did better in a test for which there were no graphical calculators available and were also more fluent at manipulating expressions and working algebraically without a context. In the Boaler (1997) study, one school educated students to take a problem-solving view of all mathematical tasks so that what students 'transferred' from one task to another was not knowledge of facts and methods but a general approach to mathematics. We described earlier how this helped them in examinations.

- There is no unique answer to the questions of why and when students can or cannot solve problems – it depends on the type of problem, the curriculum aim, the tools and resources, the experience, and what the teacher emphasises.

How can students become more systematic at identifying variables and applying operations and inverses to solve problems? One aspect is to be clear about whether the aim is for a formal method of solution or not. Another is experience so that heuristics can be used flexibly because of exposure to a range of situations in which this has to be done – not just being given equations to be solved; not just constructing general expressions from sequences; etc. The value of repeated experience might be what is behind a finding from Blume and Schoen (1988) in which 27 14-year-old students who had learnt to programme in Basic were tested against 27 others in their ability to solve typical mathematical word problems in a pen and paper environment. Their ability to write equations was no different but their ability to solve problems systematically and with frequent review was significantly stronger for the Basic group. Presumably the frequent review was an attempt to replicate the quick feedback they would get from the computer activity. However, another Basic study which had broader aims (Hatfield and Kieren, 1972) implied that strengths in problem-solving while using Basic as a tool were not universal across all kinds of mathematics or suitable for all kinds of learning goal.

A subset of common problem-solving heuristics are those that relate specifically to modelling, and modelling can be used as a problem-solving strategy. Verschaffel and De Corte (1997) working with 11-year-olds show that rather than seeing modelling-of and modelling-for as two separate kinds of activity, a combination of the two, getting learners to frame real problems as word problems through modelling, enables learners to do as well as other groups in both 'realistic' mathematical problem-solving and with word problems when compared to other groups. Their students developed a disposition towards modelling in all situational problems.

- Students may understand the modelling process better if they have to construct models of situations which then are used as models for new situations.

- Students may solve word problems more easily if they have experience of expressing realistic problems as word problems themselves.

Using ICT

Students who are educated to use available handheld technology appear to be better problem solvers. The availability of such technology removes the need to do calculations, gives immediate feedback, makes reverse checking less tedious, allows different possibilities to be explored, and gives more support for risk taking. If the purpose of complex tasks is to show assessors that students can 'do' calculations than this result is negative; if the purpose is to educate students to deal with non-routine mathematical situations, then this result is positive.

Evidence of the positive effects of access to and use of calculators is provided by Hembree and Dessart (1986) whose meta-analysis of 79 research studies showed conclusively that students who had sustained access to calculators had better pencil-and-paper and problem-solving skills and more positive attitudes to mathematics than those without. The only years in which this result was not found was grade 4 in the United States, and we assume that this is because calculator use may make students reluctant to learn some algorithmic approaches when this is the main focus of the curriculum. In the United Kingdom, these positive results were also found in the 1980s in the CAN project, with the added finding that students who could choose which method to use, paper, calculator or mental, had better mental skills than others.

We need to look more closely at why this is, what normal obstacles to learning are overcome by using technology and what other forms of learning are afforded? Doerr and Zangor (2000) recognized that handheld calculators offered speed and facility in computation, transformation of tasks, data collection and analysis, visualisation, switching representations, checking at an individual level but hindered communication between students. Graham and Thomas (2000) achieved significant success using graphical calculators in helping students understand the idea of variable. The number of situations, observation of variation, facility for experimentation, visual display, instant feedback, dynamic representation and so on contributed to this.

- Students who can use available handheld technology are better at problem solving and have more positive views of mathematics.

We do not know if it is only in interactive computer environments that school students can develop a deep, flexible and applicable knowledge of functions, but we do know that the affordances of such ICT environments allow all students access to a wide variety of examples of functions, and gives them the exploratory power to see what these mean in relation to other representations and to see the effects on one of changing the other. These possibilities are simply not available within the normal school time and place constraints without hands-on ICT. For example, Godwin and Beswetherick (2002) used graphical software to enhance learners' understanding of quadratic functions and point out that the ICT enables the learning environment to be structured in ways that draw learners' attention to key characteristics and variation. Schwarz and Hershkowitz (1999) find that students who have consistent access to such tools and tasks develop a strong repertoire of prototypical functions, but rather than being limited by these can use these as levers to develop other functions, apply their knowledge in other contexts and learn about the attributes of functions as objects in themselves.

Software that allows learners to model dynamic experiences was developed by Kaput (1999) and the integration of a range of physical situations, represented through ICT, with mental modelling encouraged very young students to use algebra to pose questions, model and solve questions. Entering algebraic formulae gave them immediate feedback both from graphs and from the representations of situations. In extended teaching experiments with upper primary students, Yerushalmy encouraged them to think in terms of the events and processes inherent in situations. The software she used emphasised change over small intervals as well as overall shape. This approach helped them to understand representations of quantities, relationships among quantities, and relationships among the representations of quantities in single variable functions (Yerushalmy, 1997). Yerushalmy claims that the shifts between pointwise and holistic views of functions are more easily made in technological environments because, perhaps, of the easy availability of several examples and feedback showing translation between graphs, equations and data sets. She then gave them situations which had more than one input variable, for example the cost of car rental which is made up of a daily rate and a mileage rate. This kind of situation is much harder to analyse and represent than those which have one independent and one dependent variable. To describe the effects of the first variable the second variable has to be invariant, and *vice versa*. In discussion, a small sample of students tried out relations between various pairs of variables and decided, for themselves, that two of the variables were independent and the final cost depended on both of them. They then tried to draw separate graphs in which one of the variables was controlled. We are not claiming that all students can do this by themselves, but that these students could do it, is remarkable. This study suggests that students for whom the ideas of variables, functions, graphs and situations are seen as connected have the skills to analyse unfamiliar and more complex situations mathematically. Nemirovsky (1996) suggests another reason is that students can relate different representations to understand the story the graph is representing. He undertook a multiple representation teaching experiment with 15- and 16-year-olds in which graphs were generated using a toy car and a motion detector. Having seen the connection between one kind of movement and the graph, students were then asked to predict graphs for other movements, showing how their telling of the story of the movement related to the graphs they were drawing. Students could analyse continuous movement that varied in speed and direction by seeing it to be a sequence of segments, then relate segments of movement to time, and then integrate the segments to construct a continuous graph. Additionally, comparing the real movement, their descriptions of it, and graphs also enabled them to correct and adjust their descriptions. Nemirovsky found that switching between these representations helped them to see that graphs told a continuous story about situations. Rather than expressing instances of distance at particular times, the students were talking about speed, an interpretation of rate from the graph rather than a pointwise use of it. Using a similar approach with nine- and ten-year-olds Nemirovsky found that these students were more likely to 'read' symbolic expressions as relationships between variables rather than merely reading them from left to right as children taught traditionally often do.

Students at all levels can achieve deep understanding of concepts and also learn relevant graphing and function skills themselves, given the power to see the effects of changes in multiple representations, taking much less time than students taught only skills and procedures through pencil-and-paper methods (Heid, 1988; Ainley and her colleagues, e.g. 1994).

- Computer-supported multiple representational contexts can help students understand and use graphs, variables, functions and the modelling process.

Recommendations

For curriculum and practice

The following recommendations for secondary mathematics teaching draw on the conclusions summarized above.

Learning new concepts

- Teaching should take into account students' natural ways of dealing with new perceptual and verbal information (see summaries above), including those ways that are helpful for new mathematical ideas and those that obstruct their learning.

- Schemes of work and assessment should allow enough time for students to adapt to new meanings and move on from earlier methods and conceptualisations; they should give time for new experiences and mathematical ways of working to become familiar in several representations and contexts before moving on.

- Choice of tasks and examples should be purposeful, and they should be constructed to help students shift towards understanding new variations, relations and properties. Such guidance includes thinking about learners' initial perceptions of the mathematics and the examples offered. Students can be guided to focus on critical aspects by the use of controlled variation, sorting and matching tasks, and multiple representations.

- Students should be helped to balance the need for fluency with the need to work with meaning.

Applications, problem-solving, modelling, mathematical thinking

- As above, teaching should take into account students' natural ways of dealing with new perceptual and verbal information (see summaries above), including those ways that are helpful for new mathematical ideas and those that obstruct their learning.

- Schemes of work should allow for students to have multiple experiences, with multiple representations, over time to develop mathematically appropriate 'habits of mind'.

- The learning aims and purpose of tasks should be clear: whether they are to develop a broader mathematical repertoire; to learn modelling and problem-solving skills; to understand the issues within the context better etc.

- Students need help and experience to know when to apply formal, informal or situated methods.

- Students need a repertoire of appropriate functions, operations, representations and mathematical methods in order to become good applied mathematicians. This can be gained through multiple experiences over time.

- Student-controlled ICT supports the development of knowledge about mathematics and its applications; student-controlled ICT also provides authentic working methods.

For policy

- These recommendations indicate a training requirement based on international research about learning, rather than merely on implementation of a new curricula.

- There are resource implications about the use of ICT. Students need to be in control of switching between representations and comparisons of symbolic expression in order to understand the syntax and the concept of functions. The United Kingdom may be lagging behind the developed world in exploring the use of spreadsheets, graphing tools, and other software to support application and authentic use of mathematics.

- The United Kingdom is in the forefront of new school mathematics curricula which aim to prepare learners better for using mathematics in their economic, intellectual and social lives. Uninformed teaching which focuses only on methods and test-training is unlikely to achieve these goals.

- Symbolic manipulators, graph plotters and other algebraic software are widely available and used to allow people to focus on meaning, application and implications. Students should know how to use these and how to incorporate them into mathematical explorations and extended tasks.

- A strong message emerging about learning mathematics at this level is that students need multiple experiences over time for new-to-them ways of thinking and working to become habitual.

For research

- There are few studies focusing on the introduction of specific new ideas, based on students' existing knowledge and experience, at the higher secondary level. This would be a valuable research area. This relates particularly to topics which combine concepts met earlier in new ways, such as: trigonometry, quadratics and polynomials, and solving simultaneous equations. (There is substantial research about calculus beyond the scope of this paper.)

- There are many studies on the development of modelling and problem-solving skills, but a valuable area for research, particularly in the new U.K. context at 14–19, would be the relationship between these and mathematical conceptual development which, as we have shown above, involves similar – not separate – learning processes if it is to be more than trial-and-error.

- There is little research which focuses on the technicalities of good mathematics teaching, and it would be valuable to know more about: use of imagery, the role of visual and verbal presentations, development of mathematical thinking, development of geometrical reasoning, how representations commonly used in secondary mathematics influence learning, and how and why some students manage to avoid over-generalising about facts, methods, and approaches.

- There is very little research on statistical reasoning, non-algebraic modelling, and learning mathematics with and without symbolic manipulators.

Endnotes

1 'Induction' here is the process of devising plausible generalisations from several examples, not mathematical inductive reasoning.

2 They claimed that the post-test was contextual because objects were used, but the relations between the objects were spurious so the objects functioned as symbols rather than as contextual tools.

3 There a little research on interpreting problems in statistical terms, but this is beyond the scope of this paper.

4 Modelling has other meanings as well in mathematics education, such as the provision or creation of visual and tactile models of mathematical ideas, but here we are sticking to what mathematicians mean by modelling.

5 Also, as is recognised in the Realistic Mathematics Education and some other projects, students are able to engage in ad hoc problem solving from a young age.

6 Meta-analysis of the studies they used is beyond the scope of this review.

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References

Ainley, J. (1994) Building on children's intuitions about line graphs. In J. da Ponte and J. Matos (eds.) *Proceedings of the 18th annual conference of the International Group for the Psychology of Mathematics Education* vol. 1, pp. 1-8, University of Lisbon, Portugal.

Ainley, J., Pratt, D., and Nardi, E. (2001). Normalising: children's activity to construct meanings for trend.' *Educational Studies in Mathematics*, 45 (1-3), 131-146.

Alexander, P., White, C. and Daugherty, M. (1997) Analogical reasoning and early mathematics learning. Pp. 117-148. In English (ed.) *Mathematical reasoning: analogies, metaphors and images*. Mahwah, NJ: Erlbaum.

Anderson, J. R., Corbett, A. T., Koedinger, K., and Pelletier, R. (1995). Cognitive tutors: Lessons learned. *The Journal of Learning Sciences, 4,* 167-207.

Ballew, Hunter; Cunningham, James W. (1982) Diagnosing strengths and weaknesses of sixth-grade students in solving word problems *Journal for Research in Mathematics Education* 13(3) 202-210

Bassler, O., Beers, M. and Richardson, L. (1975) Comparison of two instructional strategies for teaching the solution of verbal problems *Journal for Research in Mathematics Education* 6(3)170-177.

Bednarz, N. and Janvier, B. (1996). Emergence and development of algebra as a problem solving tool: Continuities and discontinuities with arithmetic. In N. Bednarz, C. Kieran and L. Lee, L. (Eds.), *Approaches to Algebra* (pp. 115-136). Dordrecht/Boston/London: Kluwer Academic Publishers.

Blume, G. and Schoen, H. (1988).Mathematical Problem-Solving Performance of Eighth-Grade Programmers and Nonprogrammers *Journal for Research in Mathematics Education* 19(2) 142-156

Boaler, J. (1997) *Experiencing School Mathematics: Teaching Styles, Sex and Setting.* Open University Press: Buckingham, England

Bock, D. De; Verschaffel, L.; Janssens, D. (1998) The Predominance of the Linear Model in Secondar y School Students' Solutions of Word Problems Involving Length and Area of Similar Plane Figures *Educational Studies in Mathematics* 35(1) 65-83

Boero, P. (2001). Transformation and Anticipation as Key Processes in Algebraic Problem Solving. In Sutherland, R., Rojano, T., Bell, A. and Lins, R.(eds.) *Perspectives on School Algebra*. pp. 99-119 Dordrecht, Kluwer.

Booth, Lesley R.(1981) Child-Methods in Secondar y Mathematics *Educational Studies in Mathematics* 12(1) 29-41

Brainerd, C. J., and Reyna, V. F. (1993). Memory independence and memory interference in cognitive development. *Psychological Review*, 100, 42-67.

Caldwell, J., Goldin, G. (1987) Variables Affecting Word Problem Difficulty in Secondary School Mathematics *Journal for Research in Mathematics Education* 18(3) 187-196.

Campbell, K.; Collis, K.; Watson, J. (1995) Visual Processing during Mathematical Problem Solving *Educational Studies in Mathematics* 28(2) 177-194

Carlson, M., Jacobs, S., Coe, E., Larsen, S. and Hsu, E. (2002) Applying Covariational Reasoning While Modeling Dynamic Events: A Framework and a Study *Journal for Research in Mathematics* 33(5) 352-378

Carpenter, T.; Ansell, E.; Franke, M.; Fennema, E.; Weisbeck, L. (1993) Models of Problem Solving: A Study of Kindergarten Children's Problem-Solving Processes *Journal for Research in Mathematics Education* 24(5) 428-441

Chik, P. and Lo. M. (2003) Simultaneity and the Enacted Object of Learning. In F. Marton and A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp.89-110) Mahwah, New Jersey: Lawrence

Clement, J. (1982) Algebra Word Problem Solutions: Thought Processes Underlying a Common Misconception *Journal for Research in Mathematics Education* 13(1) 16-30

Clements, M. (1980) Analyzing children's errors on written mathematical tasks. *Educational Studies in Mathematics*, 11 (1), 1-21

Confrey, J. and Smith, E. (1994) Exponential Functions, Rates of Change, and the Multiplicative Unit *Educational Studies in Mathematics* 26(3) 135-164

Cooper, B. and Dunne, M) (2000) *Assessing Children's Mathematical Knowledge: Social Class, Sex and Problem-Solving*, Maidenhead: Open University Press

Cooper, B. & Harries, T. (2002) Children's Responses to Contrasting 'Realistic' Mathematics Problems: Just How Realistic Are Children Ready to Be? *Educational Studies in Mathematics* 49(1) 1-23

Cuoco, A., Goldenberg, E. P., and J. Mark. (1997). 'Habits of Mind: an organizing principle for mathematics curriculum.' *Journal of Mathematical Behavior*, 15(4), 375-402.

Dickson, L. (1989) Equations. In D. Johnson (ed.) *Children's Mathematical Frameworks 8 to 13: a study of classroom teaching.* (p.151-190) Windsor: NFER-nelson

Doerr, H. and Zangor, R. (2000) Creating meaning for and with the graphing calculator *Educational Studies in Mathematics* 41(2) 143-163

Eade, F. and Dickinson, P. (2006a) *Exploring Realistic Mathematics Education in English Secondary Schools*. Proceedings PME July 2006

Eade, F., Dickinson, P., Hough S. and Gough S (2006b) *Developing Mathematics through Contexts: year 2*. Annual Report September 2006 www.partnership.mmu.ac.uk/cme

English, L. and Halford, G. (1995) *Mathematics Education: Models and Processes*. Mahwah, NJ: Lawrence Erlbaum Associates

English, L. and Sharry, P. (1996) Analogical Reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics*, Vol. 30, No. 2.135-157.

Fischbein, E. (1987) *Intuition in science and mathematics : an educational approach*. Dordrecht; D. Reidel

Foxman, D., Ruddock, G., Joffe, L., Mason, K., Mitchell, P., and Sexton, B. (1985) *A Review of Monitoring in mathematics 1978 to 1982,* Assessment of Performance Unit, Department of Education and Science, London.

Freudenthal, H. (1991) *Didactical Phenomenology of Mathematical Structures*, Dordrecht: D. Reidel.

Friel, S., Curcio, F. and Bright, G. (2001) Making Sense of Graphs: Critical Factors Influencing Comprehension and Instructional Implications *Journal for Research in Mathematics Education* 32(2), 124-158.

Godwin,S. and Beswetherick, R. (2002) Reflections on the role of task str ucture in learning the properties of quadratic functions with ICT - a design initiative. In S. Pope (ed.) *Proceedings of the British Society for Research into Learning Mathematics day conference* 16 November, Nottingham University, British Society for Research into Learning Mathematics. pp. 43-48.

Goldin, G. (2002) Representation in mathematical learning in English, L (ed.) *Handbook of international research in mathematics education* (pp. 197-218) Mahwah, NJ: Erlbaum.

Graham, A. and ; Thomas, M. (2000) Building a versatile understanding of algebraic variables with a graphic calculator *Educational Studies in Mathematics* 41(3) 265-282

Gravemeijer, K. and Doorman, M. (1999) Context Problems in Realistic Mathematics Education: A Calculus Course as an Example *Educational Studies in Mathematics* 39(1-3) 111-129

Halford, G. (1999) The Development of Intelligence Includes the Capacity to Process Relations of Greater Complexity.in M. Anderson (ed.) *The Development of Intelligence* pp. 193-213 Hove: Psychology Press

Hart, K. (Ed.) (1981) *Children's Understanding of Mathematics 11-16*, Murray, London..

Hatfield, L. and Kieren, T. (1972) Computer-Assisted Problem Solving in School Mathematics *Journal for Research in Mathematics Education* 3(2) 99-112

Heid, M. (1988) Resequencing Skills and Concepts in Applied Calculus Using the Computer as a Tool *Journal for Research in Mathematics Education* 19(1) 3-25

Hembree, Ray (1992) Experiments and Relational Studies in Problem Solving: A Meta-Analysis *Journal for Research in Mathematics Education* 23(3) 242-273

Hembree, Ray; Dessart, Donald J. (1986) Effects of Hand-Held Calculators in Precollege Mathematics Education: A Meta-Analysis *Journal for Research in Mathematics Education* 17(2) 83-99

Hershkowitz, R., Schwarz, B.and Dreyfus, T. (2001) Abstraction in Context: Epistemic Actions. *Journal for Research in Mathematics Education*, 32 (2) 195-222

Heuvel-Panhuizen, M. van den (2003) The Didactical Use of Models in Realistic Mathematics Education: An Example from a Longitudinal Trajectory on Percentage *Educational Studies in Mathematics* 54(1) 9-35

Holyoak, K. and Thagard, P. (1995) *Mental Leaps: Analogy in Creative Thought.* Cambridge, MA: Bradford Book, MIT Press

Huntley, M., Rasmussen, C., Villarubi, R., Sangtong, J. and Fey, J. (2000) Effects of Standards-Based Mathematics Education: A Study of the Core-plus Mathematics Project Algebra and Functions Strand *Journal for Research in Mathematics Education* 31(3) 328-361

Inhelder, B. & Piaget, J. (1959). *The Growth of Logical Thinking From Childhood to Adolescence*. NY: Basic Books

Johnson, D.C., Adhami, M. and Shayer, M. (1998). Does CAME work? Summary report on phase 2 of the Cognitive Acceleration in Mathematics Education, CAME, project. In *Proceedings of the British Society for Research into Learning Mathematics Day Conference, Bristol, 15 Nov. 1997*. London: BSRLM, 26-31.

Kaminski, J., Sloutsky, V. and Heckler, A. (2006) Do Children Need Concrete Instantiations to Learn an Abstract Concept? *Proceedings of the XXVIII Annual Conference of the Cognitive Science Society*, 1167-1172. Mahwah, NJ: Erlbaum

Kaminski, J., Sloutsky, V. and Heckler, A. (2008) The Advantage of Abstract Examples in Learning Math *Science* 320 (454-455).

Kantowski, M. (1977) Processes Involved in Mathematical Problem Solving *Journal for Research in Mathematics Education* 8(3) 163-180

Kapa, E. (2001) A Metacognitive Support during the Process of Problem Solving in a Computerized Environment *Educational Studies in Mathematics* 47(3) 317-336

Kaput, J. (1999) *Teaching and Learning a New Algebra With Understanding* Dartmouth: University of Massachusetts.

Kaput, J. J. (1991). Notations and representations as mediators of constructive processes. In E. von Glasersfeld (Ed.), *Radical Constructivism in Mathematics Education* (pp. 53-74). Netherlands: Kluwer Academic Publishers.

Kieran, C. (1992). The learning and teaching of algebra, in D.A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning,* Macmillan, New York, p390-419.

Kieran, C., Boileau, A.and Garancon, M. (1996) Introducing algebra by means of a technology-supported, functional approach. In Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. Pp. 257-294 Kluwer, Dordrecht

Kintsch, W. (1986) Learning from text. *Cognition and Instruction* 3(2).

Knuth, E. J. (2000). Student understanding of the cartesian connection: An exploratory study. *Journal for Research in Mathematics Education,* 31(4), 500-508.

Kouba, V. L. (1989). Children's solution strategies for equivalent set multiplication and division problems. *Journal for Research in Mathematics Education, 20*, 147-158.

Kramarski, B. (2004) Making sense of graphs: Does metacognitive instruction make a difference on students' mathematical conceptions and alternative conceptions? *Learning and Instruction* 14, 593-619.

Kramarski, B., Mevarech, Z. and Arami, M. (2002) The Effects of Metacognitive Instruction on Solving Mathematical Authentic Tasks *Educational Studies in Mathematics* 49(2) 225-250

Krutetskii, j, V. (1976). *The psychology of mathematical abilities in school children.* Chicago: University of Chicago Press.

Lamon, S. (1998) Algebra: Meaning through modeling. In A. Oliver and K. Newstead (eds.) Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education, vol. 3, pp.167-174, University of Stellenbosch: Stellenbosch.

Lee, K. (1982) Fourth Graders' Heuristic Problem-Solving Behavior *Journal for Research in Mathematics Education* 13(2) 110-123

Lesh, R. and Doerr, H. (2003). Foundations of a models and modelling perspective on mathematics teaching, learning and problem solving. In R. Lesh and H. Doerr (Eds.) *Beyond constructivisim: A models and modeling perspectives on mathematics problem solving, learning and teaching.*. Mahwah, NJ: Lawrence Earlbaum.

Lucas, J. (1974) The Teaching of Heuristic Problem-Solving Strategies in Elementary Calculus *Journal for Research in Mathematics Education* 5(1) 36-46.

MacGregor, M. and Stacey, K. (1993) Cognitive Models Underlying Students' Formulation of Simple Linear Equations *Journal for Research in Mathematics Education,* Vol. 24, No. 3.,217-232.

Mason, J. Burton L. and Stacey K. (1982) *Thinking Mathematically*, Addison Wesley, London.

Nathan, M. and Koedinger, K. (2000) Teachers' and Researchers' Beliefs about the Development of Algebraic Reasoning *Journal for Research in Mathematics Education* 168-190

Nemirovsky, R. (1996) Mathematical narratives. In N. Bednarz, C. Kieran and L. Lee, L. (Eds.), *Approaches to Algebra* (pp.197-223) Dordrecht/Boston/London: Kluwer Academic Publishers.

Nesher, P. and M. Sukenik (1991). The Effect of Formal Representation on the Learning of Ratio Concepts. *Learning and Instruction* 1: 161 - 175.

Nesher, P., Hershkovitz, S. and Novotna, Jarmila (2003) Situation Model, Text Base and What Else? Factors Affecting Problem Solving *Educational Studies in Mathematics* 52(2) 151-176

NMAP, National Mathematics Advisory Panel (2008) downloaded April 2008 from: http://www.ed.gov/about/bdscomm/list/mathpanel/index.html

O'Callaghan, B. (1998) Computer-Intensive Algebra and Students' Conceptual Knowledge of Functions *Journal for Research in Mathematics Education* 29(1) 21-40

Polya, G. (1957) *How to Solve It*, 2nd ed., Princeton University Press.

Pugalee, D. (2004) A Comparison of Verbal and Written Descriptions of Students' Problem Solving Processes Educational Studies in Mathematics 55(1) 27-47

Runesson, U., and Mok, I. A. C. (2004). Discernment and the Question, 'What can be learned?' In F. Marton and A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 63-87). Mahwah, New Jersey: Lawrence Erlbaum Associates, Inc., Publishers.

Ryan, J. and Williams, J. (2007) *Children's Mathematics 4 -15: learning from errors and misconceptions* Maidenhead Open University Press.

Schoenfeld, A. (1979) Explicit Heuristic Training as a Variable in Problem-Solving Performance *Journal for Research in Mathematics Education* 10(3) 173-187.

Schoenfeld, A. (1982) Measures of Problem-Solving Performance and of Problem-Solving Instruction. *Journal for Research in Mathematics Education* 13 (1) 31-49

Schwarz, B., Hershkowitz, R> (1999) Prototypes: Brakes or Levers in Learning the Function Concept? The Role of Computer Tools *Journal for Research in Mathematics Education* 30(4) 362-389

Senk, S. L. & Thompson, D. R. (*2003*). *Standards-based school mathematics curricula: What are they? What do students learn?* Mahwah, NJ: Lawrence Erlbaum

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sfard, A. and Linchevski, L. (1994) The Gains and the Pitfalls of Reification: The Case of Algebra *Educational Studies in Mathematics* 26(2-3) 191-228

Shayer, M., Johnson, D.C. and Adhami, M. (1999). Does CAME work (2)? Report on Key Stage 3 results following the use of the Cognitive Acceleration in Mathematics Education, CAME, project in Year 7 and 8. *In Proceedings of the British Society for Research into Learning Mathematics Day Conference, St. Martin's College, Lancaster, 5 June* London: BSRLM, 79-84.

Silver, E. (1981) Recall of Mathematical Problem Information: Solving Related Problems *Journal for Research in Mathematics Education* 12(1) 54-64

Slavit, D. (1997) An Alternate Route to the Reification of Function *Educational Studies in Mathematics* 33(3) 259-281

Squire, S., Davies, C. and Bryant, P. (2004) Does the cue help? Children's understanding of multiplicative concepts in different problem contexts *British Journal of Educational Psychology*, 515-532

Steele, D. and Johanning, D. (2004) A Schematic-Theoretic View of Problem Solving and Development of Algebraic Thinking *Educational Studies in Mathematics* 57(1) 65-90

Stillman, G. and Galbraith, P. (1998) Applying Mathematics with Real World Connections: Metacognitive Characteristics of Secondary Students *Educational Studies in Mathematics* 36(2) 157-195

Swafford, J. and Langrall, C. (2000).Preinstructional Use of Equations to Describe and Represent Problem Situations *Journal for Research in Mathematics Education* 31(1) 89-112

Swan, M. (2006) *Collaborative Learning in Mathematics: A Challenge to Our Beliefs and Practices* London: National Institute of Adult Continuing Education.

Sweller, J. and Leung-Martin, L. (1997) Learning from equations or words. *Instructional Science,* 25(1) 37-70.

Tall, D. O., (2004), Thinking through three worlds of mathematics, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Bergen, Norway.

Tall, D.O. & Vinner S. (1981), Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics*, 12, 151-169.

Tanner, H. and Jones, S. (1994) Using Peer and Self-Assessment to Develop Modelling Skills with Students Aged 11 to 16: A Socio-Constructive View *Educational Studies in Mathematics* 27(4) 413-431

Thompson, D. and Senk, S. (2001) The Effects of Curriculum on Achievement in Second-Year Algebra: The Example of the University of Chicago School Mathematics Project *Journal for Research in Mathematics Education* 32(1) 58-84

Tirosh, D. and Stavy, R. (1999) Intuitive rules: A way to explain and predict students' reasoning *Educational Studies in Mathematics* 38(1-3) 51-66

Treffers, A. (1987) *Three dimensions. A model of goal and theory description in mathematics instruction – The Wiskobas project.* Dordrecht: Reidel

Trelinski, G (1983) Spontaneous mathematization of rsituations outside mathematics *Educational Studies in Mathematics* 14(3) 275-284

Trzcieniecka-Schneider, I. (1993) Some remarks on creating mathematical concepts *Educational Studies in Mathematics* 24(3) 257-264.

Van Dooren, W., De Bock, D., Janssens, D. and Verschaffel, L. (2007) Pupils' over-reliance on linearity: A scholastic effect? *British Journal of Educational Psychology*, 77 (2) 307-321.

Van Dooren, W.; De Bock, D., Weyers, D. and Verschaffel, L. (2004) The predictive power of intuitive rules: A critical analysis of the impact of 'More A-More B' and 'Same A-Same B' *Educational Studies in Mathematics* 56(2/3) 179-207

Vergnaud, G. (1983). Multiplicative structures. In R. Lesh and M. Landau (Eds.), *Acquisition of mathematics concepts and processes*. New York: Academic Press.

Verschaffel, L. and De Corte, E. (1997) Teaching Realistic Mathematical Modeling in the Elementary School: A Teaching Experiment with Fifth Graders *Journal for Research in Mathematics Education* 28(5) 577-601

Verschaffel, L., De Corte, E. and Vierstraete, H. (1999) Upper Elementary School Pupils' Difficulties in Modeling and Solving Nonstandard Additive Word Problems Involving Ordinal Numbers *Journal for Research in Mathematics Education* 30(3) 265-285

Vicente, S., Orrantia, J. and Verschaffel, L. (2007) Influence of situational and conceptual rewording on word problem solving *British Journal of Educational Psychology*, 77(4) 829-848

Vinner, S. (1997) The Pseudo-Conceptual and the Pseudo-Analytical Thought Processes in Mathematics Learning *Educational Studies in Mathematics* 34(2) 97-129

Vos, K. (1976) The Effects of Three Instructional Strategies on Problem-Solving Behaviors in Secondary School Mathematics *Journal for Research in Mathematics Education* 7(5) 264-275

Watson, A. & De Geest, E. (2005) Principled teaching for deep progress: improving mathematical learning beyond methods and materials. *Educational Studies in Mathematics,* 58 (2), 209-234.

Webb, N. (1979) Processes, Conceptual Knowledge, and Mathematical Problem-Solving Ability *Journal for Research in Mathematics Education* 10(2) 83-93.

White, P. and Mitchelmore, M. (1996) Conceptual Knowledge in Introductory Calculus *Journal for Research in Mathematics Education* 27(1) 79-95.

Wollman, W. (1983) Determining the Sources of Error in a Translation from Sentence to Equation *Journal for Research in Mathematics Education* 14(3) 169-181

Yerushalmy, M. (1997) Designing representations: Reasoning about functions of two variables. *Journal for Research in Mathematics Education* 28(4), 431-466

Yerushalmy, M. and Gilead, S. (1999) Structures of Constant Rate Word Problems: A Functional Approach Analysis *Educational Studies in Mathematics* 39(1-3) 185-203

Yerushalmy, M.. and Schwartz, J. (1993), Seizing the Opportunity to Make Algebra Mathematically and Pedagogically Interesting, in T. Romberg, E. Fennema and T. Carpenter, (Eds.), *Integrating Research on the Graphical Representation of Functions*, Lawrence Erlbaum Associates, Hillsdale, New Jersey.

Zazkis, R. (1999).Intuitive rules in number theory: example of the more of A, the more of B rule implementation. *Educational Studies in Mathematics*, 40(2)197-209

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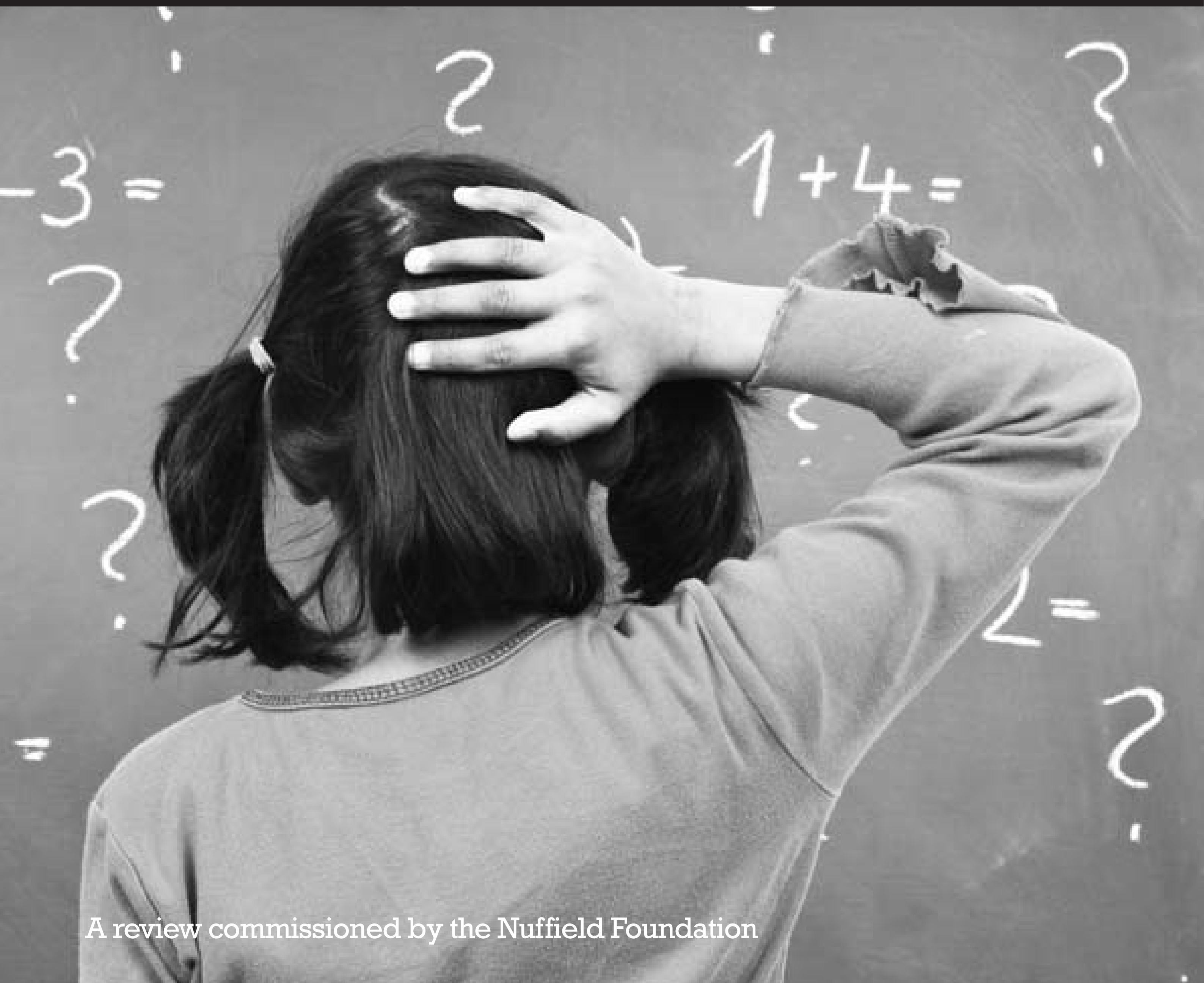
8

Key understandings in

mathematics learning

Paper 8: Methodological appendix

By Terezinha Nunes, Peter Bryant, and Anne Watson, University of Oxford



A review commissioned by the Nuffield Foundation

About this review

In 2007, the Nuffield Foundation commissioned a team from the University of Oxford to review the available research literature on how children learn mathematics. The resulting review is presented in a series of eight papers:

Paper 1: Overview
Paper 2: Understanding extensive quantities and whole numbers
Paper 3: Understanding rational numbers and intensive quantities
Paper 4: Understanding relations and their graphical representation
Paper 5: Understanding space and its representation in mathematics
Paper 6: Algebraic reasoning
Paper 7: Modelling, problem-solving and integrating concepts
Paper 8: Methodological appendix

Papers 2 to 5 focus mainly on mathematics relevant to primary schools (pupils to age 11 years), while papers 6 and 7 consider aspects of mathematics in secondary schools.

Paper 1 includes a summary of the review, which has been published separately as *Introduction and summary of findings*.

Summaries of papers 1-7 have been published together as *Summary papers*.

All publications are available to download from our website, www.nuffieldfoundation.org

Contents

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About the Nuffield Foundation

The Nuffield Foundation is an endowed charitable trust established in 1943 by William Morris (Lord Nuffield), the founder of Morris Motors, with the aim of advancing social well being. We fund research and practical experiment and the development of capacity to undertake them; working across education, science, social science and social policy. While most of the Foundation's expenditure is on responsive grant programmes we also undertake our own initiatives.

Appendix to papers 1 to 7

This review was conceived as standing between a research synthesis and a theoretical review. 'Research syntheses focus on empirical studies and seek to summarize past research by drawing overall conclusions from many separate investigations that address related or identical hypotheses. The research synthesis hopes to present the state of knowledge concerning the relation(s) of interest and to highlight important issues that research has left unresolved' (Cooper, 1998, p. 3). In a theoretical review, the aim is to present theories offered to explain a particular phenomenon and to compare them in breadth, internal consistency, and the empirical support that they find in empirical studies. 'Theoretical reviews will typically contain descriptions of critical experiments already conducted or suggested, assessments of which theory is most powerful and consistent with known relations, and sometimes reformulations or interactions or both of abstract notions from different theories.' (Cooper, 1998, p. 4).

It was quite clear to us that a review that aims to answer the question 'how children learn mathematics, ages 5 to 16' could not be treated as a straightforward research synthesis. The aim of a research synthesis is usually more restricted than this. For example, a research synthesis in education might try to examine the effect of one variable on another (e.g. the effect of reading aloud on children's literacy learning; Blok, 1999; Bus, van IJzendoorn, and Pellegrini, 1995) or the conditions under which a particular educational practice can be said to work (e.g. the effect of phonological or morphological instruction on literacy learning; Bus, and van IJzendoorn, (1999); Ehri, Nunes, Stahl, and Willows, 2001; Reed, 2008). Such searches start from previously defined variables, the incorporation of which in a study can easily be identified in a search through the literature. A review of the literature that starts with a much broader question cannot use the same conception of how the literature search will be carried out. The variables to be analysed are not conceived from the start and one of the aims of addressing such a broad question is in fact to clarify how mathematics learning could be conceptualised.

Theoretical syntheses have broader aims, which are in some ways similar to the aims adopted in this synthesis, but the current conception of theoretical syntheses can only be partially adopted in this review. Although there are occasionally alternative views of how a particular aspect of children's mathematics learning can be explained, the notion of critical experiments to assess which theory is more powerful cannot easily be met when we try to understand how children learn mathematics. The very conception of what it is that one is trying to explain varies even when the same words are used to describe the focus of the research. In the second paper in this review, we try to show exactly this. There are two alternative theories about children's understanding of number in developmental psychology but the phenomenon that they are trying to explain is not the same: Piaget's theory focuses on children's understanding of relations between quantities and Gelman's theory on children's counting skills. For older children, the problem becomes even more complex because there are alternative views of the nature and content of mathematical learning, and the role of pedagogy makes the notion of critical experiment either impossible or inapplicable. This is true of all research into secondary mathematics and reflects a change from seeing mathematics as the formalisation and extension of children's quantitative and spatial development to seeing learning mathematics as coming to understand abstract tools which can provide new formal and analytical perspectives on the world.

We did not approach this synthesis as a systematic review but as an attempt to summarise and develop some of the main ideas that are part of research and theory about how children learn mathematics. Within this perspective, we defined some inclusion and exclusion criteria from the outset.

Inclusion criteria

1 Theoretical explanations regarding how children learn mathematics which have been supported by research. There are theoretical explanations in the domain of mathematics learning which were proposed without their authors providing systematic empirical evidence. We did not consider these latter theories in the review except as frameworks to structure the approach in the absence of other explanations.

2 Research about children's mathematics learning in the age range 5 to 11 was considered when it focused on the four domains defined as the focus of this research: children's understanding of natural and rational numbers, relations between quantities and functions, and space and its representation. These were considered the cornerstones for further mathematics learning in the domains of algebra, modelling and applications to higher mathematical concepts; the focus of these two papers was on students aged 12 to 16. For algebra the available research on learning focuses on identifying typical errors, hence showing critical aspects of successful learning but not how that learning might take place. Further than this we looked at teaching experiments showing how students respond to different pedagogical approaches designed to overcome these typical difficulties. For modelling we intended to follow a similar approach but little was available except small-scale teaching experiments.

3 Research published in books and book chapters, journals and refereed conference proceedings which aim at understanding how children learn mathematics. Considering the constraints of time, the search in journals was limited to those available electronically and otherwise in the University of Oxford. A list of journals and their aims and scope is appended. The refereed conference proceedings of the International Group for the Study of the Psychology of Mathematics Education will be the only proceedings included in the review.

Exclusion criteria

1 There are domains of research, such as history of mathematics, mathematics teacher development, neuropsychological studies of adults with brain damage who have developed mathematics difficulties, and studies of mathematical abilities in animals and infants, which have not been so far connected to a theory of how children learn mathematics between 5 and 16 years. These domains of research are excluded.

2 Research that focused on learning how to use specific technologies rather than on how technologies are used by students to learn mathematics. There is a relatively large number of publications on how students learn to use particular tools that are relevant to mathematics (e.g. calculators, number line, spreadsheets, LOGO and Cabri). Considering our aim of understanding how children learn mathematics, we will only refer to research that uses these tools when the focus is on mathematics learning (e.g. using spreadsheets to help students understand the concept of variable).

We did not use methodological criteria in the choice of papers. Descriptive as well as experimental research, qualitative or quantitative studies were considered when we went through the search. In view of the brevity of the period dedicated to this synthesis, we did exclude materials that could neither be obtained by electronic means or in the libraries of the University of Oxford. There is, therefore, a bias towards papers published in English language journals, even though we could have read publications in three other languages.

The search process was systematic. We used the British Educational Index as a starting point for the search of papers in the four chapters about children in the age range 5 to 11. Three searches were carried out, one for natural and rational numbers, one for geometry and one for understanding relations and functions. We included in these searches three sets of key-words, the first defining the domain of research (mathematics education and other key words from the thesaurus), the second defining the topic area (e.g. natural number, rational number and other options from the thesaurus), and the third defining the age parameters (through schooling levels). Theses and one-page abstracts were excluded from the output list of references at this point. The references were then checked for availability and to see whether they reported research results and excluded if they were not available or did not report any research results. We repeated this search process using Psych-info, a data base which includes psychological research, which had been poorly represented in the previous data base. Finally, this initial search was complemented by a journal by journal search of the titles listed at the end of this note. This search seemed to yield mostly repeated references so we considered this the end of the process of search. We also consulted books and book chapters of works that are recognised in the literature and previous syntheses presented in the Handbook of Research on Mathematics Teaching and Learning. Two the Task Group Reports of the National Mathematics Advisory Panel, USA, were also consulted: the reports on learning processes and on conceptual knowledge. These were used as sources of references rather than for their conclusions. In the end, approximately 200 papers were downloaded and read by the authors. However, not all of these papers are cited in the chapters. The references used are those which did contribute to the development of the concepts and empirical results used in the synthesis.

For algebra, we conducted a systematic search in electronic journals in English for refereed research articles using algebra as the keyword. Journals are listed below. We did not define an age range since we were interested in how algebraic understanding develops throughout school, although this happens mainly in secondary education. We also used refereed innovation studies, which show what it is possible for learners to do, given particular kinds of teaching or technology; this tells us about possibilities. We restricted our use of these to studies for which the learning aims clearly relate to a broad view of algebra given above. For example, we did not include self-referential studies in which, for example, it is assumed that pattern-spotting is an important aspect of algebra, so teaching and learning pattern-spotting is researched, but we would for example include a study of teaching pattern-spotting where students' ability to use pattern-spotting for a higher level algebraic purpose was discussed as an outcome. We also used refereed studies of students' typical errors and methods (see below). These tell us what needs to be learnt and hence describe the development of algebraic understanding, but not how successful students learn it. We also drew on significant overviews and compilations of research on algebra. These reviews were used as gateways to other research literature. We excluded studies which focus only on short-term fluent performance of algebraic procedures in familiar situations unless this was linked specifically to the development of algebraic reasoning. Most of the studies we used base their claims to success on the complementary needs both to act fluently with symbolic expressions and to understand them. We accessed 174 papers plus 78 references in books in addition to the reviews and studies mentioned above. Of these, about 95 were read but not all are included as reference. Some of these overlapped in their conclusions, or added nothing or only a little to the main references.

For *Paper 7, Modelling, problem-solving and integrating concepts*, an initial search using U.S. and U.K. spellings gave very few relevant results. We therefore broadened the search to include: modelling, problem-solving, realistic, real-life, variable and word problems. This process was iterative as the search for explanations for what could be inferred about students' learning led us into other related areas. Later we did further searches on some other terms which emerged as important: linearity, linear assumption, equation. Finally, we searched for papers which addressed how students learned combinations of concepts which build on elementary concepts, such as trigonometry. In all we located over 3200 references using British Education Index, ERIC and other sources. Fortunately many of these were not research-based, or used the terms in irrelevant ways, or addressed the focus in limited ways related to young children. The final relevant list consisted of 125 papers and a journal special issue. We used these papers to point to other sources. Most of these papers were reports of teaching experiments. Teaching experiments usually have a particular commitment to the nature of an aspect of mathematics and how it is best learnt. The experiment is constructed to see if students will be able to do X in certain circumstances, and X is measured as an outcome but in this process knowledge of how X is learnt, and what can go wrong, can be found. In reading this literature we found an overall coherence about students' learning of higher mathematics and the final version of the paper was constructed to show these similarities. A list of journals accessed is included in this appendix. There were only four reviews of research used, two meta-analyses by Hembree, (1986; 1992) used as summaries of literature and the U.S. Task Panel (NMAP 2008) was used as a gateway to other sources.

List of journals consulted for Papers 2 to 5

British Journal of Developmental Psychology
British Journal of Educational Psychology
Child Development
Cognition and Instruction
Educational Studies in Mathematics
Eurasia Journal of Mathematics, Science and Technology Education
International Electronic Journal of Mathematics Education
International Journal for Mathematics and Learning
International Journal of Science and Mathematics Education
Journal for Research in Mathematics Education
Journal for Research in Mathematics Education Monograph
Journal of Educational Psychology
Learning and Instruction

List of journals consulted for Papers 6 and 7

British Journal of Developmental Psychology
British Journal of Educational Psychology
Child Development
Cognition and Instruction
Educational Studies in Mathematics
International Journal of Mathematical Education in Science and Technology
Journal for Research in Mathematics Education
Learning and Instruction
Mathematical Thinking and Learning
Proceedings of International Group for the Psychology of Mathematics Education

Reviews and collections used for algebra

Bednarz, N., Kieran C., Lee, L. (eds.) *Approaches to algebra: perspectives for research on teaching*. Kluwer, Dordrecht

Chick, H., Stacey, K., Vincent, J. and Vincent, J. (eds.) *Proceedings of the 12th ICMI study conference: The future of the teaching and learning of algebra*. University of Melbourne, Australia, Dec 9-14, 2001.

Greenes, C. and Rubenstein, R. (eds.) *Algebra and Algebraic Thinking in School Mathematics.* 70th Yearbook. Reston, VA: NCTM.

Kaput, J., Carraher, D. and Blanton, M. (eds.) *Algebra in the early grades.* New York: Erlbaum

Mason, J. and Sutherland, R. (2002), *Key Aspects of Teaching Algebra in Schools*, QCA, London

Nickson, M. (2000) *Teaching and learning mathematics: a teacher's guide to recent research and its applications.* London: Cassell

NMAP, National Mathematics Advisory Panel (2008) downloaded April 2008 from: http://www.ed.gov/about/bdscomm/list/mathpanel/index.html

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Large-scale studies used for Papers 6 and 7

Concepts in Secondary Mathematics and Science Project (CSMS) (see Hart *et al.,* 1981)
Diagnostic tests derived from clinical interviews with 30 children age 11 to 16. In these interviews the test items were trialled and revised, and students' own methods and typical errors were observed. Common errors and methods were found across schools which were not teacher-taught but had arisen through students' own reasoning. The sample for testing was from urban, rural and city areas across England. It was selected from volunteer schools according to IQ distributions in order to represent the country as a whole. About 3000 students took the Algebra test.

Strategies and Errors in Secondary Mathematics Project (SESM) focused on a small number of errors arising in the CSMS study. There used a large number of individual interviews and some teaching experiments involving several classes of students.

Ryan and Williams
Ryan and Williams randomly-sampled 13 000 English school children from ages 4 to 15 using diagnostic tests designed to reveal typical errors and child-methods, as CSMS, but with the express purpose of identifying progress made by students in mathematics. They found little progress made between ages 11 to 14, and that many errors were similar to those found by Hart *et al.* 20 years earlier. See Ryan, J. and Williams, J. (2007) *Children's Mathematics 4-15: learning from errors and misconceptions,* Maidenhead: Open University Press. More details of the tests can be found in *Mathematics Assessment for Learning and Teaching,* (2005) London: Hodder and Stoughton.

Mollie MacGregor and Kaye Stacey
A series of pencil and paper tests were administered to 2000 students from a representative sample of volunteer schools in Years 7–10 (ages 11 –15) in 24 Australian secondary schools.

Assessment of Performance Unit (APU) test results of 1979 (Foxman *et al*., 1981). These tests involved a cohort of 12 500 students age 11 to 15 and were designed to track development of mathematical understanding by sampling across schools and regions.

Children's Mathematical Frameworks study (CMF) (Johnson, 1989), 25 classes in 21 schools in the United Kingdom were tested to find out why and how students between 8 and 13 cling to guess-and-check and number-fact methods rather than new formal methods offered by teachers.

References for appendix

Bus, A. G., and van IJzendoorn, M. H. (1999). Phonological awareness and early reading: A meta-analysis of experimental training studies. *Journal of Educational Psychology, 91*, 403-414.

Bus, A. G., van IJzendoorn, M. H., and Pellegrini, A. D. (1995). Joint book reading makes for success in learning to read: A meta-analysis on intergenerational transmission of literacy. *Review of Educational Research, 65*, 1-21.

Blok, H. (1999). Reading to young children in educational settings: A meta-analysis of recent research. *Language Learning, 49*, 343-371.

Cooper, H. M. (1998). *Synthesizing research : a guide for literature reviews*. Thousand Oaks, CA: Sage Publications.

Ehri, L. C., Nunes, S. R., Stahl, S. A., and Willows, D. M. (2001). Systematic Phonics Instruction Helps Students Learn to Read: Evidence from the National Reading Panel's Meta-Analysis. *Review of Educational Research, 71*, 393-447.

Reed, D. K. (2008). A Synthesis of Morphology Interventions and Effects on Reading Outcomes for Students in Grades K–12. *Learning Disabilities Research and Practice, 23*, 36-49.

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